Worcester Polytechnic Institute
Mechanical Engineering Department

Design of Machine Elements
ME-3320, B’2022

Lecture 10-11
November 2022
Static failure theories

Accepted failure theories that apply to **ductile** materials:

- Total strain energy theory
- Distortion energy theory
- Pure shear-stress theory
- Maximum shear-stress theory
- Maximum normal stress theory (limited application)

Accepted failure theories that apply to **brittle** materials:

- Maximum normal stress theory (even material)
- Maximum normal stress theory (uneven material)
- Coulomb-Mohr theory
- Modified Mohr theory
Static failure theories
Ductile materials

Safety factors:

Distortion energy theory:

\[ SF = N = \frac{S_y}{\sigma} \]

Yield strength of the material
von Mises effective stress

(Obtained from)

Distortion energy theory (pure shear):

\[ SF = N = \frac{S_{ys}}{\tau_{max}} \]

Max. shear-stress

Max. shear-stress theory:

\[ SF = N = \frac{S_{ys}}{\tau_{max}} \]

Maximum shear-stress

\[ S_{ys} = 0.577S_y \]

\[ S_{ys} = 0.5S_y \]
Static failure theories
Ductile materials

\[ \sigma_2 \text{ is assumed to be the zero stress} \]

for pure torsion
\[ S_{ys} = 0.5 \ S_y \]

\textbf{FIGURE 5-5}
The 2-D Shear-Stress Theory Hexagon Inscribed Within the Distortion-Energy Ellipse
Static failure theories: experimental verifications
Ductile & brittle materials

Yielding ($\sigma_c = \sigma_y$)
- ○ Ni-Cr-Mo steel
- ● AISI 1023 steel
- □ 2024-T4 Al
- ■ 3S-H Al

Fracture ($\sigma_c = \sigma_{ut}$)
- △ Gray cast iron

**FIGURE 5-8**
Experimental Data from Tensile Tests Superposed on Three Failure Theories (Reproduced from Fig. 7.11, p. 252, in Mechanical Behavior of Materials by N. E. Dowling, Prentice-Hall, Englewood Cliffs, NJ, 1993)
Static failure theories
Brittle materials

**FIGURE 2-5**
A Tensile Test Specimen of Brittle Cast Iron After Fracture

Mohr’s circle: pure tension

**FIGURE 2-4**
Stress-Strain Curve of a Brittle Material
Static failure theories

Brittle materials

Mohr’s circle: pure shear

FIGURE 2-8
Torsion Test Specimens After Failure (a) Ductile Steel (b) Brittle Cast Iron

Pure shear condition

Pure shear condition
Static failure theories
Brittle materials: even and uneven materials

(a) An even material — $S_{uc} = -S_{ut}$

(b) An uneven material — $|S_{uc}| > |S_{ut}|$

**Figure 5-10**
Mohr’s Circles for Both Compression and Tensile Tests Showing the Failure Envelopes for (a) Even and (b) Uneven Materials
Static failure theories
Brittle materials: Coulomb-Mohr, modified-Mohr, and normal stress theories

**FIGURE 5-11**

Coulomb-Mohr, Modified-Mohr, and Maximum Normal-Stress Theories for Uneven Brittle Materials
Static failure theories: brittle materials

Coulomb-Mohr, modified-Mohr, and normal stress theories

Experimental observations

**Figure 5-12**

Static failure theories: brittle materials

**Modified-Mohr theory: quadrants of interest**

![Graph](image)

**Figure 5-13**

Modified-Mohr Failure Theory for Brittle Material
Static failure theories: brittle materials

**Modified-Mohr theory**

**Safety factor: zone I:**

\[
SF = N = \frac{S_{ut}}{\sigma_1}
\]

- Ultimate strength of the material in tension
- Max. principal normal stress

Modified-Mohr theory: applicable inside this area
Static failure theories: brittle materials

*Modified-Mohr theory*

**Safety factor: zone II**

\[
SF = N = \frac{S_{ut} \cdot |S_{uc}|}{|S_{uc}| \cdot \sigma_1 - S_{ut} \cdot (\sigma_1 + \sigma_3)}
\]

Modified-Mohr theory: applicable inside this area
Static failure theories: brittle materials

**Modified-Mohr theory**

**Safety factor: zone II**

**Modified-Mohr theory:**

\[
SF = N = \frac{S_{ut} |S_{uc}|}{|S_{uc}| \sigma_1 - S_{ut} (\sigma_1 + \sigma_3)}
\]

**EC: derive expression for the SF in Zone II**

Understand: state of stresses at points A, B, and C.

What do points A', B', and C' represent?

**Figure 5-13**

Modified-Mohr Failure Theory for Brittle Material
Static failure theories: brittle materials

Effective stress: Dowling indexes

(Similar concept as the equivalent von Mises stress in ductile materials)

\[
C_1 = \frac{1}{2} \left[ |\sigma_1 - \sigma_2| + \frac{2S_{ut} - |S_{uc}|}{-|S_{uc}|} (\sigma_1 + \sigma_2) \right]
\]

\[
C_2 = \frac{1}{2} \left[ |\sigma_2 - \sigma_3| + \frac{2S_{ut} - |S_{uc}|}{-|S_{uc}|} (\sigma_2 + \sigma_3) \right]
\]

\[
C_3 = \frac{1}{2} \left[ |\sigma_1 - \sigma_3| + \frac{2S_{ut} - |S_{uc}|}{-|S_{uc}|} (\sigma_1 + \sigma_3) \right]
\]
Static failure theories: brittle materials

**Modified-Mohr theory: effective stress**

**Safety factor:**

\[ SF = N = \frac{S_{ut \sigma}}{\tilde{\sigma}} \]

Ultimate strength of the material in tension

Effective stress. Obtained as:

\[ \tilde{\sigma} = MAX (\sigma_1, \sigma_2, \sigma_3, C_1, C_2, C_3) \]

and

\[ \tilde{\sigma} = 0 \text{ if } MAX < 0, \text{ use a different approach} \]
Determine the safety factors for the bracket rod shown considering: (a) ductile; and (b) brittle materials.

**Ductile case:**
Al 2024-T4 (consult Appendix C)

\[ S_y = 47 \text{ kpsi} \]

**Brittle case:**
Class 50 gray cast iron (consult Appendix C)

\[ S_{ut} = 52.5 \text{ kpsi}, \quad S_{uc} = 164 \text{ kpsi} \]
Uses of the **bracket model** configuration: **suspension system**
Uses of the **bracket model** configuration: transmissions
Static failure theories
Ductile materials

Safety factors:

Distortion energy theory:

\[ SF = N = \frac{S_y}{\sigma} \]

Yield strength of the material
von Mises effective stress

Distortion energy theory (pure shear):

\[ SF = N = \frac{S_{ys}}{\tau_{max}} \]

Max. shear-stress

Max. shear-stress theory:

\[ SF = N = \frac{S_{ys}}{\tau_{max}} \]

Maximum shear-stress

\[ S_{ys} = 0.577S_y \]

\[ S_{ys} = 0.5S_y \]
Static failure theories: brittle materials

*Modified-Mohr theory: effective stress*

**Safety factor:**

\[
SF = N = \frac{S_{ut}}{\tilde{\sigma}}
\]

**Effective stress:** Obtained as:

\[
\tilde{\sigma} = MAX (\sigma_1, \sigma_2, \sigma_3, C_1, C_2, C_3)
\]

and

\[
\tilde{\sigma} = 0 \quad \text{if } MAX < 0, \text{ use a different approach}
\]
Review
Example

A circular rod is subjected to combined loading consisting of a tensile load $P = 10$ kN and a torque $T = 5$ kN·m. Rod is 50 mm in diameter.

1) Draw stress element (cube) at the most highly stressed location on the rod, and
2) draw corresponding Mohr’s circle(s).
A piece of chalk is subjected to combined loading consisting of a tensile load $P$ and a torque $T$, see figure. The chalk has an ultimate strength $\sigma_u$ as determined by a tensile test. The load $P$ remains constant at such a value that it produces a tensile stress of $0.51 \cdot \sigma_u$ on any cross-section. The torque $T$ is increased gradually until fracture occurs on some inclined surface.

Assuming that fracture takes place when the maximum principal stress $\sigma_1$ reaches the ultimate strength $\sigma_u$, determine the magnitude of the torsional shearing stress produced by the torque $T$ at fracture and determine the orientation of the fractured surface.
Reading assignment

- Chapters 5 of textbook: Sections 5.2 to 5.5
- Review notes and text: ES2501, ES2502

Homework assignment

- Author's: As indicated in website of our course
- Solve: As indicated in website of our course