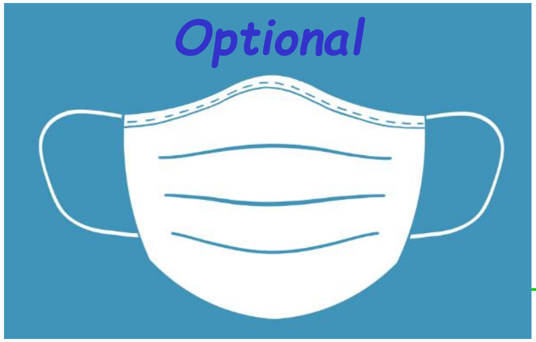
WORCESTER POLYTECHNIC INSTITUTE
MECHANICAL ENGINEERING DEPARTMENT

DESIGN OF MACHINE ELEMENTS
ME-3320, B'2024

Lecture 09-10

November 2024



Static failure theories

Loads are assumed to *not* vary over time

Failure theories that apply to:

- *Ductile materials*
- *Brittle materials*

Why do we need different theories ??

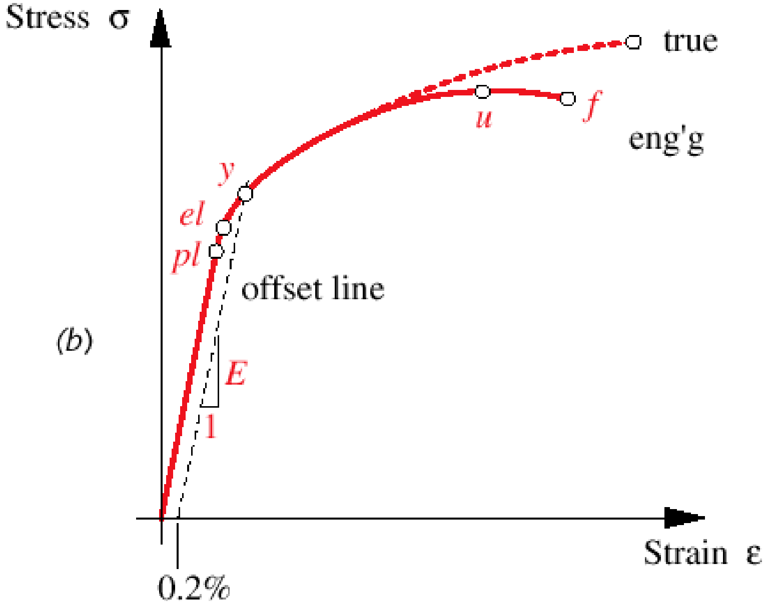


FIGURE 2-2
Stress-strain curve of a ductile material

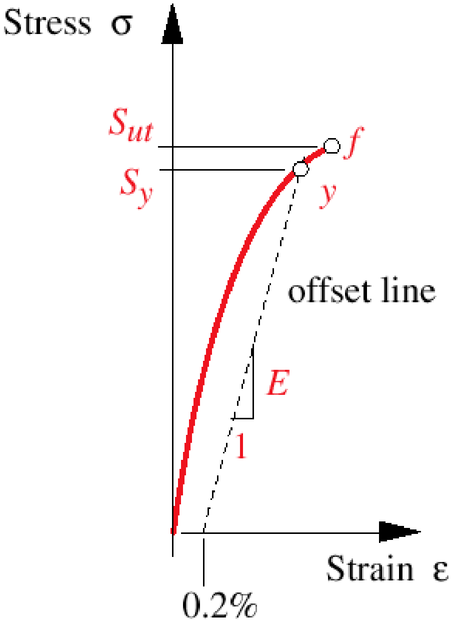


FIGURE 2-4
Stress-Strain Curve of a Brittle Material

Static failure theories

Tension test

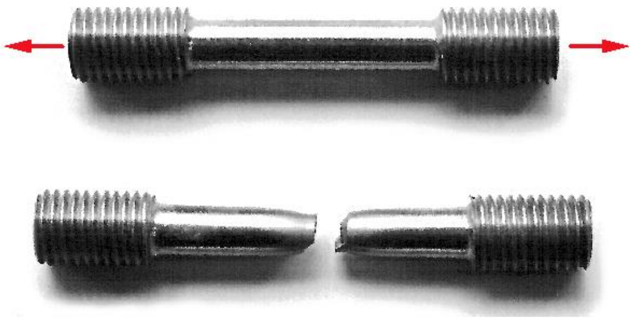


FIGURE 2-3

A Tensile Test Specimen of Mild, Ductile Steel After Fracture

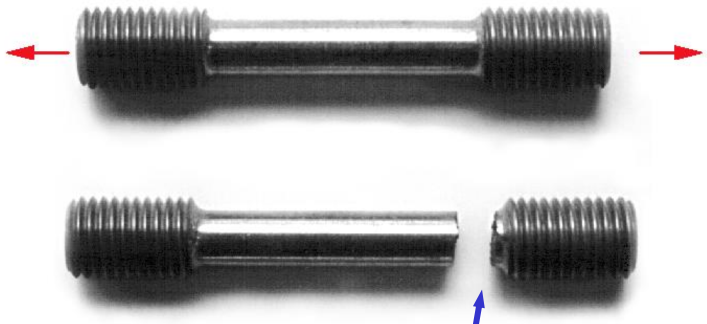
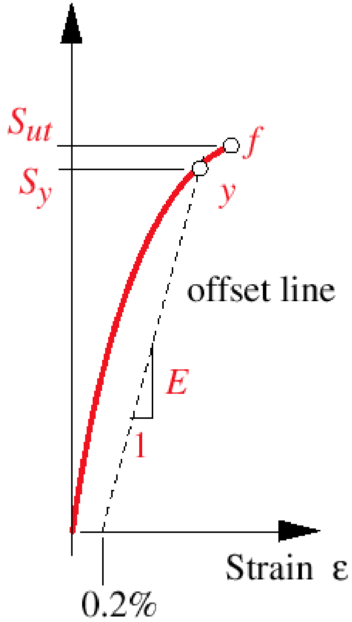


FIGURE 2-5

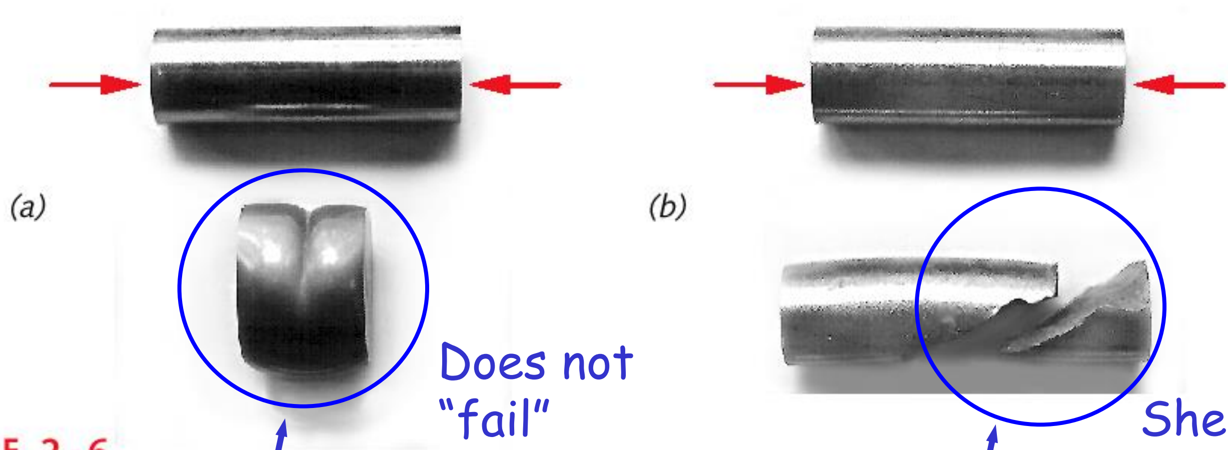
A Tensile Test Specimen of Brittle Cast Iron After Fracture



Why nearly 0° ??

Static failure theories

Compression test



Does not "fail"

Shear failure

FIGURE 2-6

Compression Test Specimens After Failure (a) Ductile Steel (b) Brittle Cast Iron

Why doesn't it fail ??

Why does it fail ??

Why nearly 45° ??

Static failure theories

Accepted failure theories that apply to **ductile** materials:

- *Total strain energy theory*
- → *Distortion energy theory*
- *Pure shear-stress theory*
- → *Maximum shear-stress theory*
- *Maximum normal stress theory (limited application)*

Accepted failure theories that apply to **brittle** materials:

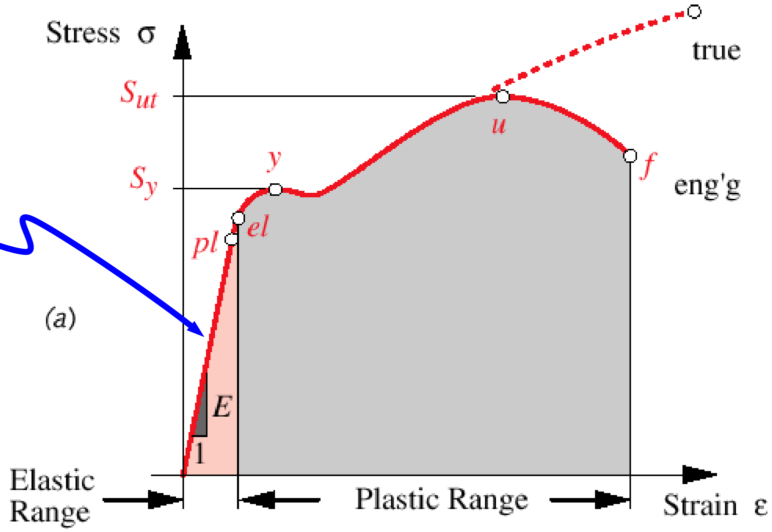
- *Maximum normal stress theory (even material)*
- *Maximum normal stress theory (uneven material)*
- → *Coulomb-Mohr theory*
- → *Modified Mohr theory*

Static failure theories

Ductile materials

Total strain energy *U*:

$$U = \frac{1}{2}\sigma\,\varepsilon$$



Elastic range

$$U = \frac{1}{2}\sigma\,\varepsilon = \frac{1}{2}(\sigma_1\,\varepsilon_1 + \sigma_2\,\varepsilon_2 + \sigma_3\,\varepsilon_3)$$

Principal stresses and strains

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\,\sigma_2 - \nu\,\sigma_3)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\,\sigma_1 - \nu\,\sigma_3)$$

$$\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu\,\sigma_1 - \nu\,\sigma_2)$$

Static failure theories

Ductile materials: total strain energy

Using previous expressions, total energy is:

$$U = \frac{1}{2}\sigma\,\varepsilon \quad = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

which can be expressed as

$$U = U_h + U_d$$

Hydrostatic energy

$$\boxed{U_h = \frac{3}{2}\frac{(1-2\nu)}{E}\sigma_h^2}$$

$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Obtained by setting:

$$U_h = U(\sigma_1 = \sigma_2 = \sigma_3 = \sigma_h)$$

Deformation energy

$$\boxed{U_d = \frac{1+\nu}{3E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3]}$$

Obtained by setting: $U_d = U - U_h$

Static failure theories

Ductile materials: distortion energy theory

$$\boxed{U_d = \frac{1+\nu}{3E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3]}$$

If **uniaxial yield** stress state (failure state):

$$\boxed{\begin{aligned} \sigma_1 &= \sigma_y \\ \sigma_2 &= 0 \\ \sigma_3 &= 0 \end{aligned}}$$

Therefore:

$$\boxed{U_d = \frac{1+\nu}{3E}\sigma_y^2}$$

Using **uniaxial yield** stress state (failure state)

Static failure theories

Ductile materials: distortion energy theory

For any other state of stresses:

$$\boxed{U_d = \frac{1+\nu}{3E}[\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_1\sigma_3]}$$

Failure criterion is obtained by setting:

$$\underbrace{\boxed{\frac{1+\nu}{3E}\sigma_y^2}}_{\text{Distortion energy: } \textit{uniaxial stress at yield}} = \underbrace{\boxed{\frac{1+\nu}{3E}[\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_1\sigma_3]}}_{\text{Distortion energy: } \textit{any other state of stresses}}$$

Static failure theories

Ductile materials: distortion energy theory - **Von Mises effective stress**

From previous equation:

$$\frac{1+\nu}{3E}\sigma_y^2 = \frac{1+\nu}{3E}[\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_1\sigma_3]$$

Therefore,

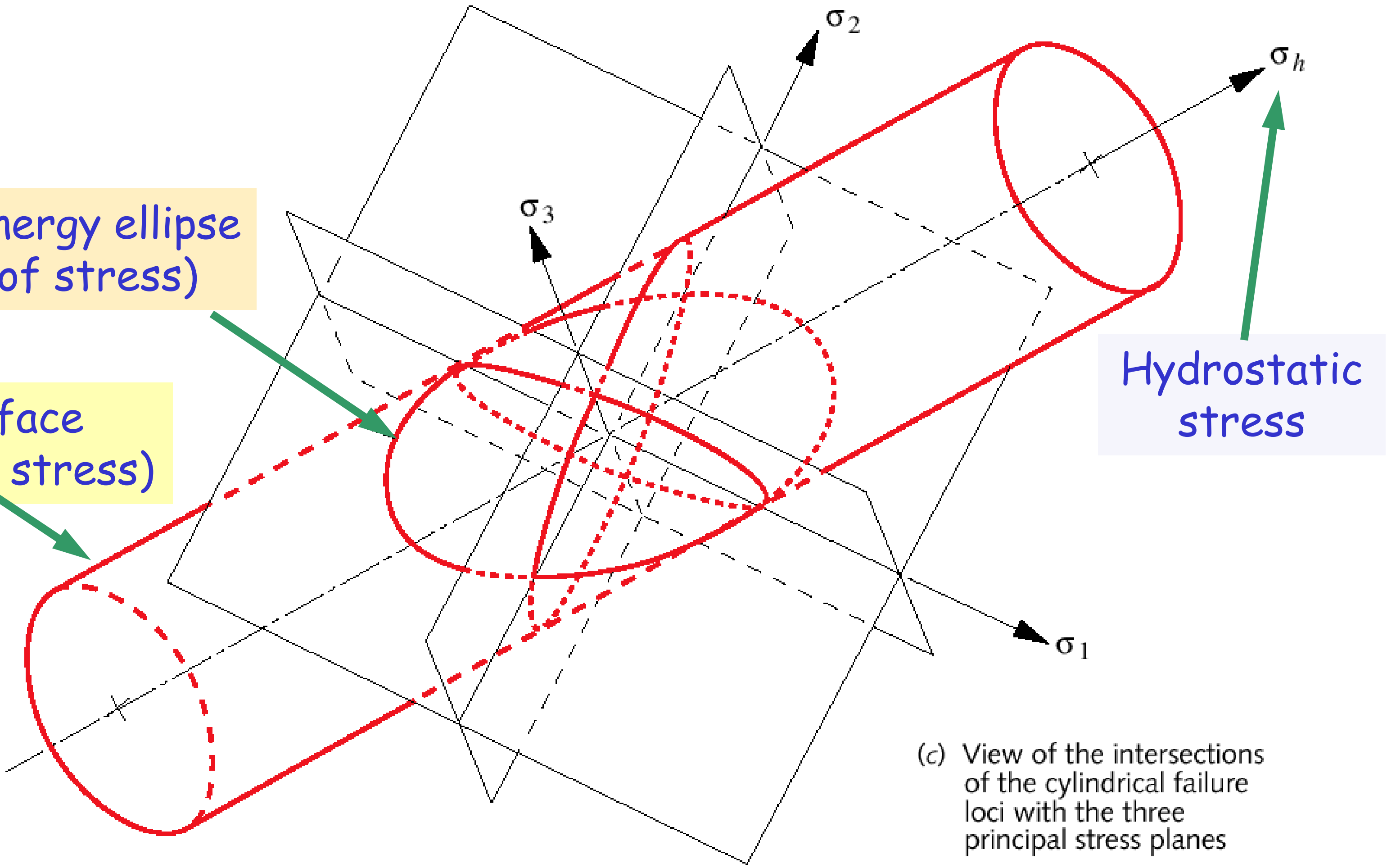
$$\sigma_y^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3$$

$$\rightarrow \sigma_y = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3}$$

(exactly at yield)

Static failure theories: plot of last equation

Ductile materials: distortion energy theory - **Von Mises effective stress**



Distortion energy ellipse (2D state of stress)

Yield surface (3D state of stress)

Hydrostatic stress

(c) View of the intersections of the cylindrical failure loci with the three principal stress planes

FIGURE 5-4

Three-Dimensional Failure Locus for the Distortion Energy Theory

Static failure theories

Ductile materials: distortion energy theory - **Von Mises effective stress**

Definition: $$\sigma' = \underbrace{\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3}}_{}$$

von Mises effective stress

Note: yield surface is reached when

$$\sigma' = S_y = \sigma_y$$

To be safe, we want to keep $\sigma' < S_y$

von Mises effective stress: *uniaxial stress that would create the same distortion energy as that created by actual combination of applied 3D/2D stresses*

Safety factor: $$\boxed{SF = N = \frac{S_y}{\sigma'}}$$

Yield strength of the material

von Mises effective stress

Static failure theories

Ductile materials: distortion energy theory

Example: pure shear load

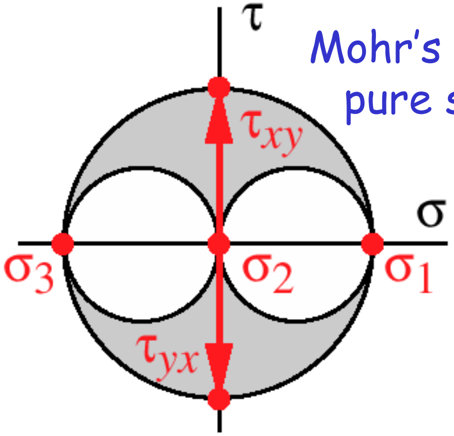
Using: $\sigma_y^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3$

Therefore:

$$\sigma_y^2 = \sigma_1^2 + (-\sigma_1)^2 - \sigma_1(-\sigma_1)$$

$$= \sigma_1^2 + (-\sigma_1)^2 - \sigma_1(-\sigma_1) = 3\sigma_1^2$$

$$= 3\tau_{\max}^2 = S_y^2$$



Mohr's circle: pure shear

and

$$\sigma_1 = \frac{1}{\sqrt{3}} S_y$$

failure occurs, in pure shear, when this holds true

Maximum stress before failure, in this case, is: $S_{ys} = \frac{1}{\sqrt{3}} S_y = 0.577 S_y$

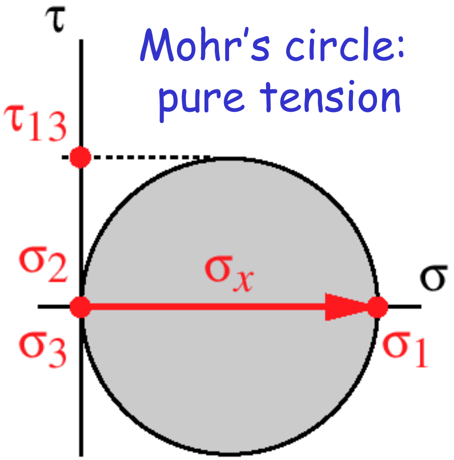
Static failure theories

Ductile materials: **maximum shear-stress theory**

This theory states that failure occurs when:

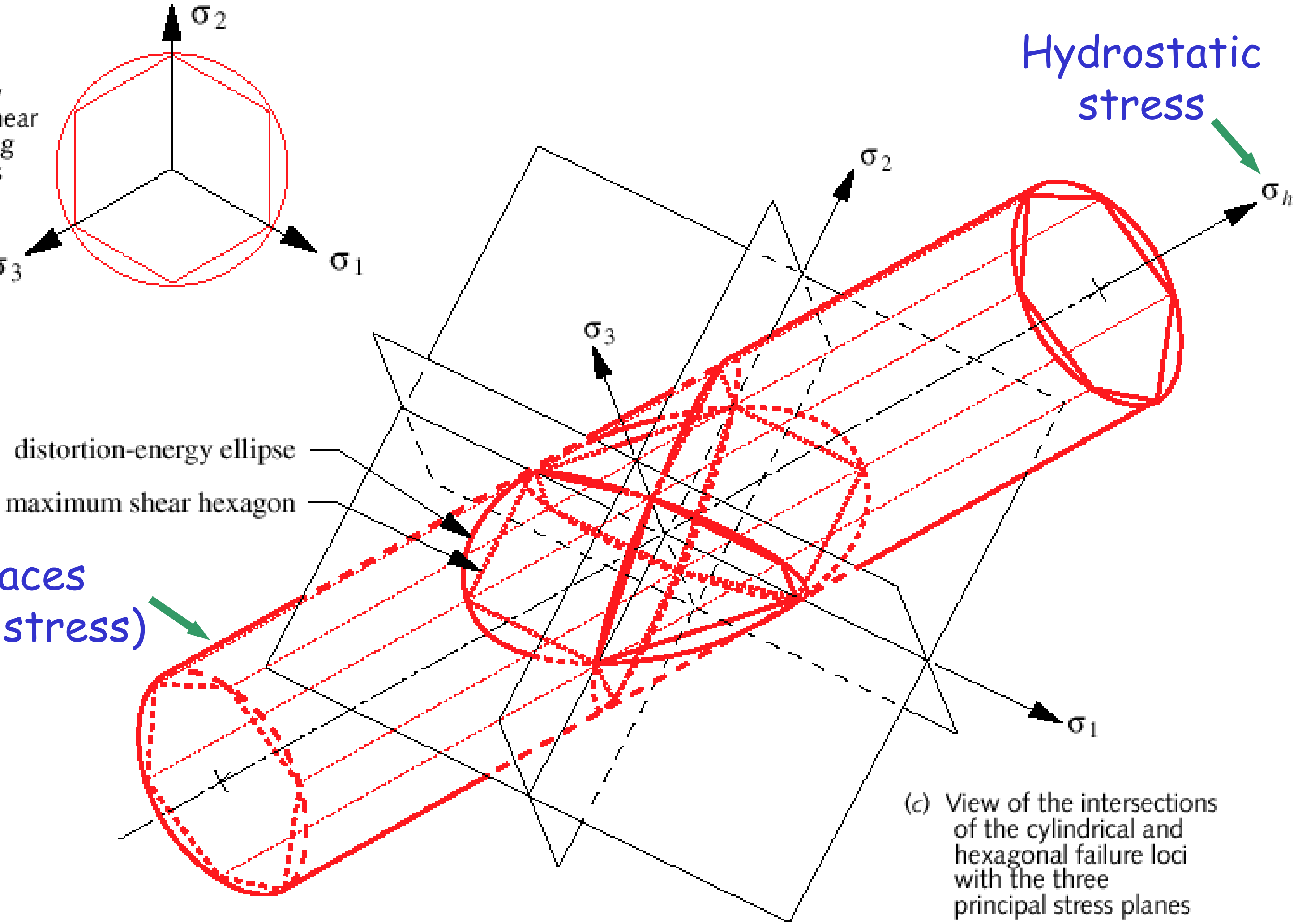
$$\boxed{S_{ys} = 0.5\, S_y}$$

(Failure occurs when maximum shear stress exceeds the shear stress at yield in pure tension)



Static failure theories

Ductile materials: **maximum shear-stress theory**



View of the distortion-energy and maximum shear failure loci looking down the σ_h axis

Hydrostatic stress

Yield surfaces (3D state of stress)

(c) View of the intersections of the cylindrical and hexagonal failure loci with the three principal stress planes

Static failure theories

Ductile materials

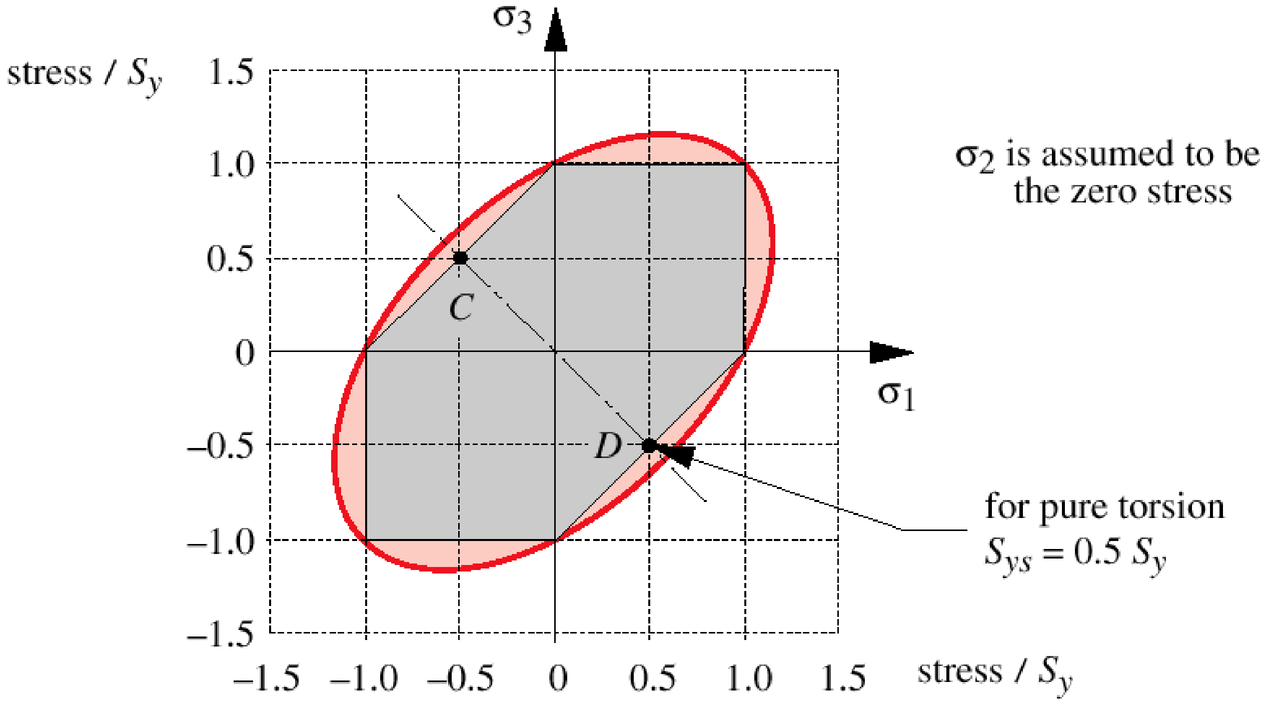


FIGURE 5-5

The 2-D Shear-Stress Theory Hexagon Inscribed Within the Distortion-Energy Ellipse

Static failure theories

Ductile materials

Safety factors:

Distortion energy theory:

$$SF = N = \frac{S_y}{\sigma'}$$

Yield strength of the material

von Mises effective stress

(Obtained from)

Distortion energy theory (pure shear):

$$SF = N = \frac{S_{ys}}{\tau_{\max}}$$

$$S_{ys} = 0.577 S_y$$

Max. shear-stress

Max. shear-stress theory:

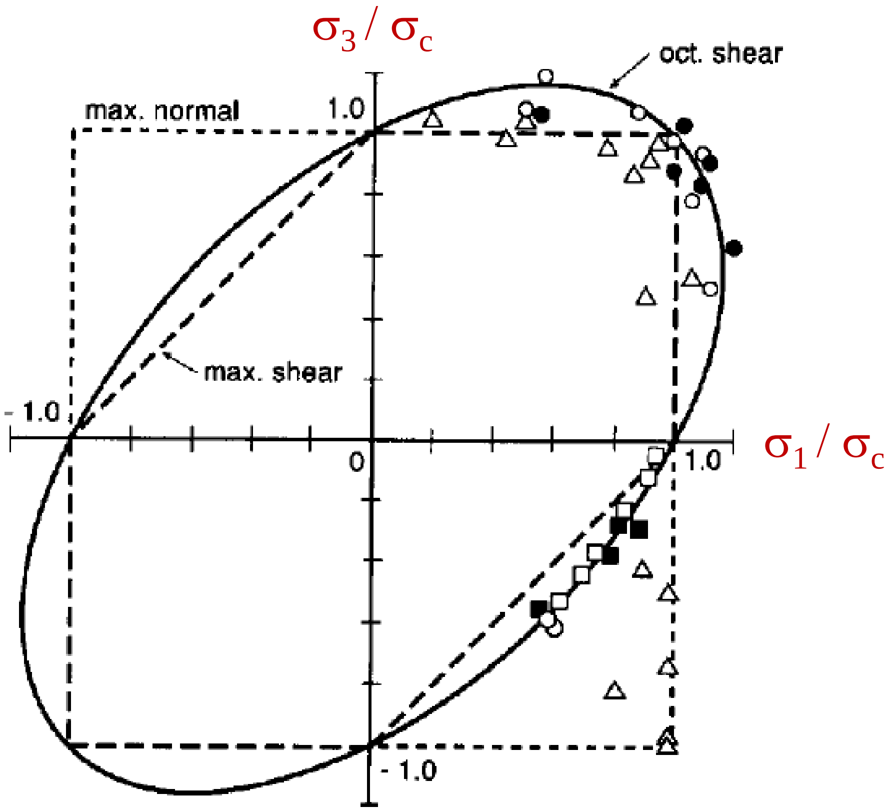
$$SF = N = \frac{S_{ys}}{\tau_{\max}}$$

$$S_{ys} = 0.5 S_y$$

Maximum shear-stress

Static failure theories: experimental verifications

Ductile & brittle materials



Yielding: $\sigma_c = \sigma_y$

- ○ Ni-Cr-Mo steel
- ● AISI 1023 steel
- ☐ 2024-T4 Al
- ■ 3S-H Al

Ductile

Yielding: $\sigma_c = \sigma_{ut}$

- △ Gray cast iron

Brittle

Experimental Data from Tensile Tests Superposed on Three Failure Theories *(Reproduced from Fig. 7.11, p. 252, in Mechanical Behavior of Materials by N. E. Dowling, Prentice-Hall, Englewood Cliffs, NJ, 1993)*

Reading assignment

- Chapters 5 of textbook: Sections 5.0 to 5.5
- Review notes and text: ES-2501, ES-2502

Homework assignment

- **Author's:** As posted in website of our course
- **Solve:** As posted in website of our course

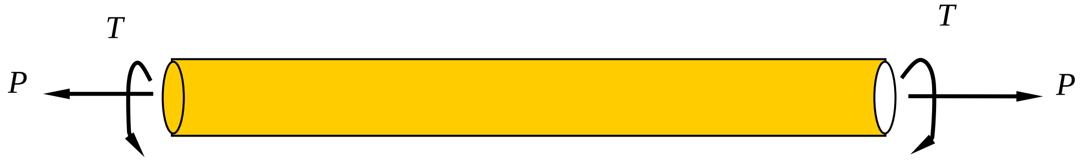
Review

Example

A circular rod is subjected to combined loading consisting of a tensile load P = 10 kN and a torque T = 5 kN·m. Rod is 50 mm in diameter.

1) Draw stress element (cube) at the most highly stressed location on the rod, and

2) draw corresponding Mohr's circle(s).



Review

Example

A piece of chalk is subjected to combined loading consisting of a tensile load ***P*** and a torque ***T***, see figure. The chalk has an ultimate strength σ_u as determined by a tensile test. The load ***P*** remains constant at such a value that it produces a tensile stress of $0.51 \cdot \sigma_u$ on any cross-section. The torque ***T*** is increased gradually until fracture occurs on some inclined surface.

Assuming that fracture takes place when the maximum principal stress σ_1 reaches the ultimate strength σ_u, determine the magnitude of the torsional shearing stress produced by the torque ***T*** at fracture and determine the orientation of the fractured surface.