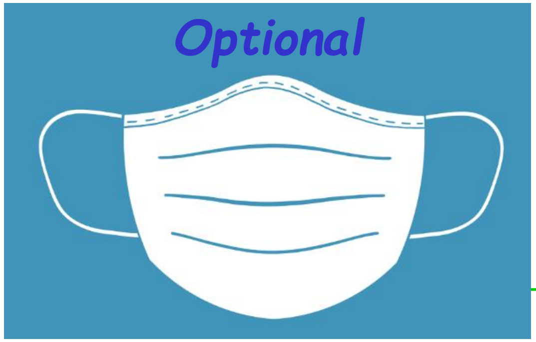
# WORCESTER POLYTECHNIC INSTITUTE
## MECHANICAL ENGINEERING DEPARTMENT

## DESIGN OF MACHINE ELEMENTS
## ME-3320, B'2024

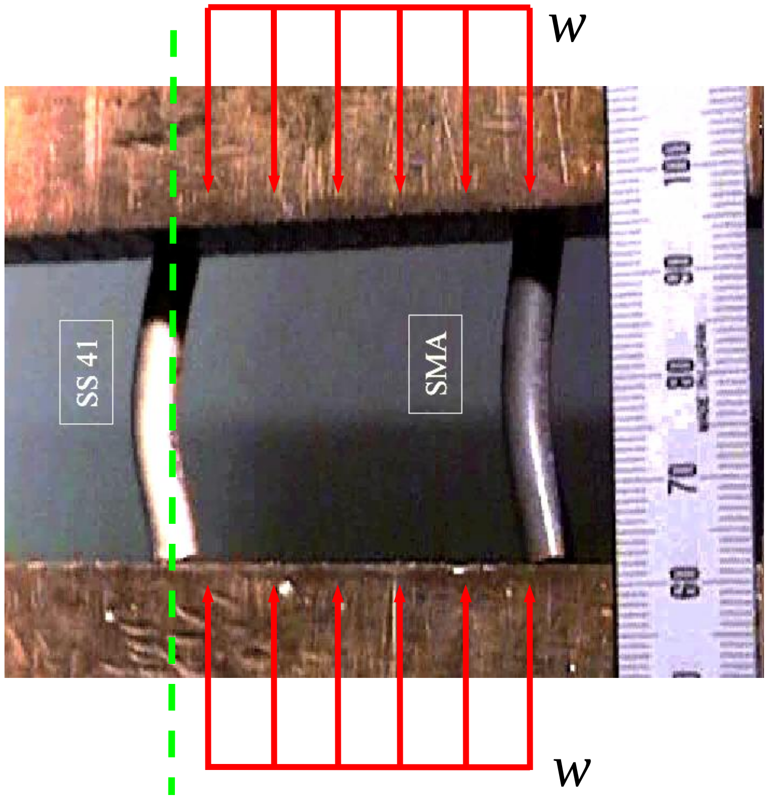
**Lecture 08-09**

**November 2024**

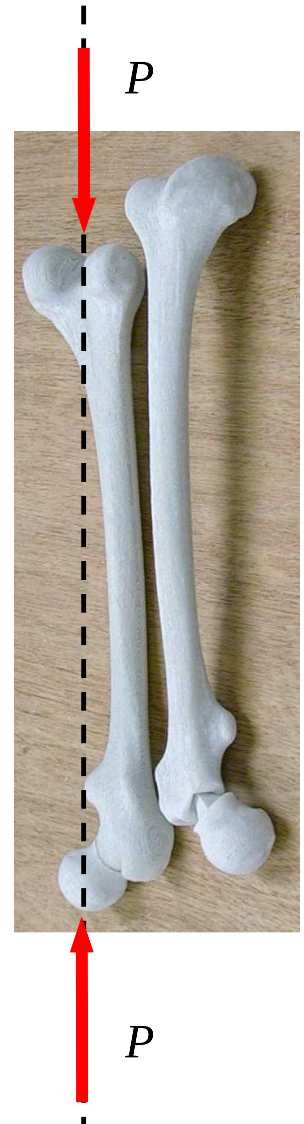


# Axial compression -- columns

## Examples



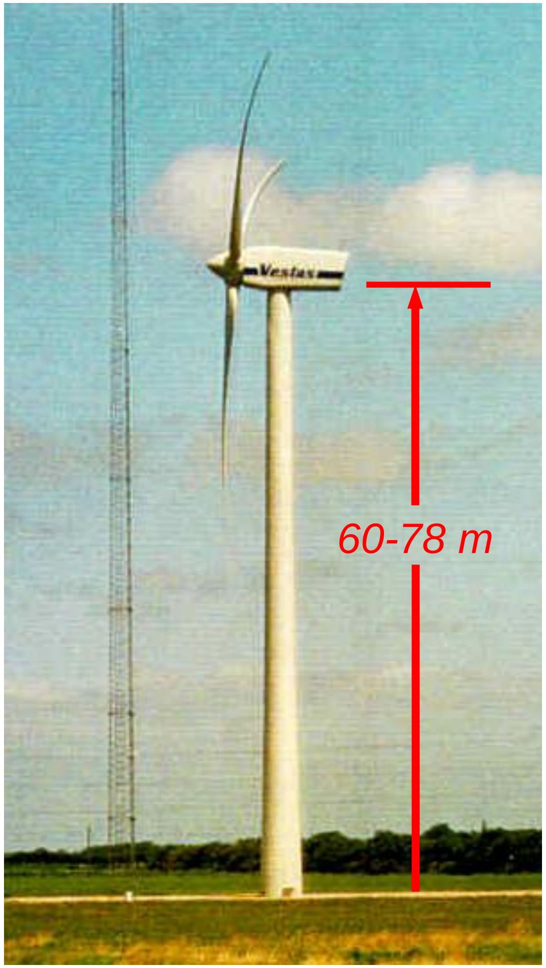

Concentric load




Eccentric load

# Axial compression -- columns
## Examples

Vestas' V80-2.0 MWatt

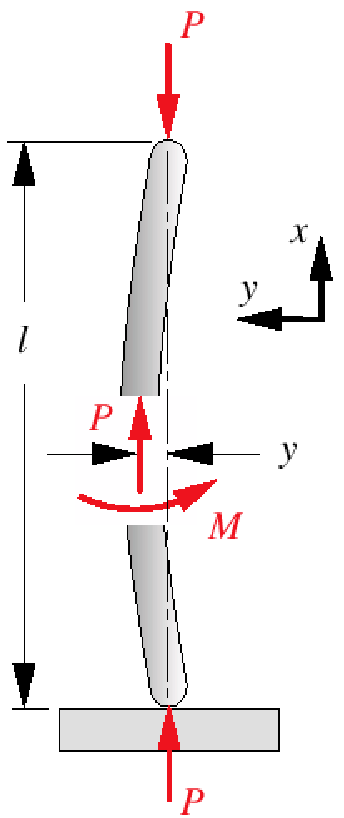



Installing a nacelle

# Axial compression -- columns

## Slenderness ratio: $S_r$

Buckling of a column




**Definitions:** $S_r = \dfrac{l}{k}$ , with $k = \sqrt{\dfrac{I}{A}}$

- *Short columns:* $S_r < 10$

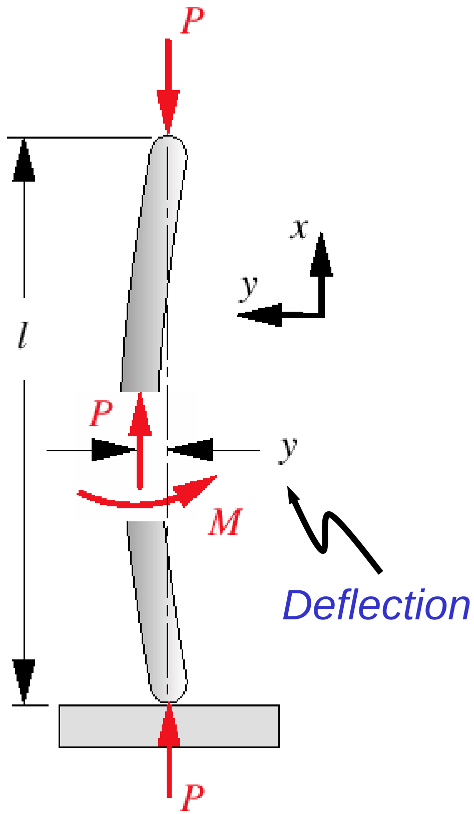
  Calculated based on compression stress criterion: $\sigma_x = \dfrac{P}{A}$

- *Intermediate/long columns:* $S_r \geq 10$

  Calculated based on critical unit load criterion: $\dfrac{P_{cr}}{A}$

# Axial compression -- columns
## Long columns: concentric load

*Buckling of a column*




Bending moment: $M = -P\,y$

For small deflections: $\dfrac{M}{EI} = \dfrac{d^2y}{dx^2}$

*(Governing ODE)* ⟹ $\dfrac{d^2y}{dx^2} + \dfrac{P\,y}{EI} = 0$

Solution (deflection) indicates:

$$y(x) = C_1 \sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

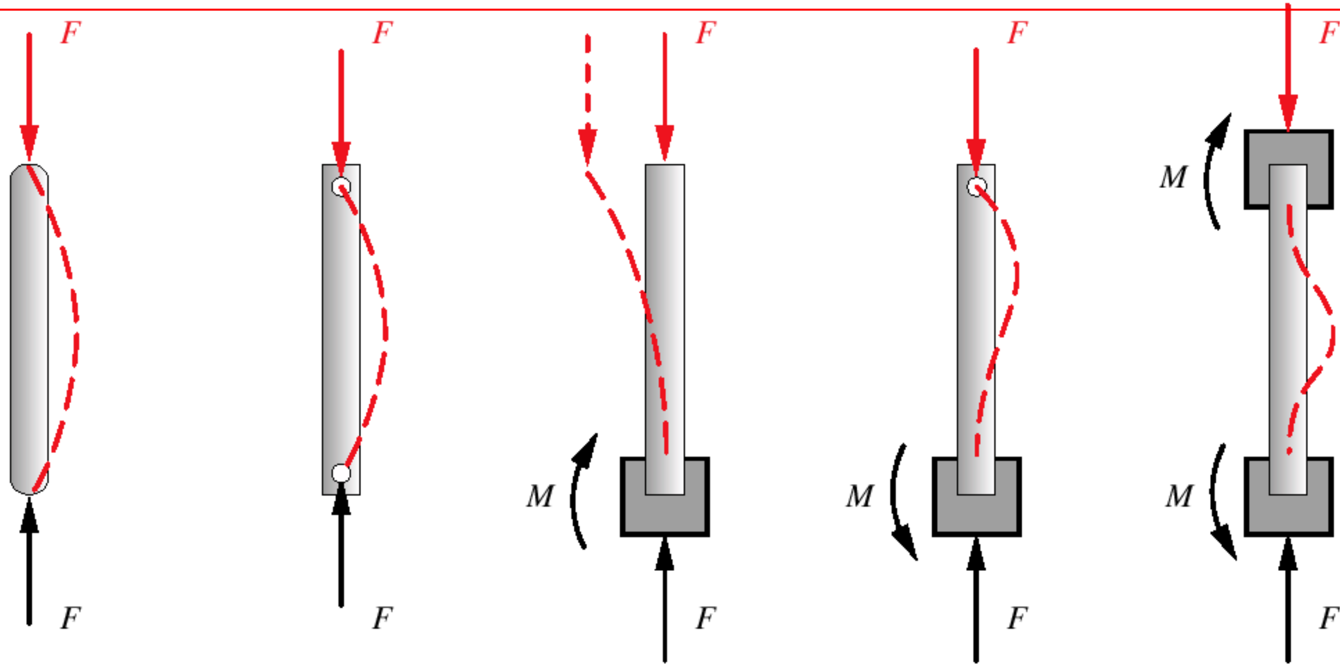
# Axial compression -- columns

## Long columns

Deflection: $$y(x) = C_1 \sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

*$C_1$ and $C_2$ are determined from*
*<u>boundary conditions</u>* (**end conditions**)

Possible end conditions (*make sure you understand BC's in terms of slope and deflection*):




(*a*) Rounded-rounded (*b*) Pinned-pinned (*c*) Fixed-free (*d*) Fixed-pinned (*e*) Fixed-fixed

# Axial compression -- columns

## Long columns: end conditions + critical load $P_{cr}$

Deflection: $$y(x) = C_1 \sin\left(\sqrt{\frac{P}{EI}}\, x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\, x\right)$$

For the rounded-rounded end conditions:

(1) $y(x=0)=0 \Longrightarrow C_2 = 0$

(2) $y(x=l)=0 \Longrightarrow C_1 \sin\left(\sqrt{\frac{P}{EI}}\, l\right) = 0$

Indicating that:

$$\sqrt{\frac{P}{EI}}\, l = n \cdot \pi; \quad n = 1,2,3....$$

*Many solutions….*

Therefore,

$$P_n = \frac{(n\cdot\pi)^2 E\, I}{l^2}; \quad n = 1,2,3....$$

*Many critical loads….*

# Axial compression -- columns

## Long columns: end conditions + critical load $P_{cr}$

Typically, designs are based on the smallest critical load. Therefore,

$$P_{cr} = \frac{\pi^2 E\, I}{l^2}; \quad \text{for } n = 1$$

*using:* $I = A k^2$ and $S_r = \dfrac{l}{k}$

$$\boxed{\frac{P_{cr}}{A} = \frac{\pi^2 E}{S_r^2}; \quad \text{for } n = 1}$$

*Critical* ***load per unit area*** *in terms of slenderness ratio*

Corresponding deflection: $y = C_1 \sin\left(\dfrac{\pi x}{l}\right)$

***For BC's:***

(1) $y(x=0) = 0$

(2) $y(x=l) = 0$

# Axial compression -- columns

## Accounting for different BC's

In order to take into account other boundary conditions, the concept of ***effective length, $l_{eff}$*** , is introduced:

$$S_r = \frac{l_{eff}}{k} \longrightarrow \boxed{\frac{P_{cr}}{A} = \frac{\pi^2 E}{S_r^2}}$$

**Table 4-3 Column End-Condition Effective Length Factors**

| End Conditions | Theoretical Value | AISC* Recommended | Conservative Value |
|---|---|---|---|
| Rounded-Rounded | $l_{eff} = l$ | $l_{eff} = l$ | $l_{eff} = l$ |
| Pinned-Pinned | $l_{eff} = l$ | $l_{eff} = l$ | $l_{eff} = l$ |
| Fixed-Free | $l_{eff} = 2l$ | $l_{eff} = 2.1l$ | $l_{eff} = 2.4l$ |
| Fixed-Pinned | $l_{eff} = 0.707l$ | $l_{eff} = 0.80l$ | $l_{eff} = l$ |
| Fixed-Fixed | $l_{eff} = 0.5l$ | $l_{eff} = 0.65l$ | $l_{eff} = l$ |

# Designing columns
## General procedure

1) Determine force to be supported and expected length of the column (*design objective and corresponding constraint(s)*). Use free-body diagram(s) and equilibrium conditions.

2) Determine cross-section parameters of a proposed column:

- *Area,* $A$
- *Moment of inertia,* $I$
- *Radius of gyration,* $k$

3) Determine slenderness ratio, $S_r = \dfrac{l_{eff}}{k}$

- *Identify boundary conditions (BC's) and apply appropriate value for the effective length* $l_{eff}$ -- *use appropriate table (Table 4-3.)*

4) Identify material to use and its corresponding *compressive yield strength*, $S_{yc}$, and *elastic modulus*, $E$.

# Designing columns
## General procedure

4) Determine slenderness ratio at half-yield: $(S_r)_D$ (***Design criterion***)

- $(S_r)_D$ *is obtained as follows:*

*(a) Set the load per unit area as:* $\dfrac{P_{cr}}{A} = \dfrac{S_{yc}}{2}$ *(Half the yield strength value in compression)*

*(b) Therefore,* $\dfrac{S_{yc}}{2} = \dfrac{\pi^2 E}{S_r^2}$

*(c) Solve for slenderness ratio -- using previous equation, (b).*
*This is the slenderness ratio at half-yield in compression.*

$$(S_r)_D = \pi \sqrt{\frac{2E}{S_{yc}}}$$

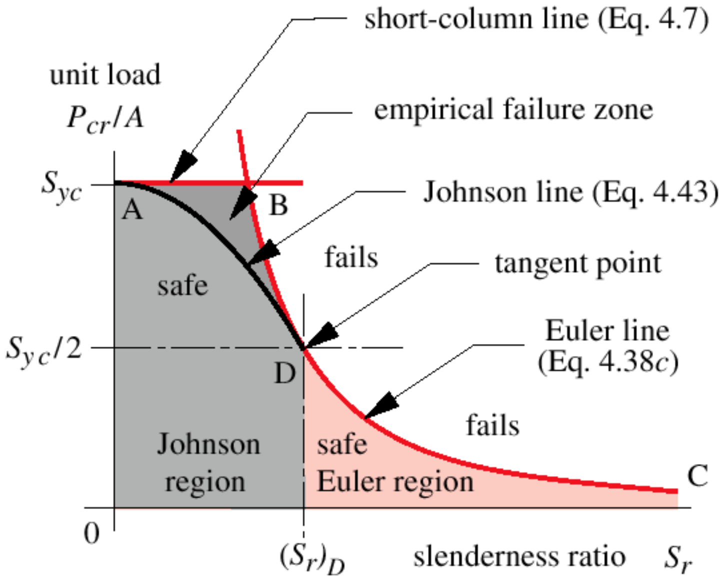
# Designing columns
## General procedure

5) Determine type of column based on proposed design:

- *Johnson*: if $S_r$ (step 3) < $(S_r)_D$ (Step 4)
- *Euler*: if $S_r$ (step 3) > $(S_r)_D$ (Step 4)

6) Determine critical load

- *Johnson*: $P_{cr} = A\left[S_{yc} - \frac{1}{E}\left(\frac{S_{yc}\,S_r}{2\pi}\right)^2\right]$
- *Euler*: $P_{cr} = A\frac{\pi^2 E}{S_r^2}$




(*a*) Construction of column failure lines

# Designing columns
## General procedure

7) Determine allowed load: $P_{allowed} = \dfrac{P_{cr}}{SF}$

- $P_{cr}$ , is the critical load (step 6)
- $SF$ , is the security factor ( > 1)

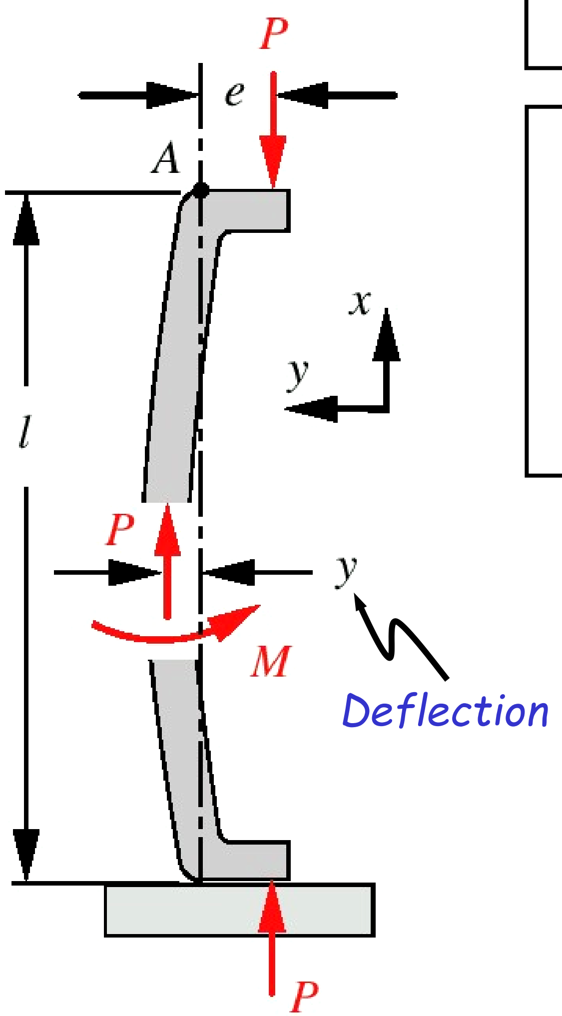
8) If $P_{allowed}$ is > than force to be supported, then, *a satisfactory design has been obtained -- not necessarily the optimal* !! *Go* to step (1) and refine your design, if possible (e.g., weight minimization)

9) If $P_{allowed}$ is < than force to be supported, then, go to step (1) and improve design. You can select a different section and/or material.

*Use of MathCad is strongly recommended !!*

# Axial compression -- columns
## Eccentrically loaded

Eccentrically loaded column




Moments: $\sum M_A = -M + P(e+y) = 0$

For small deflections: $\dfrac{M}{EI} = \dfrac{d^2 y}{dx^2}$

(Governing ODE) $\Longrightarrow$ $\dfrac{d^2 y}{dx^2} - \dfrac{P\,y}{EI} = \dfrac{P\,e}{EI}$

For BC's: $y(x=0) = 0;\quad \left.\dfrac{dy}{dx}\right|_{x=\frac{l}{2}} = 0$

$\Longrightarrow$ $y\left(x = \dfrac{l}{2}\right) = e\left[\sec\left(\dfrac{l}{2}\sqrt{\dfrac{P}{EI}}\right) - 1\right]$

# Axial compression -- columns

## Eccentrically loaded

Maximum moment: $$M_{\max} = P \cdot \left( e + y\big|_{x=\frac{l}{2}} \right) = P \cdot e \cdot \sec\left( \frac{l}{2} \sqrt{\frac{P}{EI}} \right)$$

Compressive stress: $$\sigma_c = \frac{P}{A} + \frac{M\,c}{I} = \frac{P}{A} + \frac{M\,c}{A k^2}$$

with the maximum stress level at $M = M_{\max}$ which yields

$$\sigma_{c,\max} = \frac{P}{A}\left[ 1 + \left( \frac{e\,c}{k^2} \right) \cdot \sec\left( \frac{l}{k} \sqrt{\frac{P}{4EA}} \right) \right]$$

setting $\sigma_{c,\max} = S_{yc}$

$$\Longrightarrow \quad \frac{P}{A} = \frac{S_{yc}}{1 + \left( \frac{e\,c}{k^2} \right) \cdot \sec\left( \frac{l_{eff}}{k} \sqrt{\frac{P}{4EA}} \right)}$$

Solve for $P$ to obtain the critical load

(Secant column formula)

## Reading assignment

- Chapters 4 of textbook: Sections 4.12 to 4.19
- Review notes and text: ES2501, ES2502

## Homework assignment

- **Author's:** as indicated on Website of our course
- **Solve:** as indicated on Website of our course