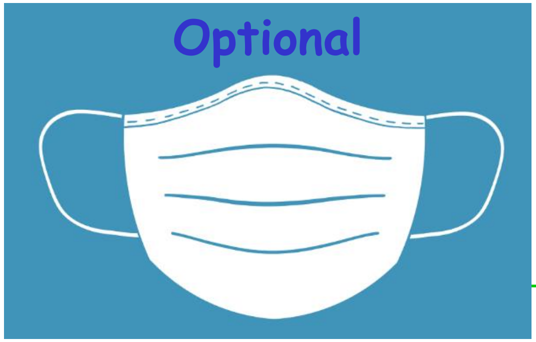
WORCESTER POLYTECHNIC INSTITUTE
MECHANICAL ENGINEERING DEPARTMENT

DESIGN OF MACHINE ELEMENTS
ME-3320, B'2024

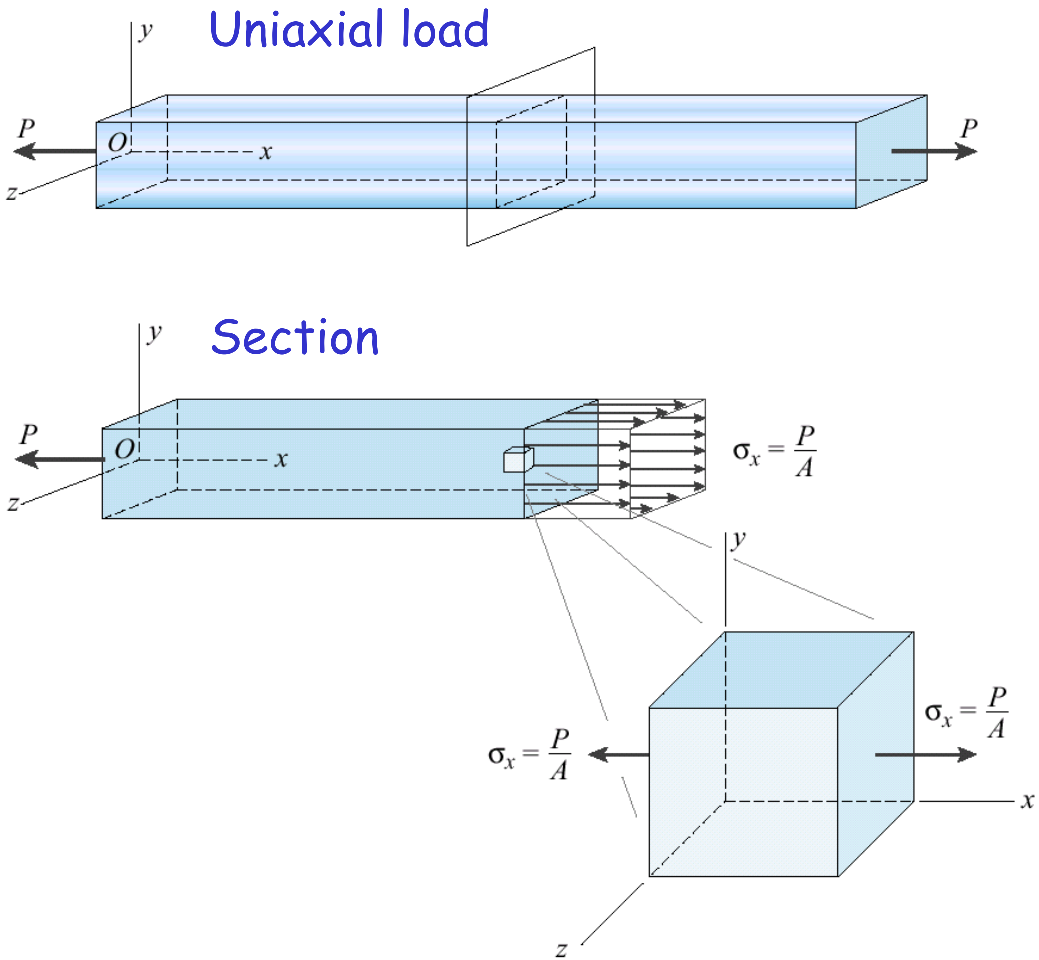
Lecture 05

October 2024



Stress at a point

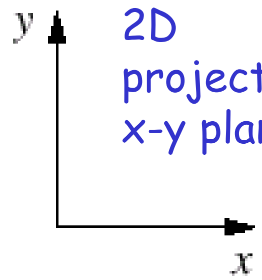
Uniaxial load



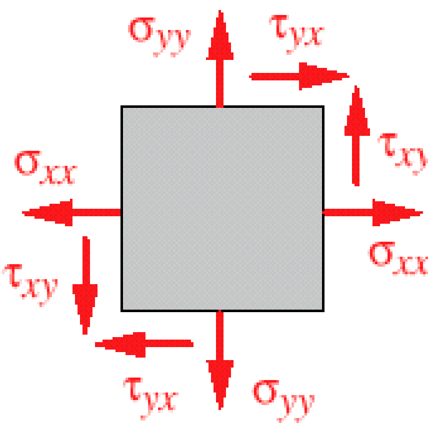
Stress cube for uniaxial stress loading

Stress at a point

General load case. Stress cube in 2D



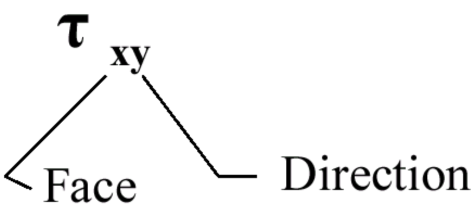
2D projection: x-y plane



Notation

σ Normal Stress

τ Shear Stress



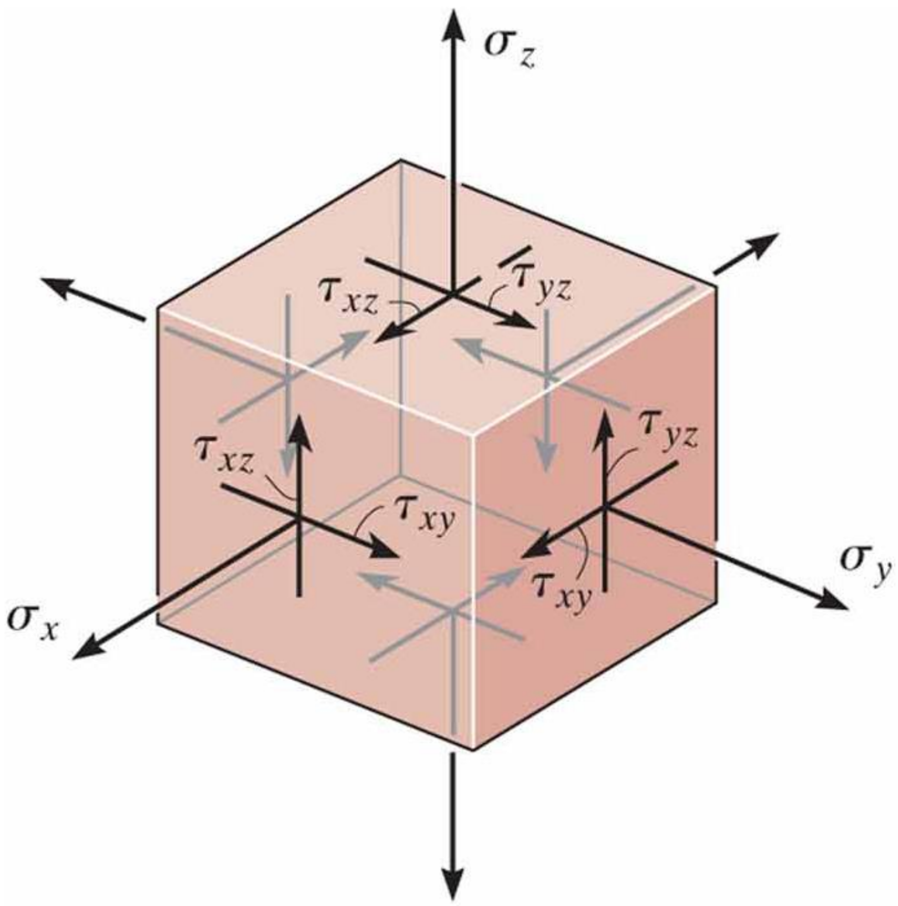
Equilibrium conditions require that

$$\tau_{xy} = \tau_{yx}$$

<u>Why?</u>

Stress at a point

General load case. Stress cube in 3D



There are 9 components of stress.

Equilibrium conditions are used to reduce the number of stress components to 6:

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

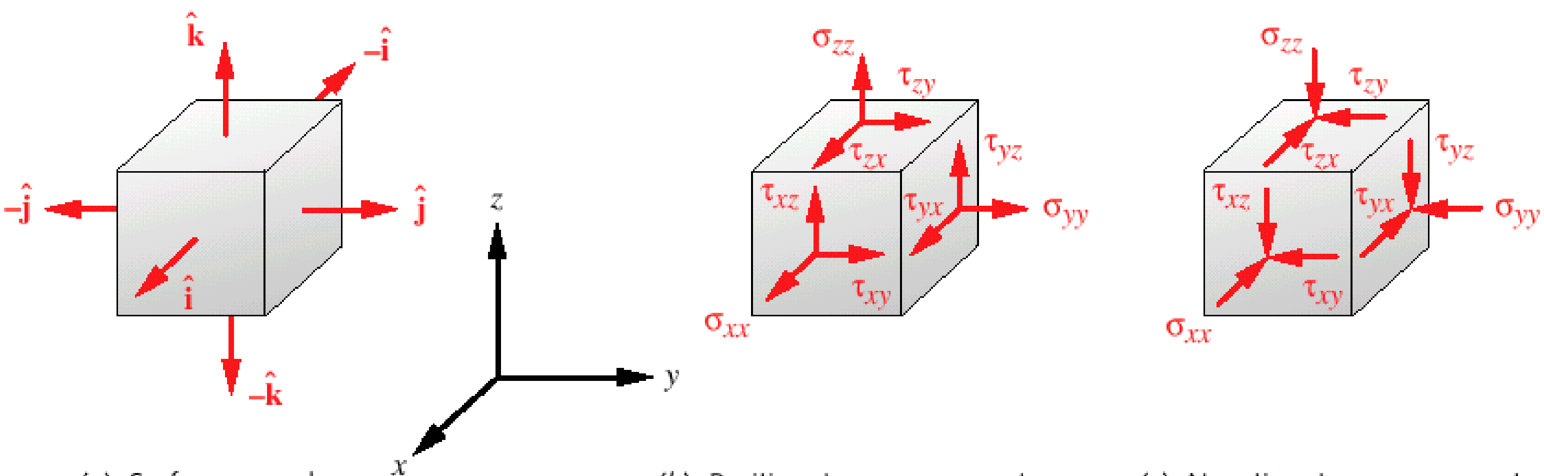
$$\tau_{yz} = \tau_{zy}$$

Stress tensor

Cauchy stress tensor

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Tensors are quantities that are *invariant* to coordinate transformations

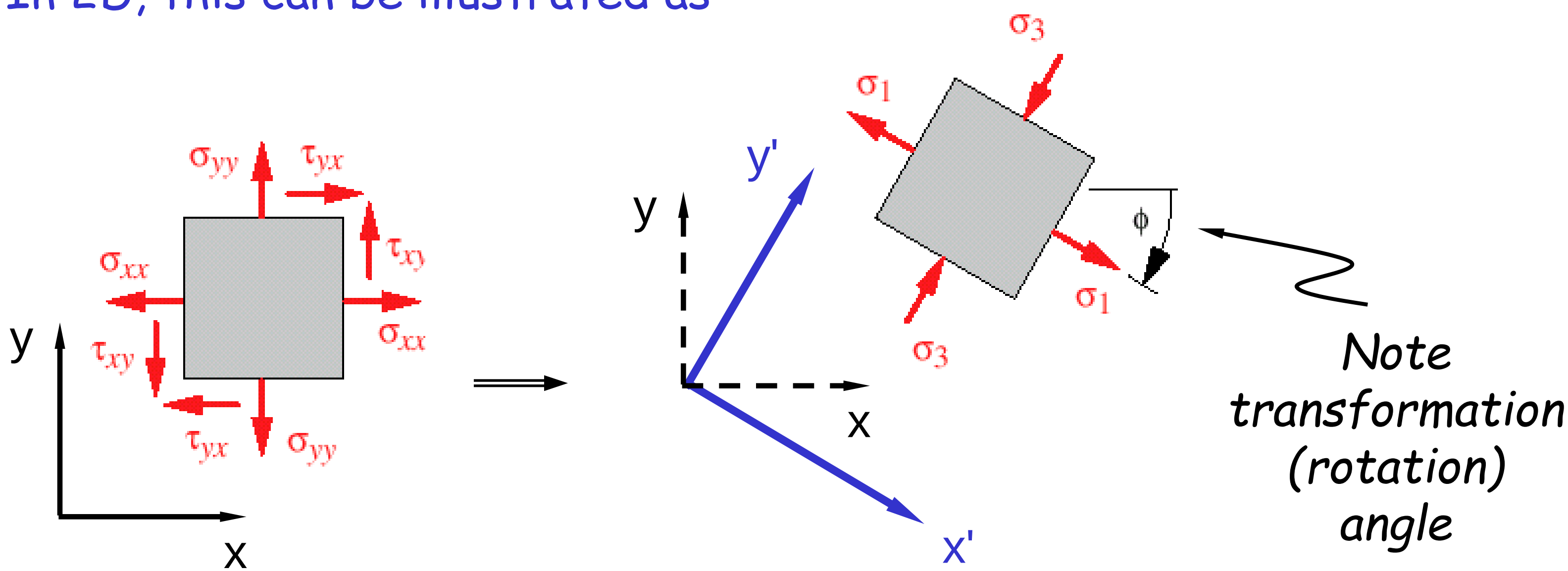


(*a*) Surface normals (*b*) Positive stress components (*c*) Negative stress components

Determination of principal stresses

Principal ***normal*** stresses

- This problem involves performing coordinate transformation, which can provide a stress tensor that does NOT contain shear stresses
- In 2D, this can be illustrated as



Stress cube in original coordinate system (x,y)

Stress cube in transformed coordinate system (x′,y′) -- only normal stresses exist: σ_1 and σ_3, in this 2D case

Determination of principal stresses

Principal ***normal*** stresses

This problem involves performing coordinate transformation, which can provide a stress tensor that does NOT contain shear stresses, that is

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \hat{\mathbf{n}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \hat{\mathbf{n}}$$

Initial stress tensor

Unit vector, normal to principal plane

Transformed stress tensor

Unit vector, normal to principal plane

Same vectors

Determination of principal stresses

Principal ***normal*** stresses

Previous equation can be written as

$$\begin{bmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz}-\sigma \end{bmatrix} \hat{\mathbf{n}} = \begin{bmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz}-\sigma \end{bmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

implying that the determinant

$$\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz}-\sigma \end{vmatrix} = 0$$

(this is an Eigenvalue problem)

Determination of principal stresses
Principal ***normal*** stresses

Expanding determinant and setting it to zero yields

$$\sigma^3 - C_2\sigma^2 + C_1\sigma - C_0 = 0$$

in which

$$C_2 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$C_1 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$C_0 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

are stress invariants (have the same magnitudes for all choices of coordinate axes (x,y,z) in which the applied stresses are measured or calculated)

The *principal normal stresses*, σ_1, σ_2, σ_3, are the three roots of the cubic polynomial -- always real and typically ordered as: $\sigma_1 > \sigma_2 > \sigma_3$

Determination of principal stresses

Principal ***shear*** stresses

Principal ***shear stresses can be found from values*** of the principal normal stresses as

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2}$$

$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2}$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

Mohr's circle: principal normal and shear stresses

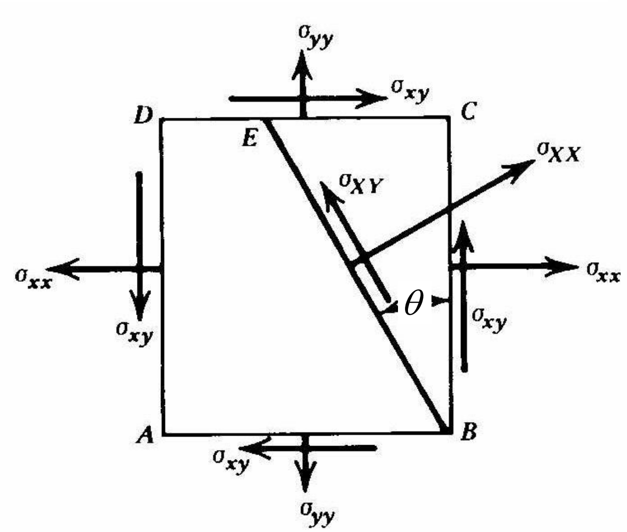
Graphical representation of previous equations: **3D**

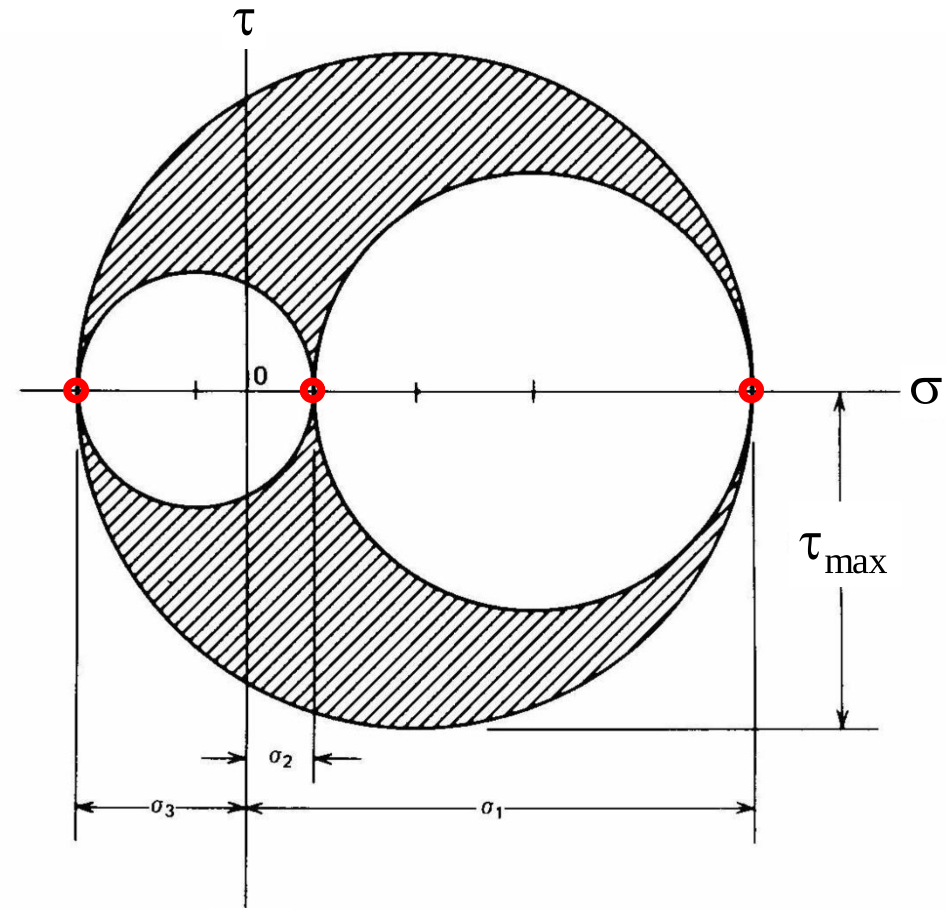
Equivalent nomenclature:

$$\sigma_x = \sigma_{xx}$$

$$\sigma_y = \sigma_{yy}$$

$$\sigma_{xy} = \tau_{xy}$$





Mohr's Circle in three dimensions.

From Boresi: Mechanics of materials

Determination of principal stresses

Principal ***normal and shear*** stresses: **2D case**

We will use these equations extensively

Principal normal stresses:

$$\sigma_1, \sigma_3 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

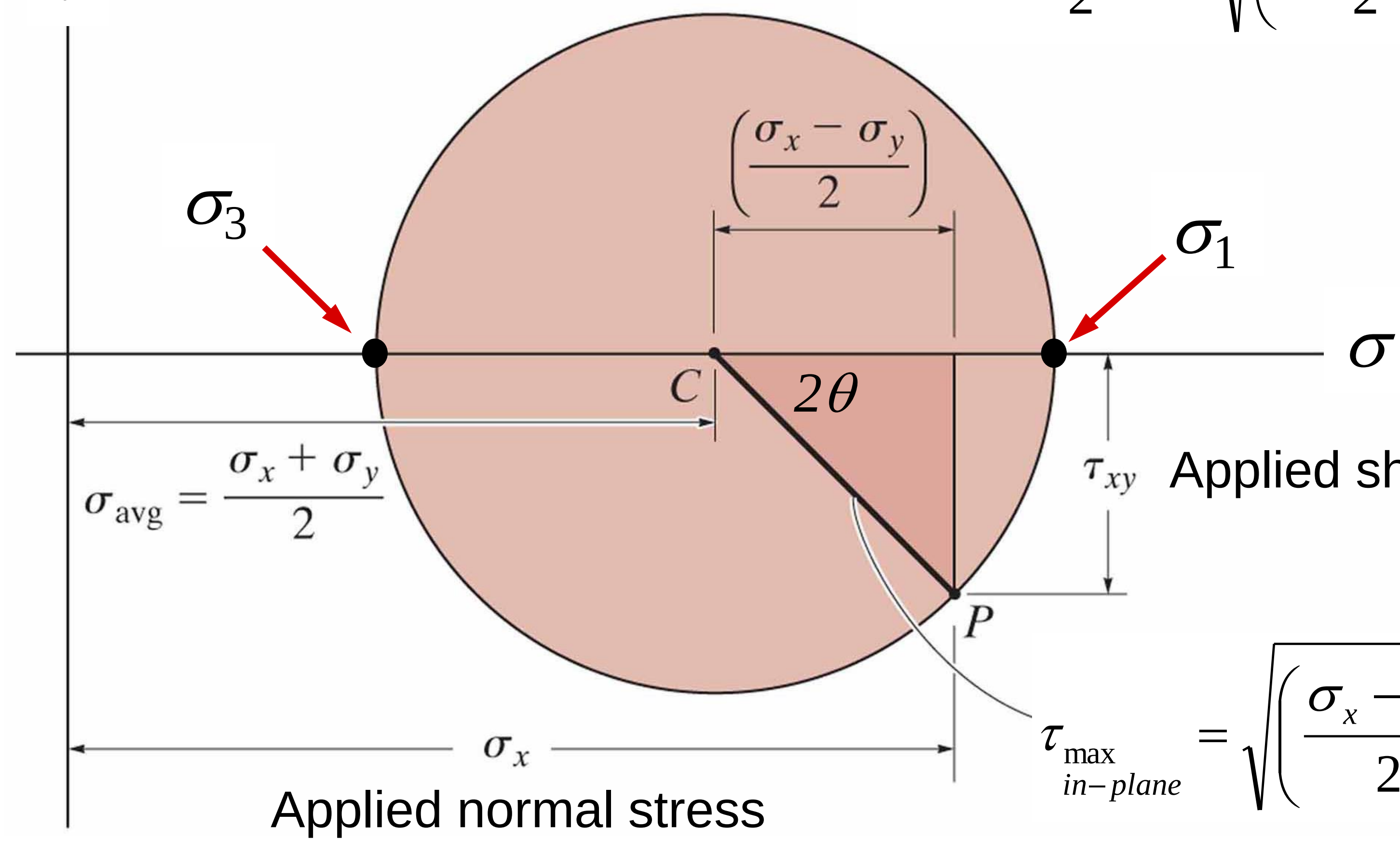
Maximum shear stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Mohr's circle: principal normal and shear stresses

Graphical representation of previous equations: **2D**

Principal stresses: $\sigma_{1,3} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$



$$\tau_{\substack{max \\ in-plane}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

Maximum shear

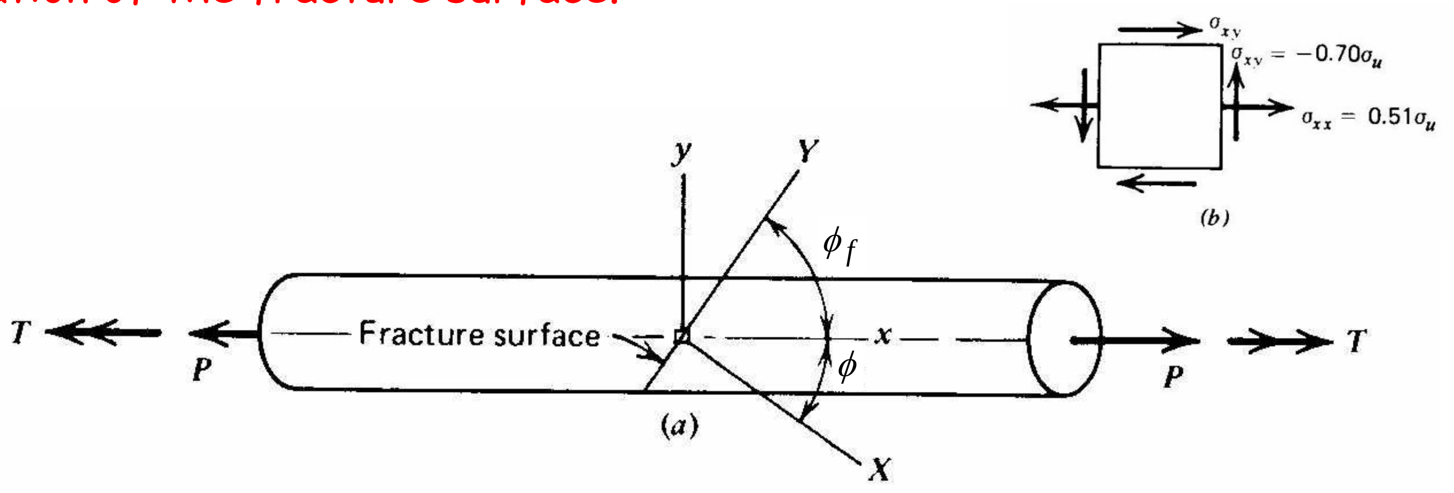
Determination of principal stresses
Review author's textbook

- Examples: 4-1, 4-2, and 4-3
- Review and master Section 4.6

Determination of principal stresses
Example: solve in class / assignment

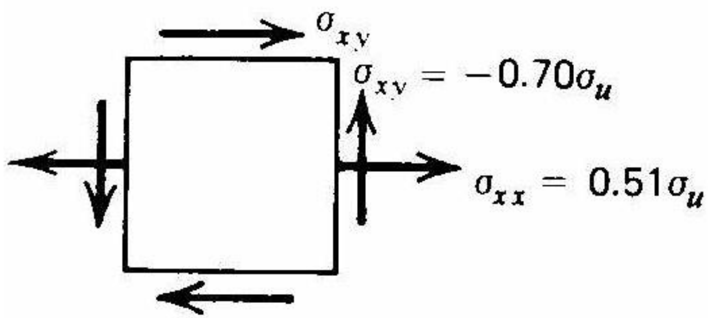
A piece of chalk is subjected to combined loading consisting of a tensile load ***P*** and a torque ***T***, see figure. The chalk has an ultimate strength σ_u as determined by a tensile test. The load ***P*** remains constant at such a value that it produces a tensile test of $0.51 \cdot \sigma_u$ on any cross-section. The torque ***T*** is increased gradually until fracture occurs on some inclined surface.

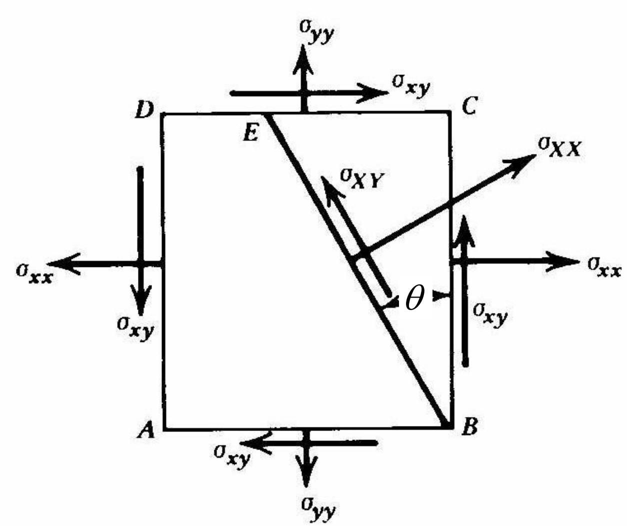
Assuming that fracture takes place when the maximum principal stress σ_1 reaches the ultimate strength σ_u, determine the magnitude of the torsional shearing stress produced by the torque ***T*** at fracture and determine the orientation of the fracture surface.

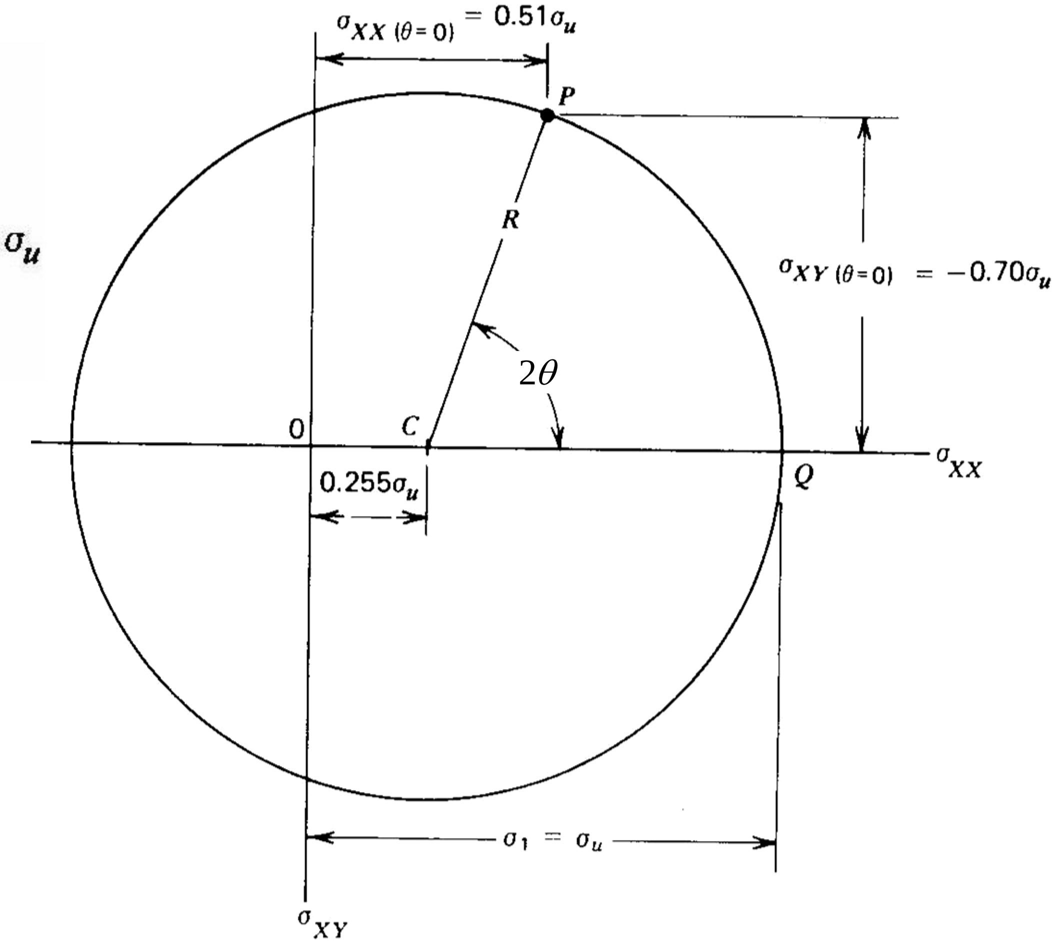


Determination of principal stresses
Example: solve in class / assignment

Stress element







Reading assignment

- Chapters 4 of textbook: Sections 4.0 to 4.6
- Review notes and text: ES2501, ES2502

Homework assignment

- Author's: as posted in Website of our course
- Solve: as posted in Website of our course