WORCESTER POLYTECHNIC INSTITUTE
MECHANICAL ENGINEERING DEPARTMENT

Optical Metrology and NDT
ME-593n/ ME-5304, C'2024

Introduction: Wave Optics
January 2024

Wave optics: light waves

Wave equation

An optical wave -- monochromatic -- can be described mathematically by the complex wavefunction

$$U(x,y,z,t) = a(x,y,z) \cdot \exp[\, j\phi(x,y,z)] \cdot \exp[\, j2\pi\nu t] \qquad (1)$$

where

x, y, z	are the components of the position vector $\boldsymbol{r}$
t	is time
$\phi(x,y,z)$	is the optical phase
$a(x,y,z)$	is the amplitude
ν	is the frequency [Hz]
	($\omega = 2\,\pi\,\nu =$ angular frequency [rad/sec])
j	is the complex quantity $\sqrt{-1}$

Wave optics: light waves

Wave equation

Equation (1) can be written as

$$U(x,y,z,t) = U(x,y,z) \cdot \exp[j2\pi\nu t] \qquad (2)$$

where the time-independent term,

$$U(x,y,z) = a(x,y,z) \cdot \exp[j\phi(x,y,z)] \qquad (3)$$

is the complex amplitude of the optical wave $U(x,y,z,t)$

Wave optics: light waves

Wave equation

Function $U(x, y, z, t)$ must satisfy the *wave equation* (in order to represent a valid wave function), therefore,

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0 \qquad (4)$$

where

$$c = \frac{c_o}{n} \quad for \quad n \geq 1 \qquad (5)$$

in which c_o is the speed of light in free-space and the wave propagates in a medium with index of refraction n.

Wave optics: light waves

Wave equation

By substituting Eq. 2 into the wave equation, Eq. 4, the following equation is obtained -- exercise in class/homework

$$(\nabla^2 + k^2)U(x, y, z) = 0 \qquad (6)$$

which is called the *Helmholtz equation*, where

$$k = \frac{2\pi\nu}{c} = \frac{\omega}{c} = \frac{2\pi}{\lambda} \qquad (7)$$

in the *wave number*, and λ is the spatial wavelength.

Note that:

$$\lambda = \frac{c}{\nu} \qquad (8)$$

Wave optics: light waves

Elementary waves

The two canonical solutions of the Helmholtz equation in a homogenous medium are*: (1) the plane wave, and (2) the spherical wave.*

(1) The plane wave

The plane wave has the complex amplitude:

$$U(x,y,z) = A\exp(-j\,\boldsymbol{k}\cdot\boldsymbol{r}) \qquad (9)$$

$$= A\exp[-j\,(k_x \cdot x + k_y \cdot y + k_z \cdot z)]$$

where

A is the amplitude, or *complex envelope*

$\boldsymbol{k} = k_x\hat{\mathrm{i}} + k_y\hat{\mathrm{j}} + k_z\hat{\mathrm{k}} = (k_x, k_y, k_z)$ is the *propagation direction vector*

$\boldsymbol{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}} = (x, y, z)$ is the *position vector*

j is the complex quantity $\sqrt{-1}$

Elementary waves

Plane waves

For Eq. 9 to satisfy the Helmholtz equation, Eq. 6, it is necessary that

$$k_x{}^2 + k_y{}^2 + k_z{}^2 = k^2 \tag{10.1}$$

that is, the magnitude of the propagation direction vector, $\boldsymbol{k}$, is equal to the wave number, k,

$$\left|\boldsymbol{k}\right| = k \tag{10.2}$$

Elementary waves

Plane waves

Since the phase, or $\arg[U(x,y,z)] = \arg[A] - \boldsymbol{k}\cdot\boldsymbol{r}$, the wavefronts are

$$k_x \cdot x + k_y \cdot y + k_z \cdot z = 2\pi q + \arg[A] \qquad (11)$$

for q = integer

Equation 11 describes the family of parallel planes that is perpendicular to the propagation direction vector, $\boldsymbol{k}$. These planes are called: wavefronts.

Planes are separated by the distance

$$\lambda = \frac{2\pi}{k} \qquad (12)$$

Elementary waves

Plane waves

If the z-axis is taken in the direction of the propagation vector, $\boldsymbol{k}$, then

$$U(z) = A\exp(-j\,k\,z) \qquad (13)$$

using Eqs 13 and 2,

$$U(z,t) = A(z)\cdot\exp[-j\,k\,z)]\cdot\exp[\,j2\pi\nu t] \qquad (14)$$

$$= A(z)\cdot\exp[\,j(2\pi\,\nu\,t - k\,z)]$$

$$= |A|\cdot\exp\{\,j(2\pi\,\nu\,t - k\,z + \arg[\,A])\} \qquad (15)$$

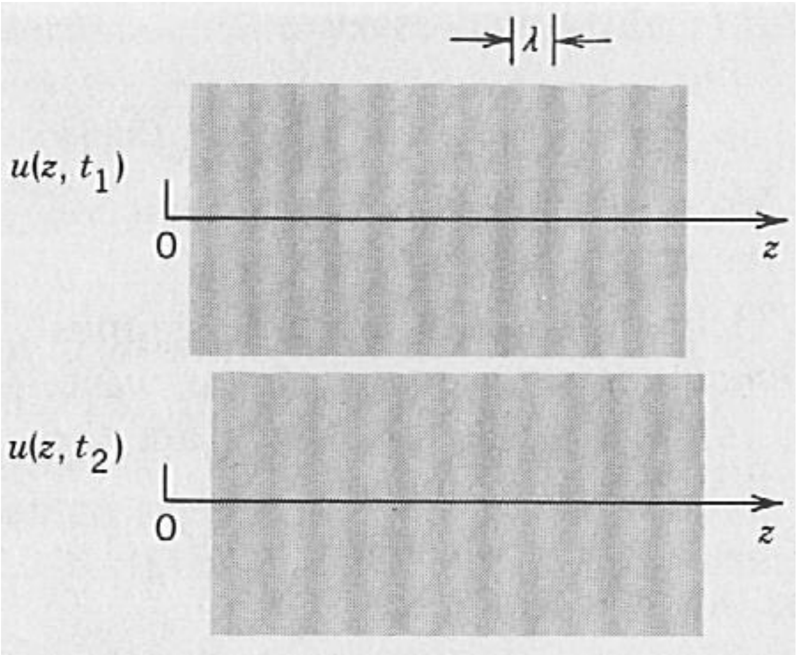
and by separating the real component of Eq. 15, it is obtained

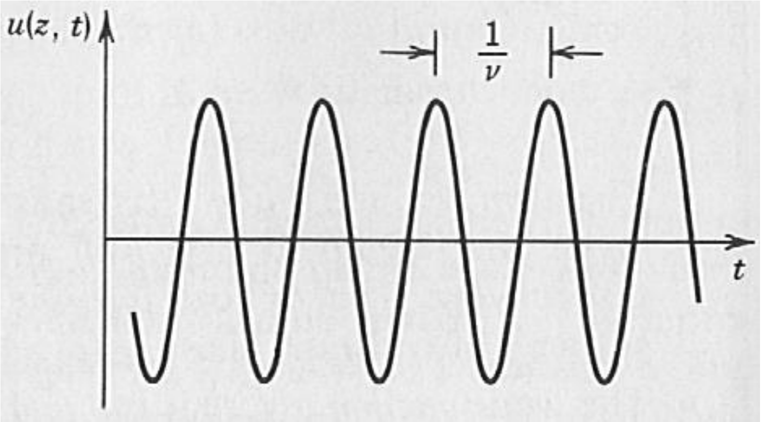
$$u(z,t) = \mathrm{Re}\{U(z,y)\} = |A|\cdot\cos(2\pi\,\nu\,t - k\,z + \arg[\,A]) \qquad (16)$$

$$= |A|\cdot\cos\{2\pi\,\nu(t - \frac{z}{c}) + \arg[\,A]\}$$

Elementary waves

Plane waves





A plane wave traveling in the z-direction is a periodic function of z with spatial period λ and a periodic function of t with temporal period $1/\nu$.

Elementary waves

Observations

- Optical phase, obtained from Eq. 16,

$$\arg[\mathrm{Re}\{U(z,t)\}] = 2\pi\nu\left(t - \frac{z}{c}\right) + \arg[A] \qquad (17)$$

varies as a function of time and position

- Optical **intensity** is determined as

$$I = |U|^2 = U \cdot U^* \qquad (18)$$

where

U^* is the complex conjugate of U

Wave optics: light waves

Elementary waves

(2) The spherical wave

The spherical wave is another canonical solution of the Helmholtz equation. Its complex amplitude is

$$U(r) = \frac{A}{r}\exp(-j\,k\,r) \qquad (19)$$

where r is the distance from the propagation origin.

$k = \frac{2\pi}{\lambda}$ is the wave number, and

$I = U \cdot U^* = \frac{|A|^2}{r^2}$ (proportional to the square of the distance from the origin)

Elementary waves

Spherical waves

Taking $\arg[A] = 0$, for simplicity,

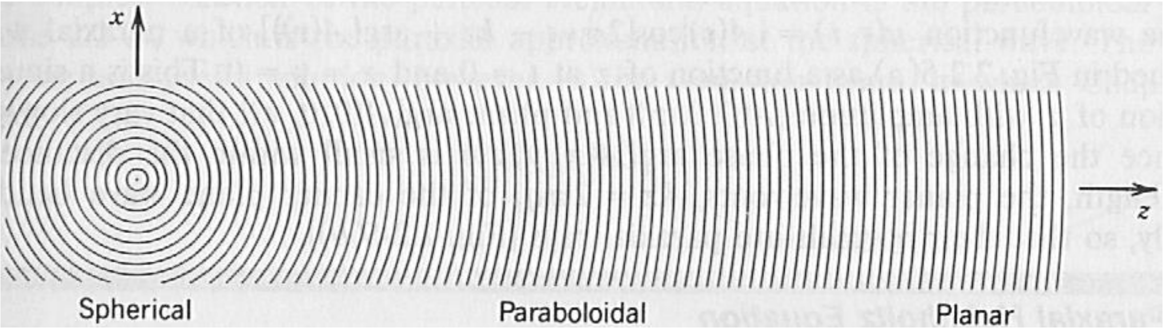
$$k \cdot r = k\sqrt{x^2 + y^2 + z^2} = 2\pi q + \arg[A] \qquad (20)$$

for q = integer

Equation 20 describes the family of concentric spheres: spherical wavefronts.

Spheres are separated by the distance

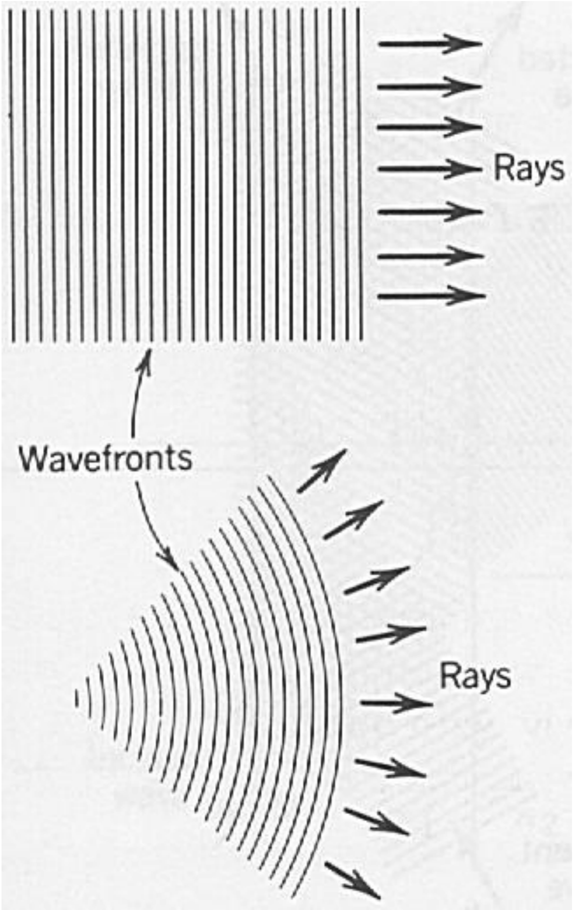
$$\lambda = \frac{2\pi}{k} \qquad (21)$$



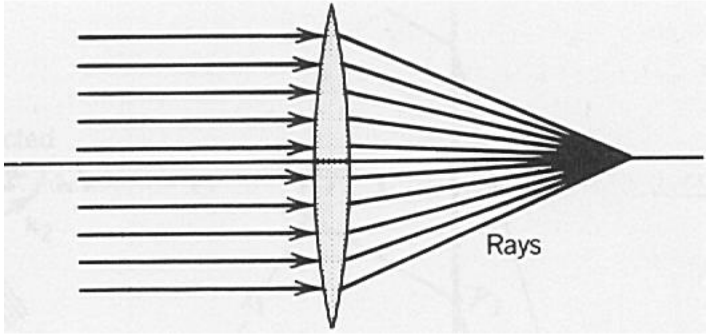
Cross-section of the wavefronts of a spherical wave

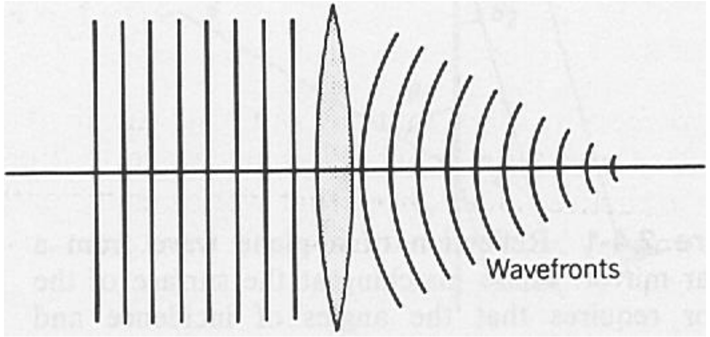
Elementary waves

Wavefronts



Effect of adding a lens



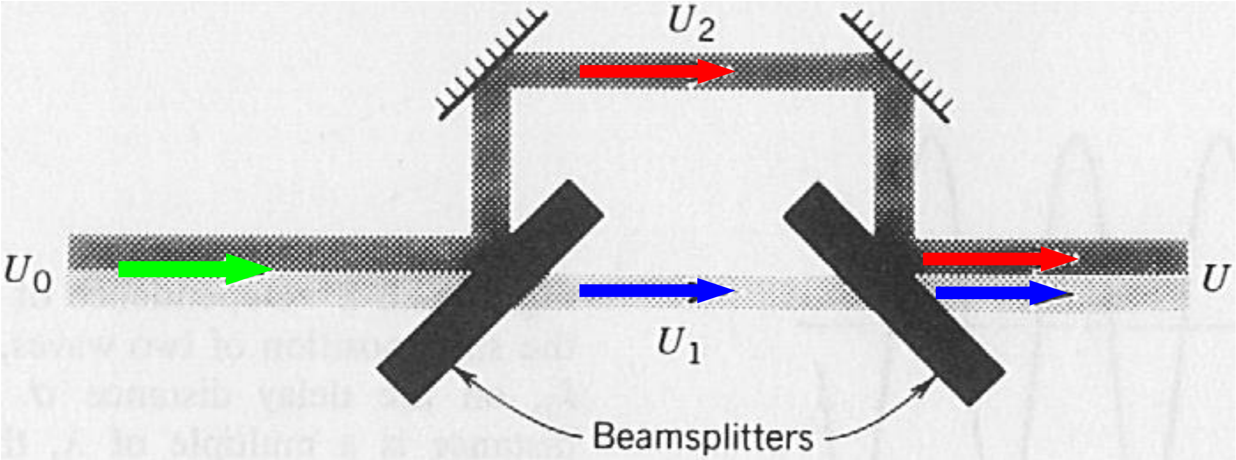


The rays of ray optics are orthogonal to the wavefronts of wave optics. Note the effect of a lens on rays and wavefronts.

Optical interference

Interferometers

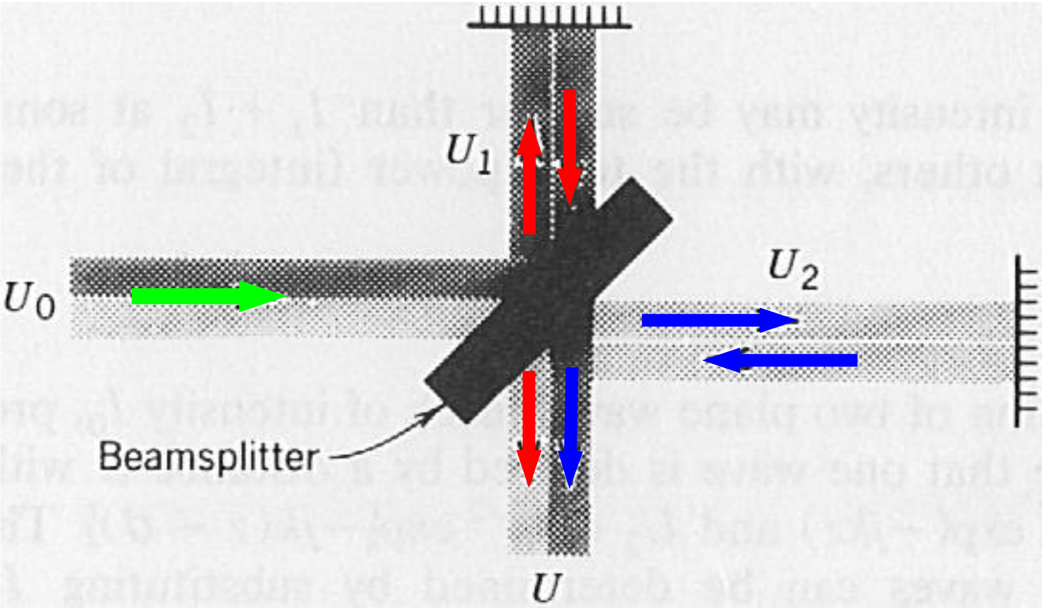
Mach-Zender



Optical interference

Interferometers

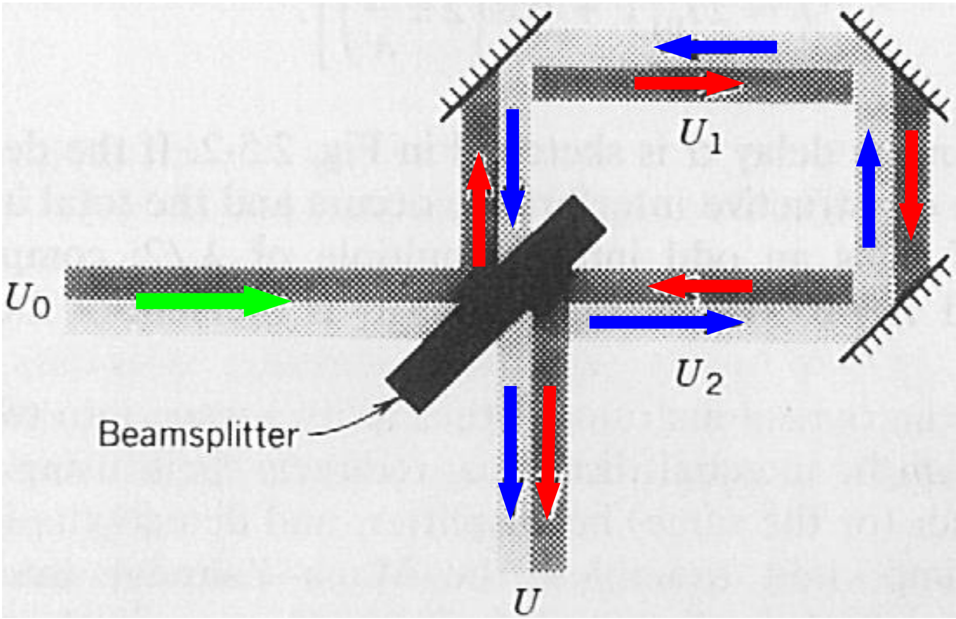
Michelson



Optical interference

Interferometers

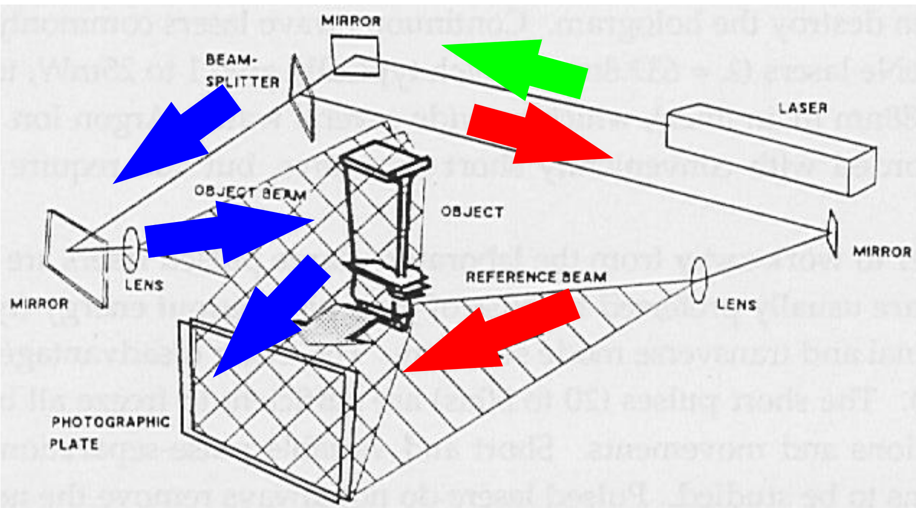
Sagnac



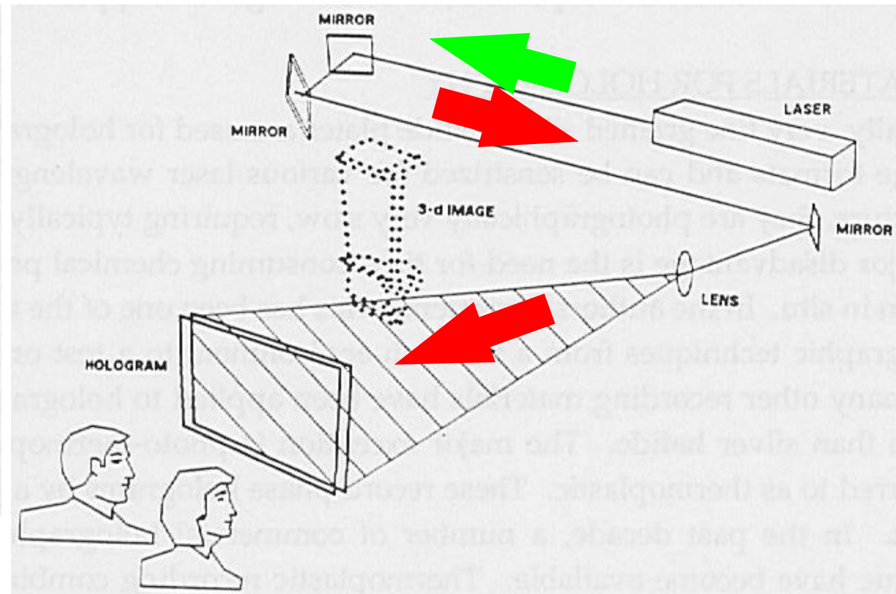
Optical interference

Holographic interferometry

Recording



Reconstruction



Interference equation

Consider the superposition of two monochromatic plane waves U_1 and U_2 from the same light source

$$U = U_1 + U_2 \quad (22)$$

The corresponding intensity is,

$$I = |U|^2 = |U_1 + U_2|^2 = |U_1|^2 + |U_2|^2 + U_1 U_2^* + U_1^* U_2 \quad (23)$$

if $\quad U_1 = A_1 \exp(-j\,\boldsymbol{k}_1 \cdot \boldsymbol{r}) = A_1 \exp(-j\,\phi_1),$

$$U_2 = A_2 \exp(-j\,\boldsymbol{k}_2 \cdot \boldsymbol{r}) = A_2 \exp(-j\,\phi_2)$$

The observed intensity, measured, is

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_2 - \phi_1)} \quad (24)$$

Interference equation, cont'd

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_2 - \phi_1) \qquad (25)$$

Defining:

$I_B = I_1 + I_2 =$ Background intensity

$I_M = 2\sqrt{I_1 I_2} =$ Modulation intensity

Intensity becomes:

$$I = I_B + I_M \cos(\Delta\phi) \qquad (26)$$