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# Binary Δ-matroids and canonical forms

Remi Cocou Avohou, **Brigitte Servatius**

# 1. Matroids vs. Δ-matroids

## $M$ is a Matroid

$E$ a finite set – the *ground set* of $M$
$\mathcal{B} \subseteq \mathcal{P}(E)$ – the *bases* of $M$

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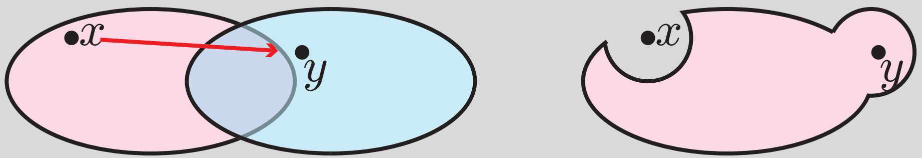
The *basis exchange axiom:*
$B_1, B_2 \in \mathcal{B}, x \in B_1 \setminus B_2 \Longrightarrow \exists y \in B_2 \setminus B_1$

$$(B_1 \cup \{y\}) \setminus \{x\} = B_1 \triangle \{x, y\} \in \mathcal{B}$$



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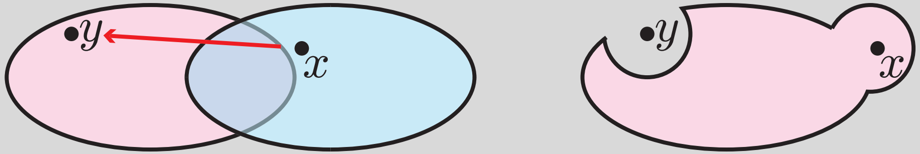

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| | | | |
|---|---|---|---|
| *Bases* | | – | $B \in \mathcal{B}$. |
| *Independent sets* | $\mathcal{I}$ | – | $I \subseteq B \in \mathcal{B}$. |
| *Dependent sets* | $\mathcal{D}$ | – | $D \notin \mathcal{I}$ |
| *Cycles (circuits)* | $\mathcal{C}$ | – | $C \in \mathcal{D}$, $C \not\subset D \in \mathcal{D}$ |
| *Spanning sets* | $\mathcal{S}$ | – | $S \supset B \in \mathcal{B}$. |

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| | | | |
|---|---|---|---|
| Whitney | 1935 | [18] | |
| W. T. Tutte | 1971 | [14] | (standard text) |
| D. J. A. Welsh | 1976 | [16] | (graph theory) |
| James Oxley | 2011 | [11] | (geometric/algebraic) |
| András Recski | 1989 | [13] | (applied approach) |
| Leonidas Pitsoulis | 2014 | [12] | |

## $D$ is a $\Delta$-matroid

The *symmetric exchange axiom:*

$$F_1, F_2 \in \mathcal{F}, x \in F_1 \triangle F_2 \implies \exists y \in F_1 \triangle F_2$$

$$F_1 \triangle \{x, y\} \in \mathcal{F}$$

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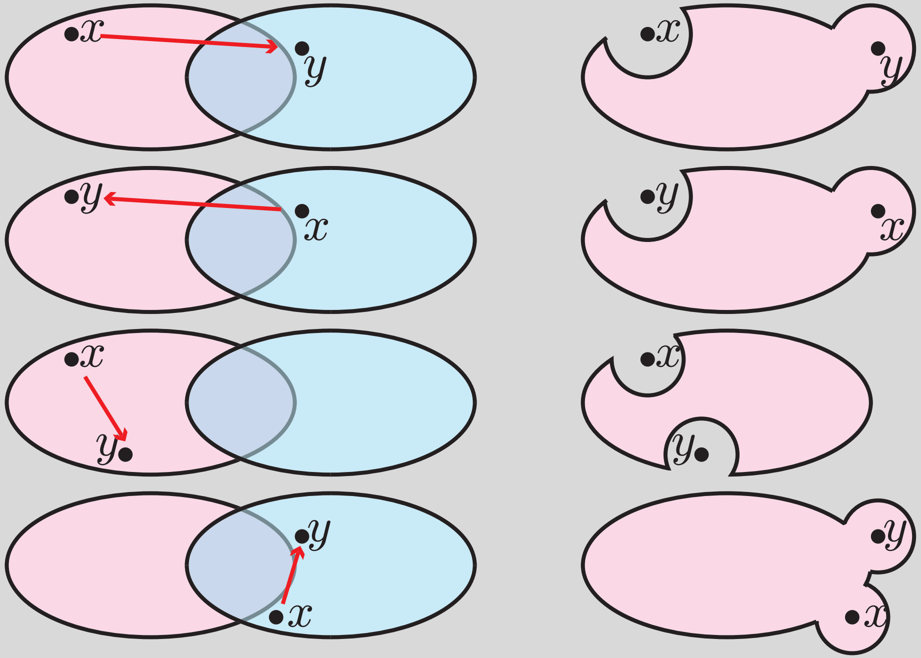
# $D$ is a Δ-matroid

The *symmetric exchange axiom:*

$F_1, F_2 \in \mathcal{F}, x \in F_1 \triangle F_2 \Longrightarrow \exists y \in F_1 \triangle F_2$

$$F_1 \triangle \{x, y\} \in \mathcal{F}$$



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$$F_1 \triangle \{x, y\} \in \mathcal{F}$$

| | | | |
|---|---|---|---|
| Bouchet | 1987 | [3] | (Δ-matroids) |
| Bouchet | 1998 | [4, 5, 7, 6] | (multimatroids) |
| Dress & Havel | 1986 | [9] | (metroids) |
| Chandrasekaran | 1988 | [8] | (pseudometroids) |

# 2. Matroids in Δ-matroids

Given a Δ-matroid $D$, we find two matroids in $D$:

$M_u$, the *upper matroid*, whose bases are the feasible sets with largest cardinality

$M_l$, the *lower matroid*, whose bases are the feasible sets with least cardinality

**Theorem 1** *Let $M = (E, \mathcal{B})$ be a matroid with independent sets $\mathcal{I}$. Then $D = (E, \mathcal{I})$ is a $\Delta$-matroid.*

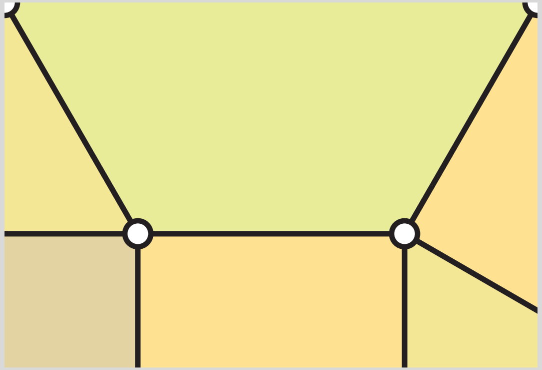
The upper matroid is $(E, \mathcal{B})$ and the lower matroid $(E, \emptyset)$.

# 3. Combinatorial Maps

## Combinatorial Maps and Δ-matroids

Tutte, in the introduction to his paper *What is a map?* [15] remarks

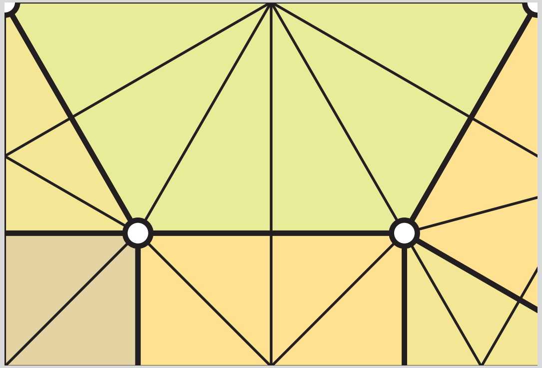
> Maps are usually presented as cellular dissections of topologically defined surfaces. But some combinatorialists, holding that maps are combinatorial in nature, have suggested purely combinatorial axioms for map theory, so that that branch of combinatorics can be developed without appealing to point-set topology.



Tutte's idea is that each edge of a map is associated with four flags, corresponding to the triangles in the barycentric subdivision.

The map can be uniquely described in terms of three perfect matchings. Two flags are matched if they differ in exactly one vertex.

Faces, Euler characteristic, and orientability can be treated combinatorially without appealing to topology.



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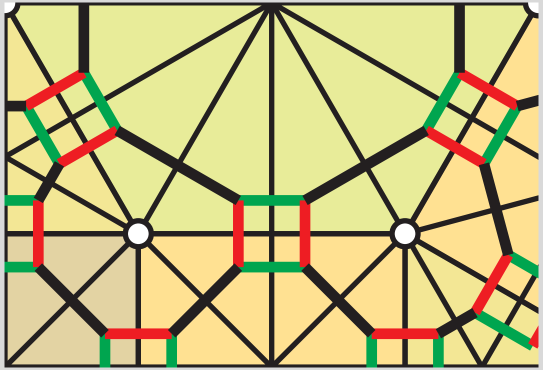

Tutte's idea is that each edge of a map is associated with four flags, corresponding to the triangles in the barycentric subdivision.
The map can be uniquely described in terms of three perfect matchings. Two flags are matched if they differ in exactly one vertex.
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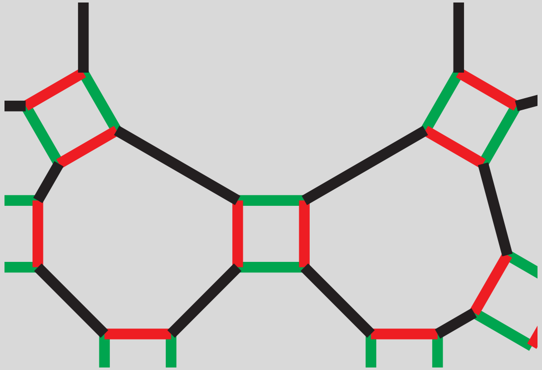

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The map can be uniquely described in terms of three perfect matchings. Two flags are matched if they differ in exactly one vertex.

Faces, Euler characteristic, and orientability can be treated combinatorially without appealing to topology.

Let $\Gamma$ be a graph whose edges are partitioned into three classes $R$, $G$, and $B$ which we color respectively red, green, and black. $\Gamma$ is called *map graph* or a *combinatorial map* if the following conditions are satisfied:

1. Each color class is a perfect matching
2. $R \cup G$ is a union of 4-cycles
3. $\Gamma$ is connected

The graph $\Gamma$ is 3-regular and edge 2-connected. $\Gamma$ may have parallel edges, although necessarily not red/green. $\Gamma$ contains 2-regular subgraphs which use all the black edges of $\Gamma$, which we call *fully black* 2-regular subgraphs; $R \cup B$ and $G \cup B$ are examples, and there always exists a fully black Hamiltonian cycle.

Proof of Hamiltonicity:

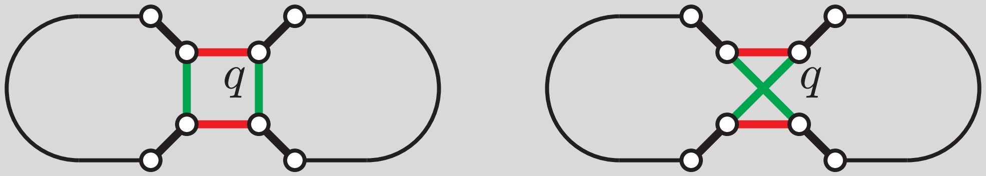
First note that a fully black 2-regular subgraph cannot contain any incident green and red edges, so every red/green quadrilateral intersects a fully black 2-regular subgraph in either two red, or two green edges.

Now consider a fully black 2-regular subgraph of $\Gamma$ with the fewest connected components.

If there is not a single component, then there is a green/red quadrilateral which intersects the subgraph in, say, two red edges which belong to two different components, and swapping red and green on that quadrilateral reduces the number of components of the subgraph, violating minimality.

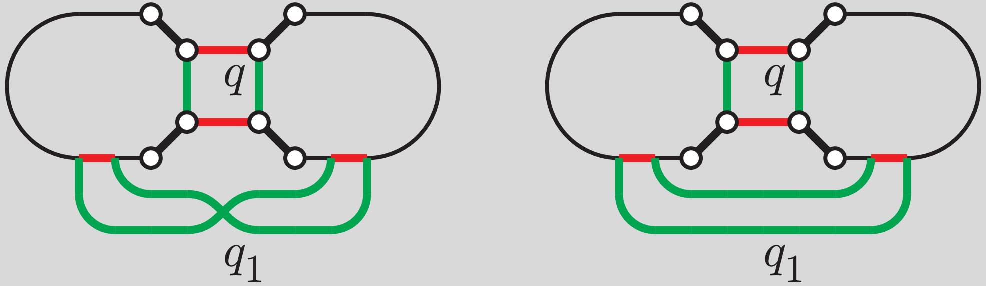
**Theorem 2** [2] *Given a combinatorial map* $\Gamma(R, G, B)$*,*
*let* $E$ *be the set of quadrilaterals of* $R \cup G$*, and*
*let* $\mathcal{F}$ *be the collection of subsets of* $E$ *corresponding to the pairs of green edges in a fully black Hamilton cycle in* $\Gamma$*.*
*Then* $(\mathcal{F}, E)$ *is a* $\Delta$*-matroid.*

We have to show the symmetric exchange property. Let $F_C$ and $F_{C'}$ be sets of quadrilaterals corresponding to fully black Hamiltonian cycles $C$ and $C'$. Let $q \in F_C \triangle F'_C$, so the edges of quadrilateral $q$ are differently colored in $C$ and $C'$, say red and green. There are two cases, either replacing in $q$ the red edges in $C$ with the green of $C'$ results in two components or one. See Figure .




If it results in just one component, then take $q' = q$, and $F_c \triangle \{q, q'\} = F_c \triangle \{q\}$ is the set of red quadrilaterals of a fully black Hamiltonian cycle, and hence feasible, as required.

Otherwise, if there are two components, the Hamiltonian cycle of $C'$ contains a non-black edge, say green, of a quadrilateral $q_1$, connecting those two components, and necessarily both red edges of $q_1$ are in $C$ and both green edges of $q_1$ connect the components, and $q' \in C \triangle C'$.




Regardless of how the green edges of $q_1$ are placed, swapping the edges of both $q$ and $q_1$ in $C$ yields a new fully black Hamiltonian cycle, so the set $Q \triangle \{q, q_1\}$ is feasible, as required.

Since $R$, $G$ and $B$ are perfect matchings, the union of any two them induces a set of disjoint cycles. Let $V$ be the set of cycles of $R \cup B$, $E$ be the set of cycles of $R \cup G$, and $V^*$ be the set of cycles of $G \cup B$. There is a graph $(V, E)$ where incidence is defined between a red-black cycle and a red-green cycle if they share an edge, and, similarly, there is a graph $(V^*, E)$ where incidence is defined between a green-black cycle and a red-green cycle if they share an edge. We say that $\Gamma$ encodes the graph $(V, E)$ and its geometric dual $(V^*, E)$.

**Theorem 3** *Let* $\Gamma(R, G, B)$ *be a combinatorial map and let* $D_\Gamma = (\mathfrak{F}, E)$ *be its associated* $\Delta$*-matroid.*
*Then*
*the lower matroid of* $D_\Gamma$ *is the cycle matroid of* $(V, E)$
*and*
*the upper matroid of* $D_\Gamma$ *is the cocycle matroid of* $(V^*, E)$.

PROOF: Given $\Gamma(R, G, B)$, recall that the feasible sets of $D$ consist of $RG$ quadrilaterals whose $R$ edges are contained in a fully black Hamilton cycle of $\Gamma$.

Any fully black Hamilton cycle $C$ of $\Gamma$ must contain the red edges corresponding to a spanning tree of $(V, E)$ as well as the green edges corresponding to a spanning tree of $(V^*, E)$. So the minimal number of red edges in $C$ is $2(|V| - 1)$, while the maximal number is $2(|E| - |V^*| + 1)$.

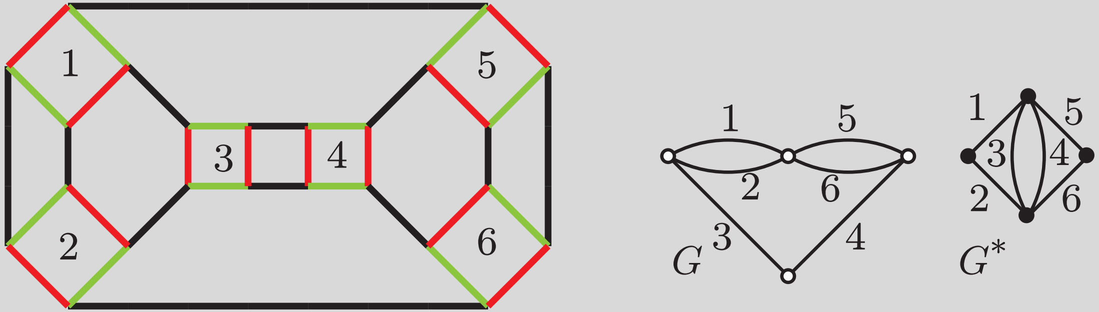
The edge sets of the spanning trees of $(V, E)$ are the bases of its cycle matroid, while the complements of edge sets of spanning trees in $(V^*, E)$ are the bases of the cocycle matroid of $(V^*, E)$. □

- The difference in rank of the upper and lower matroid of $(\mathfrak{F}, E)$ is given by

$$(|E| - |V^*| + 1) - (|V| - 1) = 2 - \chi,$$

– $\chi$ is the Euler characteristic.

•If $\Gamma$ is bipartite, all feasible sets of $D_\Gamma = (\mathfrak{F}, E)$ must have the same parity – since exchanging a red and green pair of edges always disconnects a Hamilton cycle of a bipartite $\Gamma$.



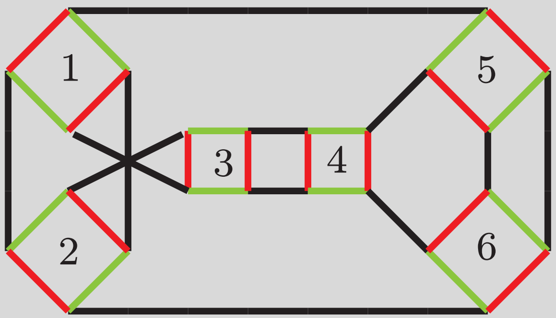

$$\mathfrak{F} = \{\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},$$
$$\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,4,6\}\}$$

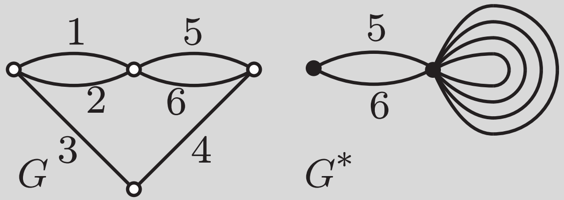
.

- $\mathfrak{F}$ is the set of spanning trees of $G$ and at the same time the set of co-trees of $G^*$ so

– all feasible sets have the same size and

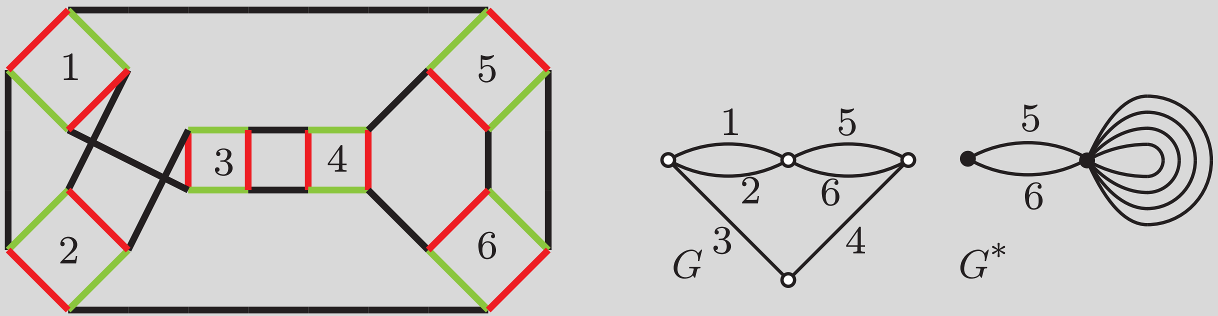
– the upper and lower matroid are identical.




$$\mathfrak{F} = \{\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},$$
$$\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,4,6\},$$
$$\{1,2,3,4,5\},\{1,2,3,4,6\}\}$$

The lower matroid is again the cycle matroid of $G$,
but the upper matroid is the co-cycle matroid of $G^*$,
The upper matroid has rank 5 and contains exactly one cycle, namely $\{5,6\}$,
which is a minimal cutset of $G^*$ and also a cycle in $G$.

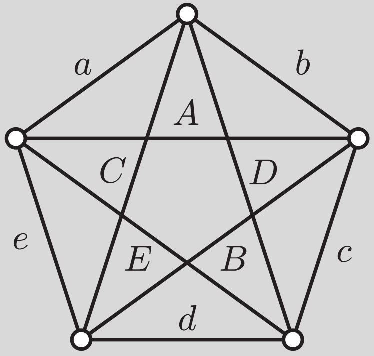


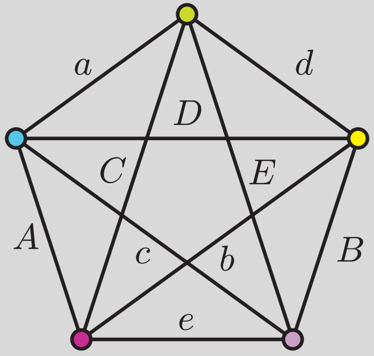

The Δ-matroid associated to this the map has,
in addition to the feasible sets of the previous example,
the feasible set $\{1, 2, 3, 4\}$, whose parity is even, while the parity of all other feasible sets is odd,
so this map is not orientable.

However, if both $G$ and $G^*$ are 3-connected, then the map is uniquely recoverable from the $\Delta$-matroid information.

**Theorem 4** *Let $D$ be the $\Delta$-matroid of a map $M$ with 2-connected upper- and lower matroid. Then $M$ is determined by $D$.*

PROOF: By Whitney's theorem [17], upper and lower matroid uniquely determine $G$ and $G^*$.
To recover $M$ from $D$, we need to specify a rotation system for each vertex $v$ of $G$.
To determine if two edges $e$ and $f$ with endpoint $v$ follow each other in the rotation about $v$, it is enough to check if $e$ and $f$ are both incident in $G^*$, since the vertex co-cycles of $G^*$ correspond to the facial cycles of the embedded $G$.
Now re-construct the map graph □

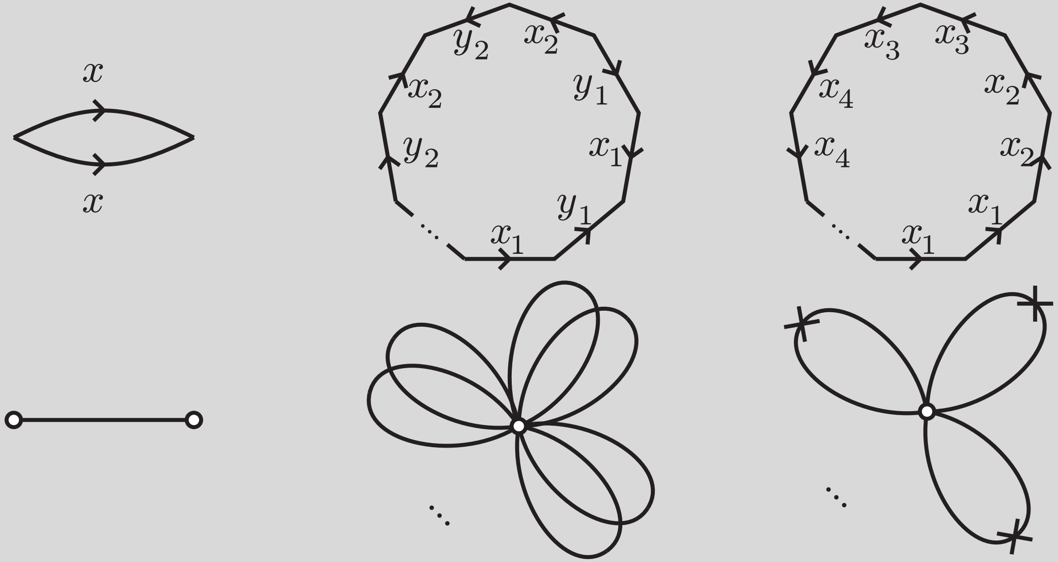



| | | | | |
|---|---|---|---|---|
| $a$ $A$ | $b$ $D$ | $c$ $B$ | $d$ $E$ | $e$ $C$ |
| $c$ $B$ | $d$ $E$ | $e$ $C$ | $a$ $A$ | $b$ $D$ |
| $e$ $C$ | $a$ $A$ | $b$ $D$ | $c$ $B$ | $d$ $E$ |
| $b$ $D$ | $c$ $B$ | $d$ $E$ | $e$ $C$ | $a$ $A$ |
| $d$ $E$ | $e$ $C$ | $a$ $A$ | $b$ $D$ | $c$ $B$ |

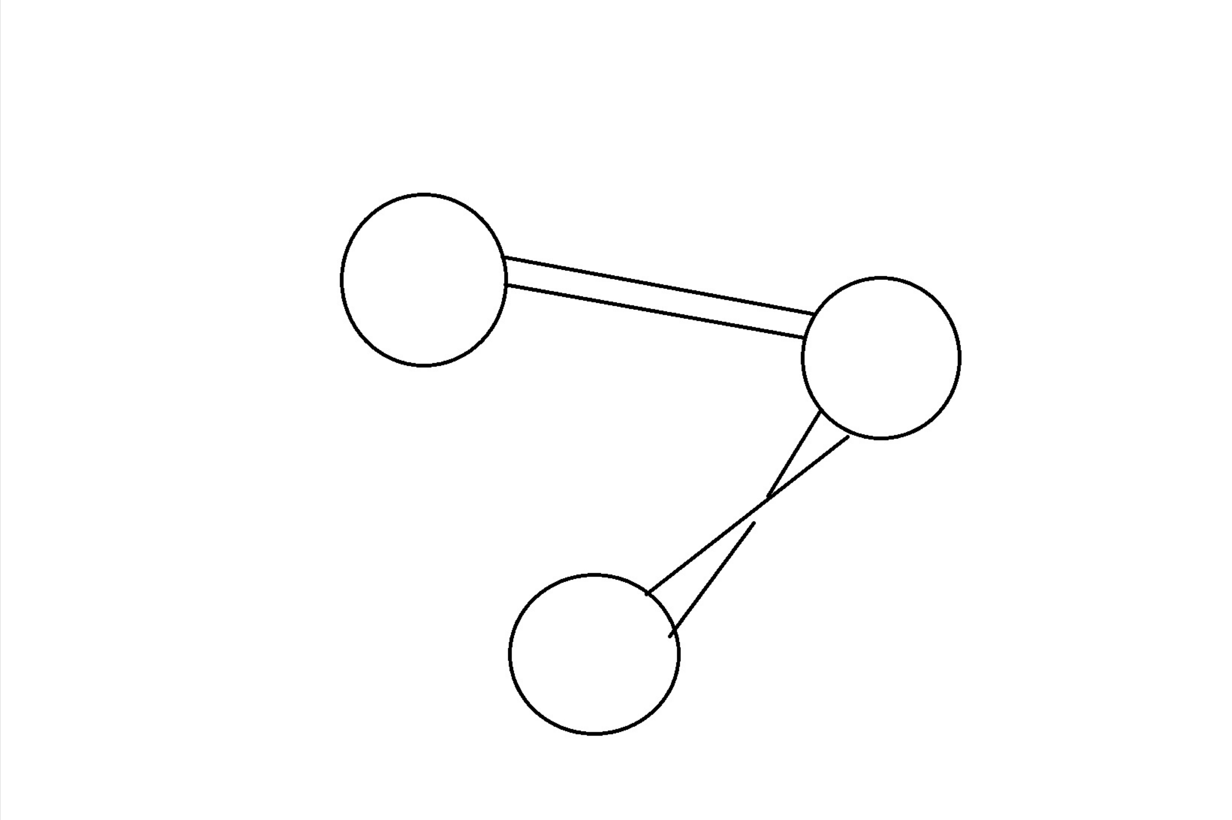
# 4. Normal Forms

| $S$ | $\chi(S)$ |
|---|---|
| sphere | 2 |
| g tori | $2-2g$ |
| g crosscaps | $2-g$ |



# 5. Handle slides

## Handle slides for ribbon graphs



# Handle slides

See [10]
Let $D = (E, \mathcal{F})$ be a set system, and $a, b \in E$ with $a \neq b$. We define $D_{ab}$ to be the set system $(E, \mathcal{F}_{ab})$ where

$$\mathcal{F}_{ab} = \mathcal{F} \Delta \{X \cup a | X \cup b \in \mathcal{F} \text{ and } X \subseteq E - \{a, b\}\}.$$

We call the move taking $D$ to $D_{ab}$
a handle slide taking $a$ over $b$.

# Example

## A handle slide on $U_{2,4}$

$$\mathcal{F} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

$$\mathcal{F}_{1,2} = \mathcal{F}\Delta\{\{1,3\},\{1,4\}\} = \{\{1,2\},\{2,3\},\{2,4\},\{3,4\}\}$$

$\mathcal{F}_{1,2}$ does not satisfy the Basis Exchange Axiom! A matroid is binary iff it does not contain $U_{2,4}$ as a minor.
The class of binary Δ-matroids is closed under handle slides [1].

## Analogue of Unitary map normal forms

See [10]

Let $D = (E, \mathcal{F})$ be a binary Δ-matroid in which the empty set is feasible. Then, for some $i$, $j$, $k$, there is a sequence of handle slides taking $D$ to $D_{i,j,0}$ if $D$ is even, or $D_{i,0,k}$, with $k \neq 0$, if $D$ is odd. Furthermore, if some sequences of handle slides take $D$ to $D_{i,j,k}$ and to $D_{p,q,r}$ then $i = p$, and so $D$ is taken to a unique form $D_{i,j,0}$ or $D_{i,0,k}$ by handle slides.

# $D_{i,j,k}$

$D_{i,j,k}$ is the direct sum of

- $i$ copies of $(e, \{\emptyset\})$,
- $j$ copies of $(\{e, f\}, \{\emptyset, \{e, f\}\})$, and
- $k$ copies of $(\{e\}, \{\emptyset, \{e\}\})$.

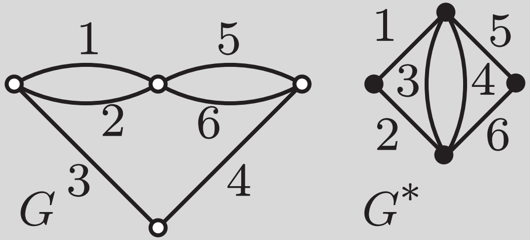
The sum is performed on isomorphic copies of these Δ-matroids with mutually disjoint ground sets.

## Normal form

$D_{i,j,k,l}$ the Δ-matroid consisting of the direct sum of $D_{i,j,k}$ with $l$ copies of the Δ-matroids isomorphic to $(e, \{\{e\}\})$.

For each binary Δ-matroid $D$, there is a sequence of handle slides taking $D$ to some $D_{i,j,k,l}$, where $i$ is the size of the ground set minus the size of a largest feasible set, $l$ is the size of a smallest feasible set, $2j + k$ is the difference in the sizes of a largest and a smallest feasible set. Furthermore, the handle slides can take $D$ to $D_{i,j,0,l}$ if $D$ is even and to $D_{i,0,k,l}$ if $D$ is odd.

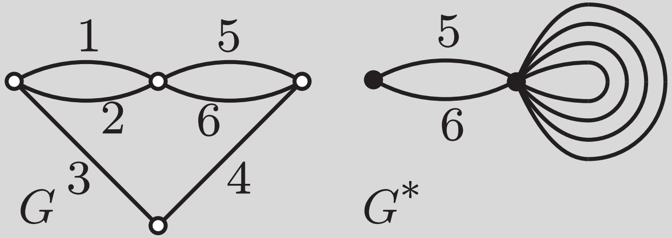
$G$ and $G^*$ on the sphere.




$$\mathcal{F}_1 = \{\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\\ \{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,4,6\}\}$$

$$D_{3,0,0,3} = (\{2\},\{\emptyset\}) \oplus (\{5\},\{\emptyset\}) \oplus (\{6\},\{\emptyset\}) \oplus (\{1\},\{\{1\}\}) \oplus (\{3\},\{\{3\}\}) \oplus (\{4\},\{\{4\}\})$$

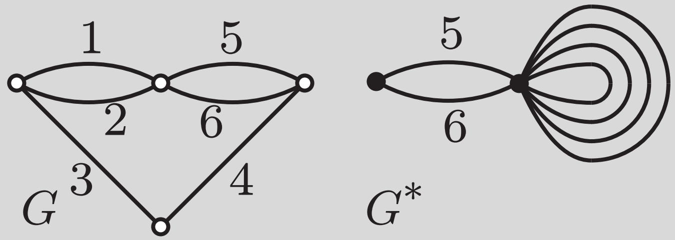
$G$ and $G^*$ on the torus.




$$\mathcal{F}_2 = \mathcal{F}_1 \cup \{\{1,2,3,4,5\},\{1,2,3,4,6\}\}$$

$$D_{1,1,0,3} = (\{6\},\{\emptyset\}) \oplus (\{2,5\},\{\emptyset,\{2,5\}\}) \oplus (\{1\},\{\{1\}\}) \oplus (\{3\},\{\{3\}\}) \oplus (\{4\},\{\{4\}\}).$$

$G$ and $G^*$ on the projective plane.




$$\mathcal{F}_3 = \mathcal{F}_2 \cup \{\{1,2,3,4\},\{1,3,4,5\}\}$$

$$D_{1,0,2,3} = (\{6\},\{\emptyset\}) \oplus (\{2\},\{\emptyset,\{2\}\}) \oplus (\{5\},\{\emptyset,\{5\}\})$$
$$\oplus(\{1\},\{\{1\}\}) \oplus (\{3\},\{\{3\}\}) \oplus (\{4\},\{\{4\}\}).$$

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