GRAPHS, DIGRAPHS, AND THE RIGIDITY OF GRIDS

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Contents

1. Rigidity of Frameworks

1.1. **Frameworks.** There is nothing like news of an earthquake to make us wonder about the strength and rigidity of buildings and bridges. For answers, we must depend upon the skill of architects and civil engineers, and they in turn, depend upon experience and mathematics.

Let's consider two very simple structures made of solid rods joined at their endpoints, one in the form of a triangle and the other a square. For the triangle, we know from geometry that the three rod lengths determine the three angles, so

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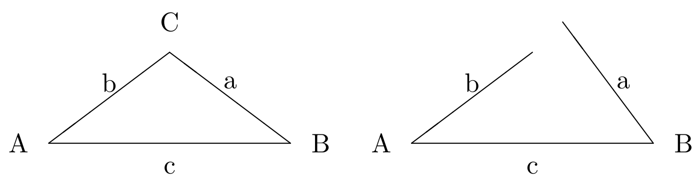
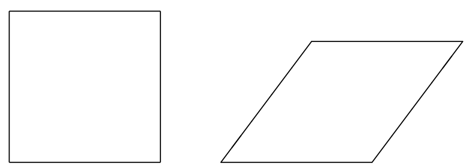
the only way a triangular structure can fail is to break a bond at one corner, or break a rod.



We say that the triangle is *rigid*.

The square, however, may be deformed into a rhombus without breaking any of its bonds or bending any of its rods.



We say the square is *not rigid*, or that the square *deforms*.

Some bonds, for instance gluing or welding, will fix angles to some extent, but this should not be relied upon. Even if the rivets and the welds are very strong, they will do a much better job keeping the objects pinned together than fixing the angles. For instance, if you nail the ends of two ordinary boards together, then it should take several hundred pounds of force to break the bond (on the order of 300lbs per nail), usually by snapping the nails or splitting the wood. On the other hand, it is not nearly so hard to bend and twist the joint since that can be done by only bending and twisting the nails. (One reason this is so easy is that the two boards each act as levers with respect to the joint.)

Since a simple joint does a relatively poor job of fixing angles, it is safer to simply assume that the joints prevent the bars from separating, but are completely flexible with regard to twisting and turning. Such a joint is called a *ball joint*. A familiar example of a ball joint is the joint in your shoulder. You also have a ball joint in your knee, however this ball joint is only 2-dimensional, and resists twisting from the side, the source of numerous sports injuries.

In this module we will study the rigidity of frameworks made of rigid rods in the plane connected by 2-dimensional ball joints.

1.2. **Walls and Grids.** A modern wall often consists of a thin light surface material covering a combination of insulations and backed by a rigid framework. Nonwasteful use of lumber or steel requires that the rods in the framework not be too long. A natural choice for a framework is a grid of squares, as pictured below. Each square in the grid is made up of four short rods joined at the four corners. Let us call the vertical rods *posts*, and the horizontal rods *beams*.

Since we are making a wall, we want the grid framework to be rigid in the plane. Rigidity in the plane is easier to achieve than rigidity in 3-space. There are many frameworks which are rigid in the plane but will fold up in 3-space. The rectangular grid of squares, however, is not even rigid in the plane. We also say that the grid

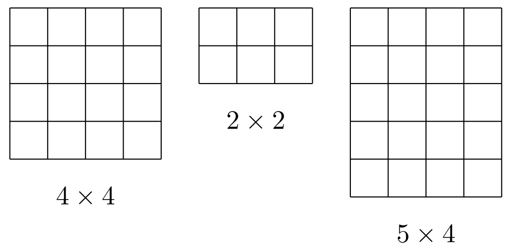
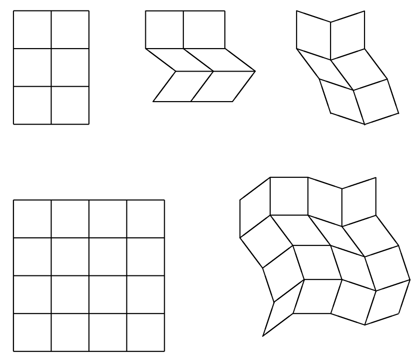


Figure 1. Grids

can be *deformed* in the plane.



Notice that, in any deformation, all the posts in the same row are parallel, since they are connected by rhombi. Similarly, all the beams in each column must be parallel in any deformation.

Fortunately, we can rigidify the grid by adding diagonal braces at each square. The fully braced grid is rigid in the plane because it is made up of triangles touching

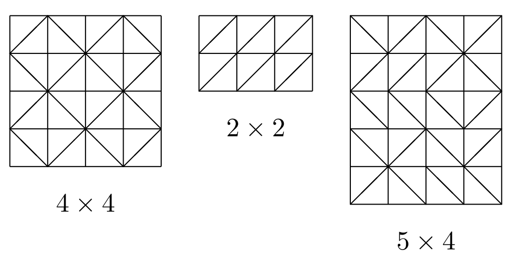


Figure 2. Fully braced grids

along edges, and triangles *are* rigid.

1.3. **Bracing a grid.** When considering a deformation of a grid, in order to specify our frame of reference, we will always fix the top left post.

We want to brace a grid so that it is rigid in the plane. To brace a single square we need to add one diagonal, so to brace a larger grid we need at most one diagonal in each square, but do we really need to brace every square? In the grids in Figure 3, every unbraced square has a corner surrounded by braced squares, and hence cannot be deformed. So the question becomes, which choices of braces rigidify a grid?

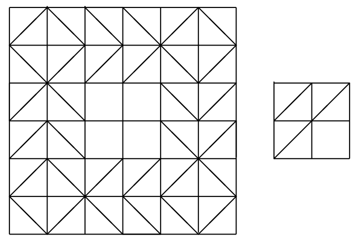


FIGURE 3. Rigid braced grids

Let the angle made with the horizontal by the posts in row i be ρ_i and the angle made with the horizontal by the beams in column j be κ_j, see Figure 4 In order

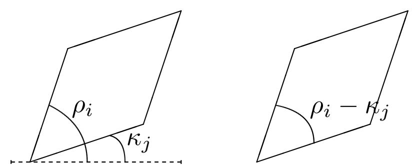


FIGURE 4. Row and column angles.

to compare the sizes of these angles, let's assume that we are looking for small deformations, say with $-20° < \kappa_j < 20°$ and $70° < \rho_i < 110°$.

Then a brace in square (i, j) forces the posts in row i perpendicular to the beams in column j. So we have the equation

$$\rho_i = \kappa_j + 90°. \tag{1}$$

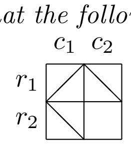
Thus every braced grid corresponds to a system of linear equations as above together with the equation

$$\rho_1 = 90$$

which comes from our choice of frame of reference.

We know the linear system has one solution, namely $\rho_i = 90$, and $\kappa_j = 0$. If this is the only solution, then we may conclude that the grid is rigid, and vice versa. This means that rigidity may be decided simply by ordinary linear algebra.

EXAMPLE 1. *We can determine that the following grid is rigid.*



The system of equations is

$$\begin{aligned} \rho_1 &= 90 \\ \rho_1 &= \kappa_1 + 90 \\ \rho_1 &= \kappa_2 + 90 \\ \rho_2 &= \kappa_1 + 90 \end{aligned}$$

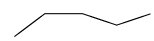
Which is easily solved to yield $\rho_1 = \rho_2 = 90$ *and* $\kappa_1 = \kappa_2 = 0$.

1.4. **The degree of freedom of a grid.** The *degree of freedom* of a grid is the number of braces required to rigidify it.

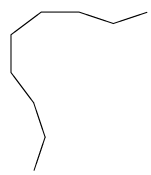
In order to find out how many braces are needed to rigidify an mxn grid, we observe that for every joint we may choose two coordinates. Fixing the top left joint leaves $(m+1)(n+1)-2$ coordinates to be determined. Each of the $m(n+1)$ posts and $n(m+1)$ beams fixes the distance between a pair of joints and hence reduces the degree of freedom of the system by 1. If we moreover insist that the top left post be vertical we see that the unbraced grid has $m+n-1$ degrees of freedom and we therefore need at least $m+n-1$ braces to achieve rigidity.

To draw a deformation of a grid, we need only to choose directions for the posts and beams and we have, leaving the top left beam fixed, $m+n-1$ free choices to do so:

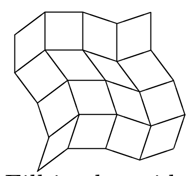
Choose the post directions



Choose the beam directions.



Make a corner



Fill in the grid

For and $n \times m$ grid there are $n+m-1$ variables, $\{\rho_2, \ldots, n\}$ and $\{\kappa_1, \ldots, \kappa_m\}$, since we are assuming that $\rho_1 = 90$. So to specify a unique solution, we require $n+m-1$ independent equations, hence at least $n+m-1$ braces. Of course, not every choice of $n+m-1$ braces will rigidify a grid, since, as is not surprising, some placements of braces do more work than others, see Figure 5. These grids each

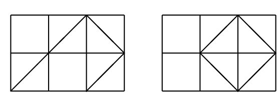
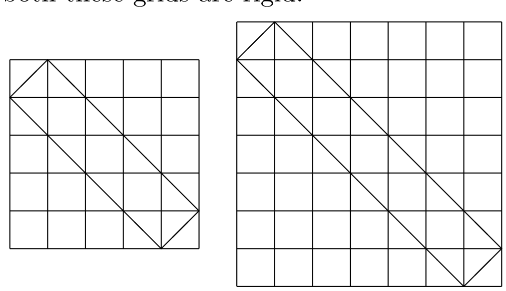


FIGURE 5. Efficient and wasteful bracings

have 4 braces, but one is rigid and the other deforms.

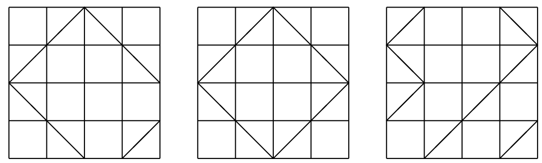
1.5. **Exercises.**

(1) Prove that both these grids are rigid.



Can you find any unnecessary braces?

(2) Determine which of the following grids are rigid.



If one is non-rigid, then draw a deformation.

(3) Find 9 braces which will make a 5×5 grid rigid.
(4) Suppose that a grid is braced so that there is some brace which is the only brace in its row, and that same brace is the only brace in its column. Prove that the grid has a deformation.
(5) Suppose that a $n \times n$ grid is braced so that there are at least two braces in every row, and at least two braces in every column. Is it true that the grid is rigid?

2. Graphs and Grids

2.1. The brace graph. Mathematically, a grid is a collection of rows of parallel posts and columns of parallel beams, and if there is a brace in square (i, j), then the posts in row i are perpendicular to the beams in column j. So the braces in a grid induce a binary relation on the set of rows and columns, which is naturally represented by a graph.

The *brace graph* has vertices $\{r_1, r_2, r_3, \ldots\}$ for the rows and vertices $\{c_1, c_2, c_3, \ldots\}$ for the columns, and an edge between r_i and c_j if there is a brace in square (i, j).

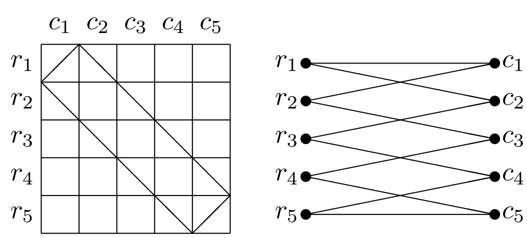


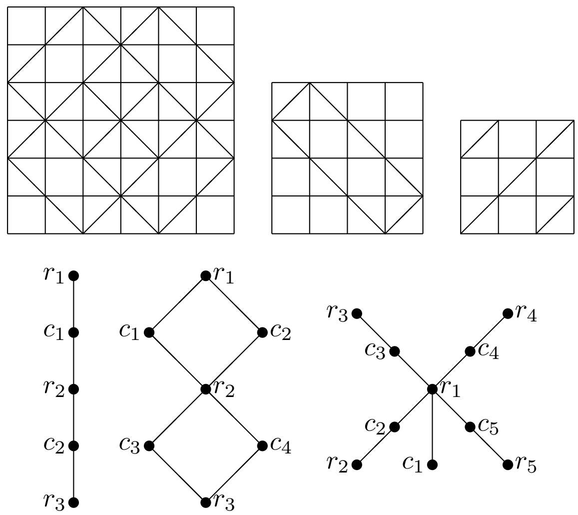
Figure 6. The Brace Graph of a Grid

The brace graph carries all the information that is contained in the picture of the braced grid, so we should be able to study the rigidity of braced grids directly from their brace graphs.

All brace graphs are *bipartite*, that is, the vertex set can be partitioned into two subsets so that all the edges join a vertex in the first set with a vertex in the second.

2.2. Exercises.

(1) Prove that the brace graph in Figure 6 can be redrawn as a decagon.
(2) Draw the brace graph for each of the following grids. See if you can find a "nice" picture for each graph.
(3) Draw the braced grids that go with the following brace graphs.
(4) Show that any bipartite graph is the brace graph of some grid.

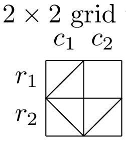


2.3. **Connected brace graphs.** We may easily detect rigidity via the brace graph.

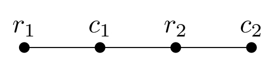
THEOREM 1. *A braced grid is rigid if and only if its brace graph is connected.*

If a brace graph is *connected*, that is, there is a path between every pair of vertices, then there is a path from r_1 to every other r_i and c_j. Since each edge in a path stands for the post-beam relation of being perpendicular, a path from r_1 to r_i means that the posts in rows 1 and i are parallel, and a path from r_1 to c_j means that the posts in row 1 are perpendicular to the beams in column j.

We can see this in the case of a 2×2 grid with three braces,

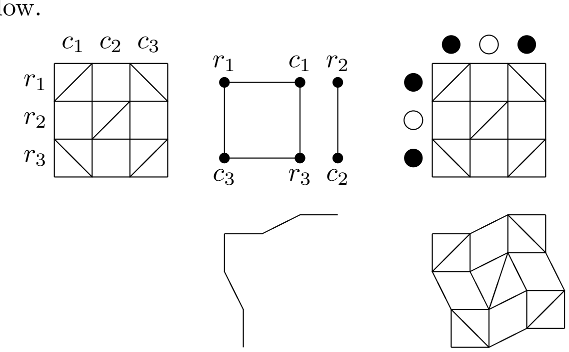


whose brace graph is a path:



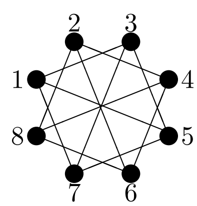
$$r_1 \perp c_1 \perp r_2 \perp c_2$$

Suppose, on the other hand, that the brace graph is *disconnected* like that of the brace graph below.

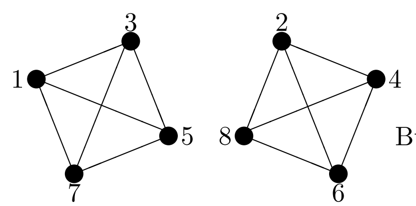


Then we would like to show that the braced grid has a deformation. Place a black dot by every row and column that is connected to r_1 in the brace graph. Place a white dot next to all the other rows and columns. Now we have to choose directions for the beams and the posts in the deformation. The directions of the rows of posts and columns of beams with black dots should be perpendicular with each other, since there may be braces between them. The same must be true for the the rows and columns with white dots, since they also may have braces between them. There is no other restriction on the directions since there is no brace connecting a black dotted row with a white dotted column or connecting a white dotted row with a black dotted column. Choose the direction of the posts in the black rows to be vertical (90°) and the posts in the white rows to be, say, 100°. Then the beams in the black columns must be horizontal, (0°) and the beams in the white columns must have angle 10°. This describes our deformation.

2.4. **An algorithm for detecting connectivity.** If a graph is not connected, or *disconnected*, then it is possible to draw the graph so that it is obvious that it is disconnected by simply drawing the vertices in clumps.



This graph looks connected . . .



But it isn't!

So, while deciding if a graph is connected or not is often quite easy by looking at the picture, we have to be careful that we do not mistake disconnected graphs for connected graphs.

The following algorithm could have been inspired by hauling in a fishing net hand over hand, dropping what we already have in the boat. If we don't get the whole net in the boat this way, then it is broken.

Step 0: Pick any vertex, circle it and call it the active vertex.

Step 1: Circle all the vertices connected by an edge to the active vertex. Put a cross on the active vertex.

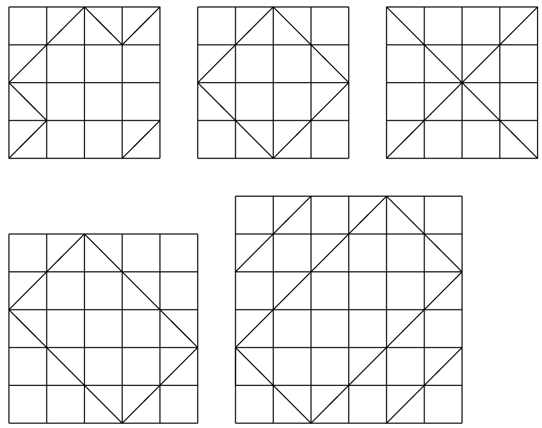
Step 2: If all the vertices are circled, then declare the graph connected and STOP.

Step 3: If all the circled vertices have crosses then declare the graph to be disconnected and STOP.

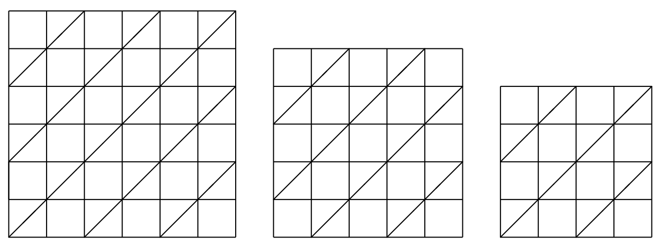
Step 4: Declare one of the uncrossed circled vertices to be the active vertex and GOTO Step 1.

2.5. **Exercises.**

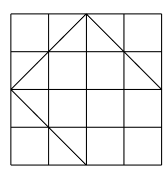
(1) Determine which of the following braced grids are rigid. Try to find a deformation for the non-rigid ones. For the rigid ones try to find any unnecessary braces.



(2) Explain why none of the following “checkerboard” braced grids are rigid.



(3) This grid needs just one more brace to make it rigid. Find all the squares in which you can add a brace to make the grid rigid.



(4) Find a graph which is not the brace graph of any grid. Can you find infinitely many?

2.6. **Trees and Efficiency.** A *cycle* in a graph is a path that starts and stops on the same vertex. If a graph is connected but has no cycles it is called a *tree.* In a tree there is exactly one path between any two vertices. Removing any edge from a tree disconnects it.

If we want to brace a grid efficiently, that is, so that there are no unnecessary braces, we want the brace graph to be connected but have no unnecessary edges. Thus a grid is efficiently braced when the brace graph is a tree.

Every connected graph has a spanning tree, that is, a subgraph which is a tree and contains all the vertices of the original graph. This means that every rigid braced graph can be made efficient by simply deleting some of the braces.

2.7. **Investigation: Fault tolerant bracings.** One problem with efficient bracings is that there is no margin for error. If any brace breaks in an efficiently braced grid, then there is a deformation. In the rigid braced grid below there is one brace

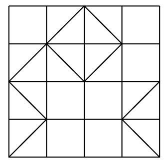
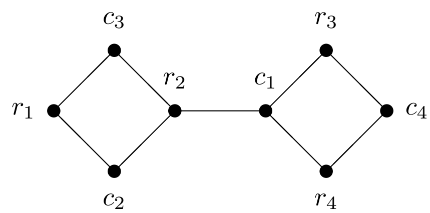


FIGURE 7. Where is the critical brace.

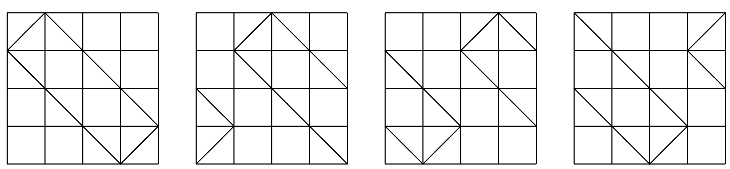
which is *critical*, that is, if it breaks, then the grid deforms. Such a brace corresponds to a *bridge* in the brace graph, i.e. an edge whose deletion disconnects the graph.



It is clear that the only bridge in the brace graph above is the edge between c_1 and r_2.

If we want the grid to remain rigid after the failure of any brace, then the brace graph must be edge 2-connected, that is, the graph must be connected, and have no bridges. One way to check 2-connectivity is to make sure that every edge is contained in a cycle. In general, the edge connectivity of the brace graph is related to the *fault tolerance* of the the grid.

2.8. **Amazing transformations.** In the following sequence of grids each grid is obtained from the previous one by tearing off a column from the right hand side and gluing it on the left.



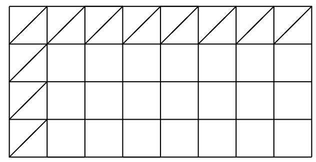
In general, rearranging the rows or columns of a grid simply relabels the vertices in the brace graph, so can have no effect on the rigidity of the grid.

On the other hand, any automorphism of a connected brace graph corresponds to a rearrangement of the rows and columns of a brace graph, together perhaps with turning the grid 90°.

2.9. **Exercises.**

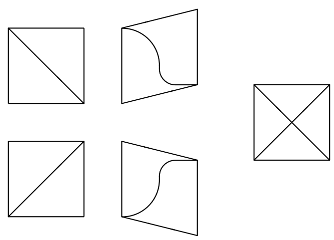
(1) Is every tree the brace graph of some grid? Can you find a relation between the shape of the tree and the dimension of the grid, for instance, what grids correspond to stars? To paths?
(2) Find the critical brace in the grid below. Show how to move exactly one brace so that the resulting grid has no critical braces.
(3) Show that the $n \times n$ grid needs only $2n$ braces in a rigid fault tolerant bracing. What is the form of the brace graph in this case? What about an 3×4 grid? What about 3×5?

(4) What is the smallest number of braces which must be added to the grid below so that no brace failure gives a non-rigid grid.

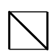


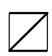
3. Directed Graphs and Tensegrity

3.1. **Bracing grids with cables.** So far we have been bracing our grids with rigid rods. A cheaper and lighter solution is to brace the grid with *cables.* This is a different mathematical problem. If we brace a single square with one cable it is *not* rigid since the cable may buckle. The cable does not stretch, however, so that the



corners of the square at which the cable is attached cannot be acute. This means that, unlike rod braces, it matters which diagonal of the square is braced with a cable, so let us distinguish these braces as N-cable braces and Z-cable braces.

 An N-cable brace

 A Z-cable brace

A square with both a Z-cable and an N-cable brace is rigid. This means that we can turn any rigid rod-braced grid into a rigid cable-braced grid by replacing every rod brace with two cables in the same square. In fact, if there were no unnecessary

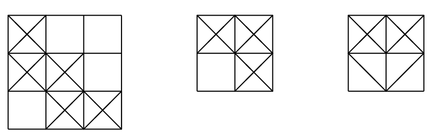
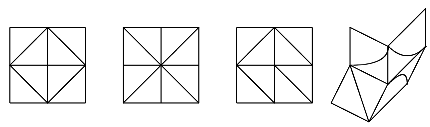


Figure 8. Some rigid cable-braced grids.

rods, it is not too difficult to verify that every cable is necessary as well - if we remove any one of them, the grid deforms. This is not the only configuration, however, in fact, as we will see shortly, the cable braced grid on the right above is also rigid.

Since the degree of freedom of a grid is $n+m-1$, we might expect that it will take at least $2(n+m-1)$ cable braces to make an $n \times m$ grid rigid. What is quite startling is that this is often false. There are rigid $n \times n$ cable-braced grids with only $2n$ cables, one more than the number of rods required, even though each cable does only half the work of a rod!

The 2×2 grid below can be rigidified with only 4 cables. The cable-braced grid



at the left is rigid because the cables prevent each of the four angles that meet in the middle from being obtuse. Since they add to 360, they must all be right angles. The same reasoning applies to the second cable-braced grid, where none of the middle angles can be acute. Notice that the cabling is working together. If one of the cables is attached the other way, as in the third grid, the effect is lost and there can be a severe deformation.

3.2. **Cables and Linear Inequalities.** Recall that ρ_i is the angle made with the horizontal by the posts in row i, and κ_j be the angle made with the horizontal by the beams in column j, see Figure 4, $70° \leq \rho_i \leq 110°$ and $-20° \leq \kappa_j \leq 20°$. As before we assume that $\rho_1 = 90°$.

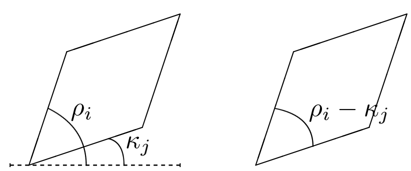
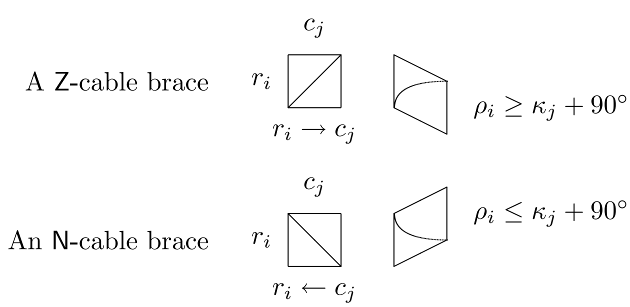


FIGURE 9. Row and column angles.

If square (i, j) has an N-cable brace. then $\rho_i - \kappa_j \leq 90°$, that is $\rho_i \leq \kappa_j + 90°$. If square (i, j) has an Z-cable brace. then $\rho_i - \kappa_j \geq 90°$, that is $\rho_i \geq \kappa_j + 90°$.



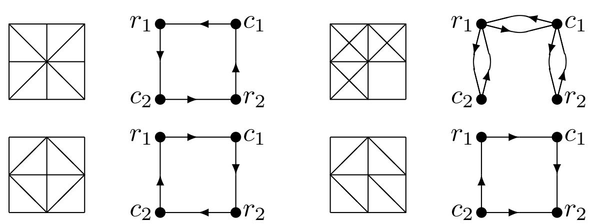
For rod braces, we obtained a system of linear equations, so the rigidity could be determined by linear algebra. The use of cable braces gives us instead a system of linear inequalities, so the rigidity of cable braced grids is a linear programming problem.

There is also a graph theoretical formulation.

3.3. **Buckminster Fuller.** The word tensegrity was coined by Buckminster Fuller, Professor at MIT. He also invented the geodesic dome, which may be interpreted as a 3-dimensional grid and considered by some the most significant structural innovation of the 20th century. The 76m geodesic dome that housed the US pavilion at the world's fair Montreal in 1976 weighs only about one thousandth of the weight of the 50m dome of St. Peter's Cathedral in Rome.

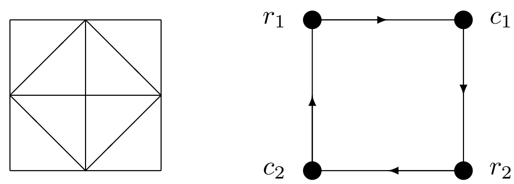
3.4. **The cable-brace graph.** In the brace graph, an edge between r_i and c_j indicates a brace in square (i, j). For cable braces we have to distinguish between the N-cable braces and the Z-cable braces. We can indicate this information in the cable-brace graph by directing the edges. We think of directed edges as arrows instead of lines. So if square (i, j) has a Z-cable brace, then we draw an arrow from r_i to c_j. If square (i, j) has an N-cable brace we will draw an arrow from c_j to r_i.

Some cable brace graphs are drawn below. Remember that we say that an

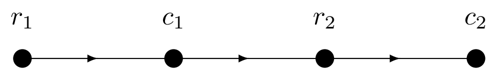


ordinary graph is *connected* if you can move from any vertex to any other vertex by moving along the edges. We say that a directed graph is connected if you can move from any vertex to any other vertex by moving along the arrows *in the directions of the arrows* - as if the arrows were one-way streets. The cable-brace graph of the last example above is not connected, since there is no path from any vertex to c_2, as well as no path from r_2 to any vertex.

The paths in the cable brace graph are related to the row and column angles. If there is a directed path from r_i to c_j in the cable brace graph, then $\rho_i \geq \kappa_j + 90°$ in any deformation. To see why let us look at one of the examples above.

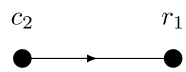


A cable-braced grid and its graph.



A path in the brace graph

$\rho_1 \geq \kappa_1 + 90^\circ \geq \rho_2 \geq \kappa_2 + 90^\circ$ Row and column angle inequalities.



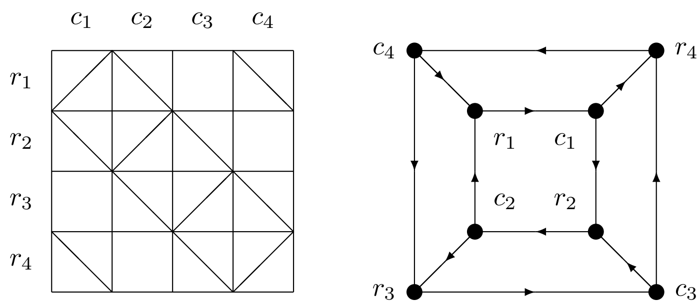
A path back from c_2 to r_1

$\kappa_2 + 90^\circ \geq \rho_1$

Conclusion: $\rho_1 = \kappa_2 + 90$ and $r_1 \perp c_2$.

Theorem 2. *A cable-braced grid is rigid if and only if its cable brace graph is connected.*

Here is an example of a cable-braced grid and its cable-brace graph. Let's check



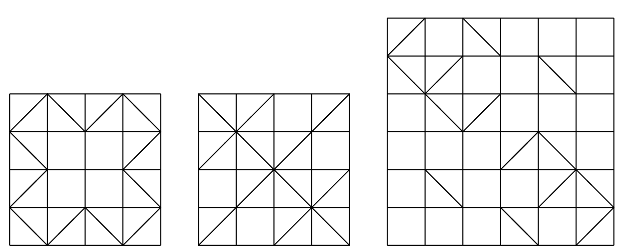
that the graph is connected. There is a path form r_1 to any vertex in the inner square by going around clockwise. Following the arrow from c_1 to the outer square we can as well get to every vertex of the outer square. We can get back to r_1 from any vertex on the outer square via the arrow from c_4 to r_1, and we can get to r_1 from any vertex on the inner square by going around clockwise.

3.5. **An Algorithm for Directed Graphs.** Determining whether or not a directed graph is connected is more difficult than the analogous problem for undirected graphs. One way to do it is to check whether there is a directed path between each pair of vertices. This can be rather tedious for large directed graphs. Here is a somewhat more efficient algorithm:

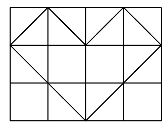
Step 0: Pick a vertex v and write $f(v) = 0$.
Step 1: If $f(x) = 0$ for every vertex x, then the directed graph is connected. STOP.
Step 2: If there is no unpicked edge with origin p such that $f(p)$ maximal, then the directed graph is disconnected. STOP.
Step 3: Choose an unpicked edge e with origin p such that $f(p)$ is maximal. Let q be the terminus of e. If $f(q)$ is undefined, set $f(q) = f(p) + 1$. If $f(q)$ is defined, then redefine every $f(x)$ for which $f(x) > f(q)$ to be $f(q)$. Mark e picked.
Step 4: GOTO Step 1.

3.6. **Exercises.**

(1) Draw the cable-brace graph for each of the following grids. Check the grids for rigidity.

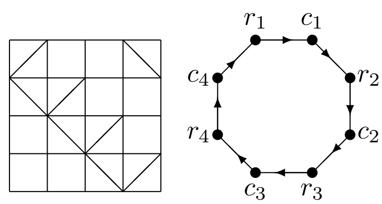


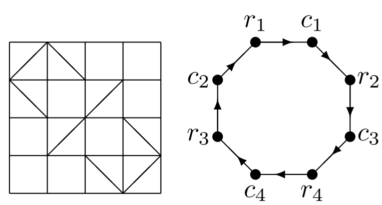
(2) The following cable-braced grid is not rigid. Draw the cable brace graph and see if you can make it rigid by switching some cables from N-cables to Z-cables .



(3) Show the following: If a cable-braced grid is rigid, then replacing all the N-cables with Z-cables and vice versa also gives a rigid cable bracing.

(4) A 4×4 grid with 8 cables is rigid if and only if the corresponding cable brace graph is a cycle. Why?





Here are two examples. How many others can you find? What can you say about them? Can you find all of them?