COMBINATORICS AND THE RIGIDITY OF FRAMEWORKS

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A wide variety of physical structures, e.g. scaffolding (rigid), or a DNA molecule (non-rigid), may be modeled as a collection of rigid rods connected to one another by idealized ball joints. For a given structure, the basic question that arises is whether it is rigid and, if not, how to describe its motion. We would like to describe some of the applications of combinatorics to this area.

By a *framework* we mean a graph $G = (V, E)$ together with an embedding $\mathbf{p}$ of V into Euclidean space. A motion of the framework is a motion of the vertices which preserves the distance between adjacent vertices, and a framework is *rigid* if the only motions which it admits arise from congruences. As an example, consider a rectangle with one diagonal in the plane. This framework is clearly rigid. Notice

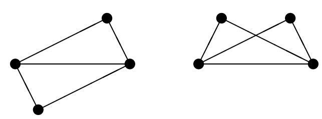

FIGURE 1.

that if we consider the same framework embedded in 3-space, it is no longer rigid since the diagonal rod acts as a hinge, so the rigidity of a framework depends upon the dimension of the space in which it is embedded. By using the diagonal as a hinge, we find two non-congruent planar frameworks with identical rod lengths and adjacencies. We say that the rectangle with one diagonal is not strongly rigid. A framework is *strongly rigid* if the underlying graph, together with specified edge lengths, determines the congruence class of the framework. Asimow and Roth [2] showed that the complete graphs are the only graphs that are (strongly) rigid in all dimensions for all embeddings.

If we consider the initial velocities, $\mathbf{p_i'}$, of the endpoints $\mathbf{p_i}$ of a single edge (i, j) under a continuous motion of a framework, then, to avoid compressing or extending the edge, it must be true that the components of those velocities in the direction parallel to the edge must be equal, i.e.

$$(\mathbf{p_i} - \mathbf{p_j}) \cdot (\mathbf{p_i'} - \mathbf{p_j'}) = \mathbf{0}, \tag{1}$$

A function assigning vectors to each vertex of the framework such that equation 1 is satisfied at each edge is called an *infinitesimal motion.* If the only infinitesimal motions are *trivial*, that is, they arise from infinitesimal translations or rotations of $\mathbb{R}^d$, then we say that the framework is infinitesimally rigid. Infinitesimal rigidity implies rigidity, see for example [7]. On the other hand, since not every infinitesimal motion is realized as the initial velocities of an actual motion of the framework, a rigid graph is not necessarily infinitesimally rigid, see Figure 2a. This framework

Newsletter of the SIAM Activity Group on Discrete Math. 4, (1), 1–5, 1993.

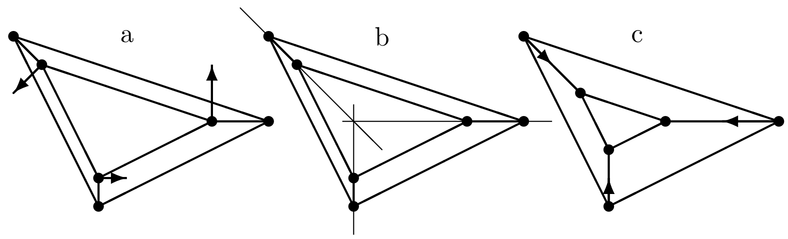

FIGURE 2.

admits an infinitesimal motion which rotates the inner triangle about the intersection of the three lines extending the rods joining the two triangles, see Figure 2b, while the outer triangle is held fixed, this infinitesimal motion is indicated by arrows in Figure 2a. It is often easier to detect an infinitesimal motion of a framework by looking for a *parallel redrawing* of the graph, Figure 2c, a dual formulation which requires an infinitesimal motion of the vertices to preserve not the length but the direction of the rods, leading to a set of conditions equivalent to Equation 1, see [26].

If the framework of Figure 2a is altered so that these lines do not all meet at a point (including the point at infinity), then the framework is infinitesimally rigid. We see that by altering slightly the positions of the vertices it is possible to change the rigidity properties of the framework (even though the vertices are in general position in $\mathbb{R}^2$). We call a framework *generic* if we can can "wiggle" the vertices a little bit without altering any of its rigidity properties. Generic embeddings form an open dense subset in the space of all embeddings. A graph G is called *generically rigid* (in dimension d) if there is a generic embedding of G in $\mathbb{R}^d$ which is rigid.

Euler [12] conjectured in 1766 that the 1-skeleton of any triangulated polyhedral surface in 3-space is rigid. In 1813, Cauchy [5] proved that Euler's conjecture holds for strictly convex polyhedra. Alexandrov [1], 1950, proved that "strictly" could be dropped from the hypothesis of Cauchy's theorem, and Gluck [13], in 1975, proved that any triangulation of a (topological) sphere is generically rigid in 3-space. 1897 Bricard [4] found a non-rigid embedding of the 1-skeleton of an octahedron. This was a quite astonishing example, but it does not disprove Euler's conjecture, since the 1-skeleton of the Bricard octahedron cannot be extended to a 2-skeleton, because some triangles intersect. Bricard's octahedron shows that the rigidity of a framework depends both upon on the combinatorial structure of the graph, as well as the geometry of the embedding of the vertices. Euler's conjecture was settled in 1977 by Connelly [6], who found a flexible sphere. Of course Connelly's surface is non-convex and non-generic.

We consider now the complete graph (V, K) on $V = \{1, \dots, n\}$. Let $\mathbf{p}$ be a fixed embedding of V into $\mathbb{R}^d$. Equation 1 defines a system of linear equations, indexed by the edges (i, j), in the variables $\mathbf{p}'_\mathbf{i}$. The matrix $R(\mathbf{p})$ of this system is a real $n(n-1)/2$ by nd matrix and is called the *rigidity matrix*. A framework $(V, E, \mathbf{p})$ is infinitesimally rigid (in dimension d) iff it the submatrix of $R(\mathbf{p})$ consisting of the rows corresponding to E has rank $dn - \binom{d+1}{2}$.

In 1864 Maxwell [20] studied the rank of $R(\mathbf{p})$ by looking at the null space of $R(\mathbf{p})^*$, called the *space of stresses*, see [3, 8, 24, 25].

Linear independence of the rows of $R(\mathbf{p})$ induces a matroid $\mathcal{F}(\mathbf{p})$ on K. Its restriction to E is called the infinitesimal rigidity matroid of the framework $(V, E, \mathbf{p})$.

If the embedding $\mathbf{p}$ is generic, then the infinitesimal rigidity matroid on the complete graph on n vertices is denoted by $\mathcal{G}_d(n)$ and called the d–dimensional *generic rigidity matroid*.

The rigidity of a framework is a property of algebraic geometry, while it's infinitesimal rigidity belongs to the realm of linear algebra, and generic rigidity is in turn a graph theoretic property. We now describe a more general combinatorial construction which comprises all of these concepts, first introduced by Graver in [14].

Consider a matroid on the edge set of a complete graph on n vertices. If E is a subset of edges of this complete graph, we denote the set of vertices which are endpoints of edges in E by $V(E)$ and $K(V)$ denotes the edge set of the complete graph on the vertex set V. Let us call an edge set rigid if its closure is complete, i.e. if $\langle E\rangle = K(V(E))$. In order for this matroid to reflect rigidity properties of a graph on n (or fewer) vertices embedded in $\mathbb{R}^d$, its closure operator should, in addition to the usual four axioms for a matroid closure operator, satisfy two rather obvious conditions: [C5] If two edge sets have fewer than d vertices in common, then the closure of their union should equal the union of their closures, since one can rotate one edge set with respect to the other about an axis in d space which contains all vertices of intersection. This motion changes the relative distance between any pair of vertices whose components are endpoints of edges in different edge sets. [C6] If two rigid sets intersect in d or more vertices, their union should be rigid.

In [14, 15] it is shown that, if $\mathbf{p}$ is general then $\mathcal{F}(\mathbf{p})$ is an abstract rigidity matroid. Defining abstract rigidity matroids via the closure operator is a natural approach because of the concept of an "implied" edge: if (i, j) lies in the span of E in $\mathcal{F}(\mathbf{p})$, then every infinitesimal motion of $(V, E, \mathbf{p})$ preserves the length of (i, j), so the closure of E consists of all edges implied by E.

The axioms imply that an edge set E containing a vertex v of degree d or less is independent if and only if $E - \mathrm{star}(v)$ is independent. This result allows us to recursively build up independent sets: Start with a complete graph on d vertices (it is independent and rigid it dimension d) and proceed to attach d–valent vertices, one at a time. It follows immediately that each successive graph is both independent and rigid, and we conclude that all d–dimensional abstract rigidity matroids on n vertices have rank $dn - \binom{n+1}{2}$.

We also see that K_{d+1} is always a cycle in a d-dimensional abstract rigidity matroid, and the star of a vertex minus $d - 1$ of its edges is always a cocycle. In fact, see [16], these two properties give an alternate definition of abstract rigidity matroids.

One could define a more general rigidity theory for any matroid by first introducing, in addition to the underlying set of the matroid, a set of "vertices", and then linking the two sets by specifying a support function. Rigidity of a set can then be defined by an appropriate operator on the support of the set.

For dimension 1, the six axioms determine a unique matroid, which must be $\mathcal{G}_1(n)$, and coincides with the well-known connectivity matroid.

By contrast, there are many non-isomorphic abstract rigidity matroids for dimension 2, in fact there exist 2-dimensional abstract rigidity matroids which do not arise as infinitesimal rigidity matroids. The following theorem characterizes $\mathcal{G}_2(n)$.

Theorem 1. *Let $\mathcal{A}_2$ be a 2-dimensional abstract rigidity matroid on n vertices. The following are equivalent:*

(1) $\mathcal{A}_2 = \mathcal{G}_2(n)$;
(2) [Graver [14]] $\mathcal{A}_2$ *has the* 1-*extendability property: an independent set on* k *vertices can be extended to an independent set on* $k+1$ *vertices by subdividing an edge with a new vertex, and attaching that new vertex to any other vertex;*
(3) [Laman [18]] *The independent sets of* $\mathcal{A}_2$ *are those sets which satisfy Laman's condition:*

(2) $$|F| \leq 2|V(F)| - 3 \text{ for all } F \subseteq E, F \neq \emptyset;$$

(4) [Crapo [9]] *The bases of* $\mathcal{A}_2$ *consist of those edge sets which are decomposable into* 3 *trees, such that every vertex is covered twice, and no subset induces a subgraph spanned by only* 2 *subtrees;*
(5) *All cycles of* $\mathcal{A}_2$ *are rigid;*
(6) *For any closed set* E *of* $\mathcal{A}_2$ *with cliques* $E_1, \ldots, E_k$, $r(E) = r(E_1) + \cdots + r(E_k)$, *where* r *is the rank function of* $\mathcal{A}_2$*;*
(7) [Lovász and Yemini [19]] $r(E) = \min \sum_{i=1}^{k}(2|V(E_i)| - 3)$, *where the minimum is taken over all collections* $\{E_i\}$ *of nonempty sets such that* $E = \cup E_i$.
(8) [Graver [14]] $\mathcal{A}_2$ *is maximal among all* 2-*dimensional abstract rigidity matroids.*

Laman's Theorem was proved in 1970 and it was this theorem that promoted the use of matroids to attack rigidity questions. A nice new proof is given by Tay [23].

Crapo's characterization is based on a theorem of Nash-Williams which was first applied to rigidity by Lovász and Yemini (see also [22]) to yield a polynomial algorithm, [11], to determine generic independence. Crapo's three tree criterion provides improvement. Lovász and Yemini also give a different algorithm based on the submodularity of the rank function.

To date no combinatorial characterization is known for $\mathcal{G}_d(n)$ for $d \geq 3$. The obvious generalizations of conditions 2–8 in Theorem 1 fail to characterize $\mathcal{G}_d(n)$ for $d \geq 4$, see for example [21, 15]. In fact 3 and 5 fail already for $d = 3$: the "double banana", Figure 3 is a non-rigid cycle satisfying Laman's condition for dimension 3, as the reader may check. Reformulations of conditions 2 and 6 that may work in

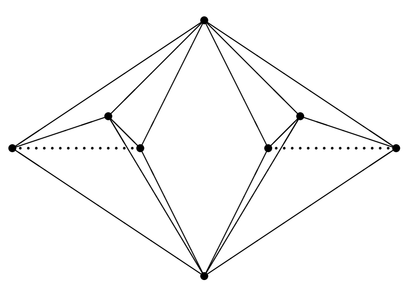

FIGURE 3. The double banana.

3-space are known respectively as Graver's Conjecture and Dress' Conjecture. It is not known if these two conjectures are equivalent. The Maximal Conjecture states that there exists a unique maximal abstract rigidity matroid in dimension 3, and that it coincides with $\mathcal{G}_3(n)$. The Maximal Conjecture implies both Graver's Conjecture and Dress' Conjecture, see [15].

Should the characterization problem be too frustrating to get hooked into research in rigidity, you may want to settle the following question: Connelly can show that if there are two realizations of the same graph, one in $\mathbb{R}^n$ and one in $\mathbb{R}^m$, non-congruent but with the same edge lengths, then there is a motion in $\mathbb{R}^{m+n}$ joining them. If $m = n = 2$ then one might expect that there is such a motion already in $\mathbb{R}^3$ (like in our first example of the rectangle with one diagonal) but Whitely has a counterexample. In general, is $n+m$ best possible? Connelly's result links strong rigidity with (infinitesimal) rigidity.

If this is still too hard for a start, here is an unsolved problem due to Connelly in dimension 2: Can all 3-connected cycles of $\mathcal{G}_2(n)$ be obtained from the tetrahedron by a sequence of 1-extensions? Henneberg [17] gave a parallel construction for independent sets.

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