SYMMETRY, AUTOMORPHISMS, AND SELF-DUALITY OF INFINITE PLANAR GRAPHS AND TILINGS

BRIGITTE AND HERMAN SERVATIUS

Abstract. We consider tilings of the plane whose graphs are locally finite and 3-connected. We show that if the automorphism group of the graph has finitely many orbits, then there is an isomorphic tiling in either the Euclidean or hyperbolic plane such that the group of automorphisms acts as a group of isometries. We apply this fact to the classification of self-dual doubly periodic tilings via Coxeter's 2-color groups.

1. Definitions

By a *tiling*, $\mathcal{T} = (G, p)$ we mean a tame embedding p of an infinite, locally finite three–connected graph G into the plane whose geometric dual G^* is also locally finite (and necessarily three-connected.) We say an embedding of a graph is *tame* if it is piecewise linear and the image of the set of vertices has no accumulation points. Each such tiling realizes the plane as a regular CW–complex whose vertices, edges and faces we will refer to indiscriminately as *cells.* It can be shown [9] that all such graphs can be embedded so that the edges are straight lines and the faces are convex, however we make no such requirements.

The boundary ∂X of a subcomplex X is the set of all cells which belong both to a cell contained in X as well as a cell not contained in X. Given a CW-complex X, the *barycentric subdivision* of X, $\mathrm{B}(X)$, is the simplicial complex induced by the flags of X, however it is sufficient for our purposes to form $\mathrm{B}(X)$ adding one new vertex to the interior of every edge and face and joining them as in Figure 1. If X is a plane tiling and each face is the convex hull of its vertices, then we can

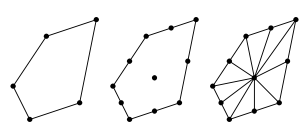

Figure 1.

take the new interior vertex to be their centroid (barycenter), however this is not necessary and there are many other topological realizations.

2. Combinatorial and metric automorphism groups.

Every tiling $\mathcal{T}$ of R^2 has many candidates for its automorphism group, depending on what one wishes to study. The most restrictive is the metric automorphism group, $\mathrm{Isom}(\mathcal{T})$ which is the group of isometries of the plane which respect the tiling. More generally, one can look at the combinatorial automorphism group, $\mathrm{Aut}(\mathcal{T})$, which is the group of permutations of the cells of the tiling which preserve dimension and incidence. Conceivably one could also just look at the automorphism group of the underlying graph of the tiling, i.e. its 1-skeleton, however since we assume the 1-skeleta of our tilings are 3-connected, Whitney's theorem, [13], implies that the combinatorial automorphism group is identical with the automorphism group of the graph of the tiling, and is realizable as a group of homeomorphisms of the plane which respect the tiling. For dealing with non-3-connected tilings see [7, 12].

Clearly the combinatorial automorphism group is not identical with the metric automorphism group, for instance the tilings indicated in Figure 2 are both com-

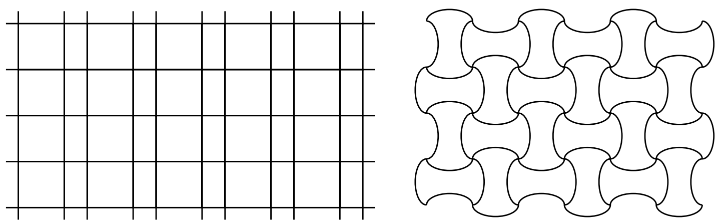

FIGURE 2.

binatorially isomorphic to the square tiling $[4,4]$ yet have metric automorphism groups strictly contained in p4m. Moreover, since the hyperbolic plane H^2 is homeomorphic to the Euclidean plane E^2, any regular hyperbolic tiling gives rise to a rather unsymmetric Euclidean tiling with large combinatorial automorphism group, and vice versa. In fact, in this case, $\mathrm{Isom}(\mathcal{T})$ is not even of finite index in $\mathrm{Aut}(\mathcal{T})$. In the Euclidean case, $\mathrm{Isom}(\mathcal{T})$ is one of the seventeen plane crystallographic groups, see [4] or [8].

A tiling $\mathcal{T}$ is *harmonious* if $\mathrm{Aut}(\mathcal{T}) = \mathrm{Isom}(\mathcal{T})$. In [2] it is conjectured that every doubly periodic normal tiling is isomorphic to a harmonious tiling. The following theorem confirms this conjecture and gives a more comprehensive result.

THEOREM 1. *Let $\mathcal{T}$ be a tiling of R^2 whose* 1*-skeleton is a tamely embedded* 3*-connected locally finite graph with locally finite dual. If Aut($\mathcal{T}$) has finitely many face orbits, then there is a tiling Ideal($\mathcal{T}$) of E^2 or H^2 which is isomorphic to $\mathcal{T}$ and such that the metric automorphism group of Ideal($\mathcal{T}$) realizes its combinatorial automorphism group.*

We sometimes say that the tiling $\mathcal{T}$ is *straightened* to $\mathrm{Ideal}(\mathcal{T})$.

Every Euclidean plane crystallographic group contains a translation subgroup which is free abelian of rank 2, and since this group does not act discretely on the hyperbolic plane, $\mathcal{T}$ can not have both Euclidian and hyperbolic idealizations, and it follows from Theorem 1 that for a tiling with finitely many face orbits we can say definitively whether it is intrinsically Euclidean or hyperbolic.

LEMMA 1. *Let $\mathcal{T}$ be a tiling of the plane with combinatorial automorphism group Aut($\mathcal{T}$). No non-trivial element of Aut($\mathcal{T}$) fixes a face of* $\mathrm{B}(\mathcal{T})$*. Furthermore, every automorphism which fixes an edge of* $\mathrm{B}(\mathcal{T})$ *also fixes its endpoints.*

Proof. If $g \in \mathrm{Aut}(\mathcal{T})$ fixes a face or edge of $\mathrm{B}(\mathcal{T})$, then since the vertices of the face, respectively the endpoints of the edge, correspond to cells of different dimensions in $\mathcal{T}$, those vertices must be fixed as well. If a face is fixed, then the vertices of the faces incident to it must also be fixed, and since $\mathrm{B}(\mathcal{T})$ is connected, g is the identity. $\square$

Note that not every automorphism of $\mathrm{B}(\mathcal{T})$ need be induced from one of $\mathcal{T}$, so there may be non-trivial automorphisms of $\mathrm{B}(\mathcal{T})$ which fix faces, see Theorem 2.

LEMMA 2. *There exists a fundamental subcomplex F of $B(\mathcal{T})$, that is a connected, simply connected, regular subcomplex containing exactly one face representing each orbit of faces under $Aut(\mathcal{T})$.*

Proof. Starting with a single triangle we may assume we have a connected subcomplex X of $\mathrm{B}(\mathcal{T})$ containing at most one representative from each face orbit. Each edge of the boundary of X is incident to one face not belonging to X. If for all boundary edges, this face belongs to the orbit of a face in X, then the union of all the orbits of X is open, and similarly for the orbits of faces not contained in X. So the orbit of X is the whole plane.

Thus, starting with a single face, we can add new faces along boundary edges representing new orbits until we have one representative from each orbit. If X is not also simply connected, then there is a simple cycle on the boundary of X which bounds a disk Y whose interior lies in the complement of X. Y is tiled by images of X, and since each image has the same properties as X, there are infinitely images of X contained in Y, contradicting the fact that $\mathcal{T}$ is locally finite and tame. □

PROOF OF THEOREM 1: Form $\mathrm{B}(\mathcal{T})$ and let X be a fundamental complex. Since $\mathcal{T}$ has finitely many face orbits, it follows that its fundamental complex X is a closed disk.

Every point of ∂X which belongs to more than two orbits of X we call a *corner point* of X, and we say the *multiplicity* of the corner point is the number of orbits of X it intersects. The corner points since they belong to ∂X and belong to at least three faces of $\mathrm{B}(\mathcal{T})$ must be vertices of $\mathrm{B}(\mathcal{T})$.

Suppose X has corner points $x_1, \ldots, x_k$ reading counterclockwise around X, with multiplicities $m_1, \ldots, m_k$. We want to construct $\mathrm{Ideal}(X)$, a polygon in the Euclidian or hyperbolic plane so that each element of the orbit of X can be replaced by $\mathrm{Ideal}(X)$ such that the plane is tiled by congruent shapes in the appropriate geometry and the action of G will be to rigidly permute the faces.

Let $\alpha = \sum(m_i - 2)$.

If $\alpha = 2$, then if we take, for each i, a regular Euclidean m_i-gon of sidelength 1, then they will just fit together cyclically about a common vertex in the Euclidean plane, see Figure 3. Take the convex hull of the centroids of the k regular polygons as $\mathrm{Ideal}(X)$.

For each $g \in \mathrm{Aut}(\mathcal{T})$ we may now equivariantly straighten gX into the shape $\mathrm{Ideal}(X)$. That is, there is a p.l. homeomorphism $\phi : \mathrm{Ideal}(X) \to X$ which matches the corner points of X with the corresponding vertices of $\mathrm{Ideal}(X)$. For each g define $\phi_g : \mathrm{Ideal}(X) \xrightarrow{\phi} X \xrightarrow{g} gX$, take the disjoint union $\sqcup_{g \in \mathrm{Aut}(\mathcal{T})} \mathrm{Ideal}(X)$ and define

$$\Phi : \bigsqcup_{g \in \mathrm{Aut}(\mathcal{T})} \mathrm{Ideal}(X) \mapsto \bigcup_{g \in \mathrm{Aut}(\mathcal{T})} gX = R^2,$$

and the quotient, $\tilde{\Phi}$, of this map contains the information how to glue together the polygons $\mathrm{Ideal}(X)$ to form a Euclidean tiling $\mathrm{Ideal}(\mathcal{T})$,

$$\tilde{\Phi} : \mathrm{Ideal}(\mathcal{T}) \mapsto \mathcal{T}$$

as well as giving the action of G on $\mathrm{Ideal}(T)$. G acts on $\mathrm{Ideal}(T)$ by rigidly permuting the tiles, i.e. isometrically, and $\tilde{\Phi}^{-1}$ gives the straightening.

If $\alpha > 2$, so that the Euclidean polygons fit around a vertex with overlap, choose instead regular hyperbolic polygons with equal sidelengths. Their vertex angles will shrink as their common side length is expanded, so there is one choice of sidelength for which they will just fit and we proceed as above. In this case $\mathrm{Aut}(\mathcal{T})$ must be a hyperbolic group.

It cannot happen that $\alpha < 2$, i.e. that the Euclidean polygons do not come together, since then we could choose instead regular polygons of equal sidelength on the sphere for some choice of sidelength, and in this case have a spherical tesselation which is necessarily finite. □

The straightening process is illustrated by Figure 3, in which a tesselation of the

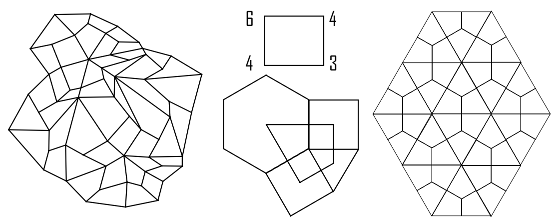

FIGURE 3. The straightening process.

Euclidean plane by quadilaterals, each of which may be taken to be the fundamental complex, is straightened.

We conjecture that Theorem 1 remains true even if we require the edges of the $\mathrm{Ideal}(\mathcal{T})$ to be straight lines and the faces to be convex. If $\mathrm{Aut}(\mathcal{T})$ is generated by reflections, then the fixed lines of these reflections are the boundary of the fundamental complex, so, up to the action of $\mathrm{Aut}(\mathcal{T})$, the fundamental complex is uniquely defined. In other cases, however, there is more freedom in choosing the fundamental region, and different choices will give non-isometric idealizations. For examples see Figures 4–7 which restraighten a harmonious (colored) tiling.

Although Theorem 1 implies an effective algorithm for straightening tilings with respect to their combinatorial automorphism group, such an algorithm requires us to have in hand the automorphism group and a means of deciding whether or not two faces belong to the same orbit, which is a lot of information to be blessed with. A better application of Theorem 1 seems to be the ability to study ordinary tilings through their harmonious idealizations, as we do in the next section.

3. Self-dual tilings.

Given a tiling $\mathcal{T}$, let $\mathcal{T}^*$ denote the dual tiling. As there are many notions of isomorphism between tilings, so are there various notions of self-duality. The weakest condition is for $\mathcal{T}$ to be *combinatorially self-dual*, that is, there is a combinatorial isomorphism between $\mathcal{T}$ and $\mathcal{T}^*$. It is easy to see that this condition is equivalent to requiring a topological homeomorphism of the plane such that the image of $\mathcal{T}$ is a tiling in dual position to $\mathcal{T}$. Both of these conditions really are associated with the underlying graph of $\mathcal{T}$. A *metric self-duality* is a rigid motion of the plane which carries $\mathcal{T}$ into dual position, and examples are known [2] of combinatorially self-dual tilings which are not metrically self-dual.

Possibly the strongest notion is that of a *harmoniously self–dual* tiling, in which we specify a tiling, an isometric dual tiling in dual position, and require that all automorphisms $\mathcal{T} \to \mathcal{T}^*$ arise from rigid motions of the plane which interchange $\mathcal{T}$ and $\mathcal{T}^*$, see Figure 45 where three generating reflection lines are shown, one of which interchanges the black and white grids.

Progress in the study of self-duality has been minimal until fairly recently when Grünbaum [8] defined the *self-dual permutation*, that is, a permutation σ of the cells of $\mathcal{T}$ which reverses dimension and preserves incidence. The existence of a self-dual permutation is equivalent to combinatorial self-duality. The self-dual permutations generate a permutation group $\mathrm{Dual}(\mathcal{T})$ in which $\mathrm{Aut}(\mathcal{T})$ is a subgroup of index 2, and the self-dual permutations comprise the other coset. Thus, using group theoretic techniques, there has been much progress studying self-dual tilings [2], polyhedra [2, 1, 3, 8, 10], spherical maps [11], finite graphs and matroids [6, 12]. The most useful classification of self-dual objects is via the *self-dual pairing* $(\mathrm{Dual}(\mathcal{T}), \mathrm{Aut}(\mathcal{T}))$ which was completed for self-dual polyhedra in [11] and for 2-connected self-dual graphs in [12].

Let $\mathcal{T}$ be a tiling. Every vertex of $\mathrm{B}(\mathcal{T})$ which corresponds to an edge of $\mathcal{T}$ is of valence 4, and since $\mathcal{T}$ is 3-connected and so has no vertices of valence 2, $\mathrm{B}(\mathcal{T})$ has no other vertices of valence 4. Let us color the vertices of $\mathrm{B}(\mathcal{T})$ black if they correspond to vertices of $\mathcal{T}$, white if they correspond to faces of $\mathcal{T}$, and gray if they correspond to edges of $\mathcal{T}$. Every face of $\mathrm{B}(\mathcal{T})$ is a triangle with one black, white and grey vertex, so $\mathrm{B}(\mathcal{T})$ is uniquely vertex 3-colorable, and every automorphism of $\mathrm{B}(\mathcal{T})$ induces a permutation of the colors. Moreover, since the gray vertices are the only vertices of valence 4, such an automorphism either fixes the colors or reverses white and black, with a black/white reversing automorphism of $\mathrm{B}(\mathcal{T})$ corresponding to a self-duality of $\mathcal{T}$. We have the following theorem.

Theorem 2. *$\mathcal{T}$ is combinatorially self-dual if and only if $Aut(\mathcal{T}) \neq Aut(B(\mathcal{T}))$. If $\mathcal{T}$ is combinatorially self-dual, then $(Dual(\mathcal{T}), Aut(\mathcal{T})) = (Aut(B(\mathcal{T})), Aut(\mathcal{T}))$.*

We note that conclusion of Theorem 2 remains true without the connectivity and finiteness assumptions on $\mathcal{T}$, with the k-gons embedded in the sphere as the only exceptions. The 2-gon, in fact, has the octahedron as its barycentric subdivision, and every permutation of colors occurs. Note also that it is impossible for $\mathrm{B}(\mathcal{T})$ to be itself self-dual since it has vertices of valence 4 and every face is a triangle.

Theorem 3. *Every combinatorially self-dual tiling with finitely many face orbits is isomorphic to a harmoniously self-dual tiling.*

Proof. Let $\mathcal{T}$ be a self-dual tiling with finitely many face orbits. Then we can form the barycentric subdivision. $\mathrm{B}(\mathcal{T})$ contains an embedding of $\mathcal{T}$ and $\mathcal{T}^*$ in dual position, and every automorphism of $\mathrm{B}(\mathcal{T})$ corresponds an element of $\mathrm{Dual}(\mathcal{T})$, either to an element of $\mathrm{Aut}(\mathcal{T})$ or a self-duality. Apply Theorem 1 to form $\mathrm{Ideal}(\mathrm{B}(\mathcal{T}))$, which we note entails taking a second barycentric subdivision of $\mathcal{T}$. The harmonious superposition of the graph of $\mathcal{T}$ and its dual is obtained by deleting from $\mathrm{Ideal}(\mathrm{B}(\mathcal{T}))$ all edges joining black and white vertices. □

The result of the straightening process described in the proof of Theorem 3 depends on the choice of fundamental complex, and a self-dual tiling may have many different harmonious incarnations. This is illustrated in Figures 4–7 where even the original tiling is harmonious. In Figure 4 the fundamental complex has five corner points of multiplicities $\{4, 3, 4, 3, 3\}$, so in Figure 5 we map it into a pentagon with angles $\{\pi/2, \pi/3, \pi/2, \pi/3, \pi/3\}$. In Figure 6 we choose a different

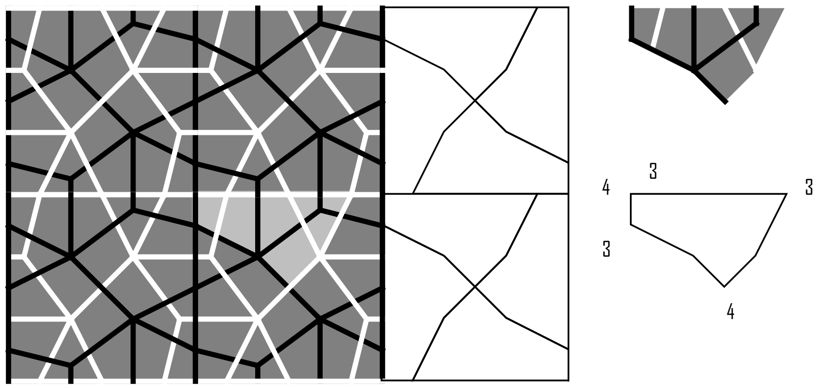

FIGURE 4. Choosing a fundamental complex.

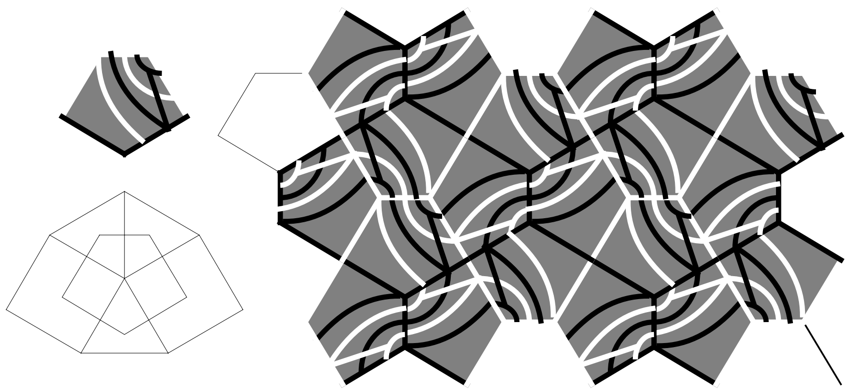

FIGURE 5. Straightening a self-dual tiling.

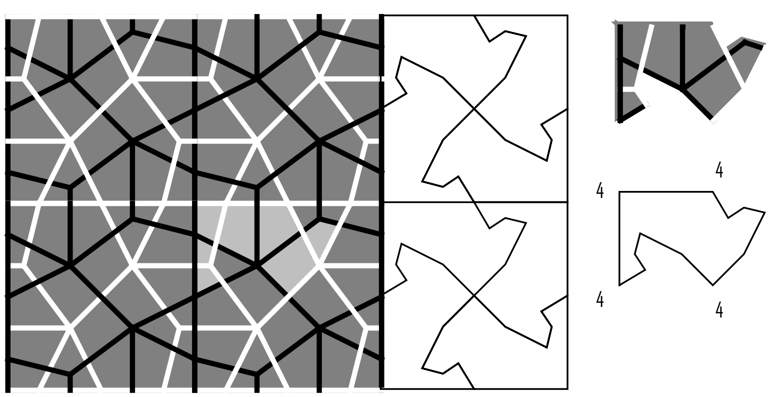

FIGURE 6. A more awkward choice.

fundemental complex with 4 corner points each of multiplicity 4, so we map the fundemental complex into a square. We see, in Figures 7 that with some choices of fundamental complex it is impossible to arrange the mapping into Ideal(X) so that

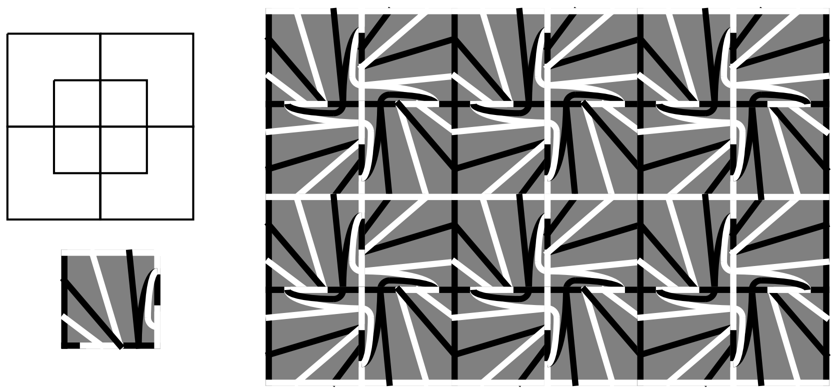

FIGURE 7. The aftermath.

the resulting tiling has straight edges or convex cells. Nevertheless, no example is known that cannot be so straightened, and we conjecture that none exists.

A doubly periodic self-dual tiling is necessarily Euclidean and has finitely many face orbits. Since every self-dual periodic tiling is isomorphic to a harmonious one, Dual($\mathcal{T}$) and Aut($\mathcal{T}$) are both one of the 17 plane crystallographic groups, and the pair [Dual($\mathcal{T}$), Aut($\mathcal{T}$)] is one of the 46 2-*color groups* enumerated by Coxeter [5], with the color preserving symmetries corresponding to Aut($\mathcal{T}$). In [11] spherical 2-color groups were used to classify finite self-dual 3-connected graphs. In the final section of this paper we will enumerate the two-color groups which correspond to self-dual doubly periodic tilings, which is summarized as follows.

THEOREM 4. *The following* 2*-color groups are the self-dual pairings of Euclidean tilings:* (cm,p1), (cm,pg), (cm,pm), (cmm,cm), (cmm,p2), (cmm,pgg), (cmm,pmg), (cmm,pmm), (p1,p1), (p2,p1), (p2,p2), (p4,p2), (p4,p4), (p4g,cmm), (p4g,p4), (p4g,pgg), (p4m,cmm), (p4m,p4m), (p4m,pmm), (pg,p1), (pg,pg), (pgg,p2), (pgg,pg), (pm,cm), (pm,p1), (pm,pg), (pm,pm)a, (pm,pm)b, (pmg,p2), (pmg,pg), (pmg,pgg), (pmg,pm), (pmg,pmg), (pmm,cmm), (pmm,p2), (pmm,pm), (pmm,pmg), (pmm,pmm).

Moreover, each 2*-color group has a realization as a self-dual pairing of a harmonious tiling with convex cells.*

Notice that the group p4g, which is generated by a 90° rotation and a reflection in a line not through the pole of the rotation, does not occur as Aut($\mathcal{T}$) for any self-dual pairing, even though it does occur on the list in [2] of possible symmetry groups of self-dual tilings. It follows that any self dual Euclidean tiling with Isom($\mathcal{T}$) = p4g can be straightened to have symmetry group p4m, for instance the tiling on the right in Figure 2.

We will need the following lemmas.

LEMMA 3. *Let g be a rotation of a harmoniously self-dual tiling which is a self-duality. Then g has order 2 or 4.*

Proof. Let B($\mathcal{T}$) be harmoniously embedded, and let g be a rotation. The center of the rotation cannot be in the interior of a triangle, since then g would cyclically permute the colors instead of interchanging white and black.

Thus the fixed point is either in the interior of the edge joining a black and white vertex, and g has order 2, or the fixed point is a grey vertex, and the rotation is of order 4 since the grey vertex is of valence 4 with neighbors black and white alternately. □

LEMMA 4. *Let $\mathcal{T}$ be a harmoniously self-dual tiling in the plane and let L_1 and L_2 be two intersecting lines of reflective symmetry in Dual($\mathcal{T}$). If both reflections are self-dualities, then they meet at right angles. If just one is a self-duality, then they either meet at right angles or* $45°$.

Proof. We may assume that Dual($\mathcal{T}$) acts on a harmonious B($\mathcal{T}$) colored as before. If the lines of two color reversing reflections intersect, their product is a color preserving rotation about that fixed point, which must be a vertex of B($\mathcal{T}$), and since the only vertices which can be fixed by a color reversing reflection are grey, the lines intersect in a valence 4 grey vertex, and they meet at right angles. If the lines of a color preserving and color reversing reflection meet, then their product is a color reversing rotation which by the previous lemma is of order 2 or 4, so they meet at right angles or $45°$. □

PROOF OF THEOREM 4: We show that the groups *p6, p3, p6m, p3m1* and *p31m* do not occur as Dual($\mathcal{T}$) in the self-dual pairing of a self-dual Euclidean tiling. The groups p3 and p6 are generated by rotations of orders 3 and 6, none of which can be self-dualities by Lemma 3. The groups p6m is generated by reflections in a right triangle with angle $60°$ and p3m1 is generated by relections in the sides of an equilateral triangle, none of these generating transformations can be self-dualities by Lemma 4. The group p31m is generated by the relections in the sides of an equilateral triangle and an order 3 rotation about its center, none of which can be self-dualities.

The remaining 12 Euclidean crystallographic groups occur as Dual($\mathcal{T}$) and the cases are enumerated in Section 4. □

4. The color groups of self-dual tilings.

The pairing symbols for 2-color groups, even the symbols for the plane crystallographic groups themselves, are rather awkward to use and a stumbling block for the uninitiated. More useful symbols, though less accessible typographically, are marked polygons which represent a fundamental region and decorated to indicate the generating transformations as follows:

- an undecorated edge is assumed to be a line of reflection.
- a rotation is indicated by a small circular arc about the center, with the order of the rotation indicated by the angle,
- a translation is indicated by an arrow in the interior of the proper length and direction,
- a glide reflection is indicated by a half-headed arrow along the reflective line.

We color reversing symmetries (self-dualities) by doubling the decoration. In the diagrams, a simple 2-color pattern is also shown which follows the marked diagram.

Ambient group $p1$**.** If the ambient group is $p1$, generated by two linearly independent translations, then, up to afinity, there is only one 2-color symmetry pairing, $(\mathrm{p1}, \mathrm{p1})$, which is realized as a self-dual tiling in Figure 8. (Note that there are

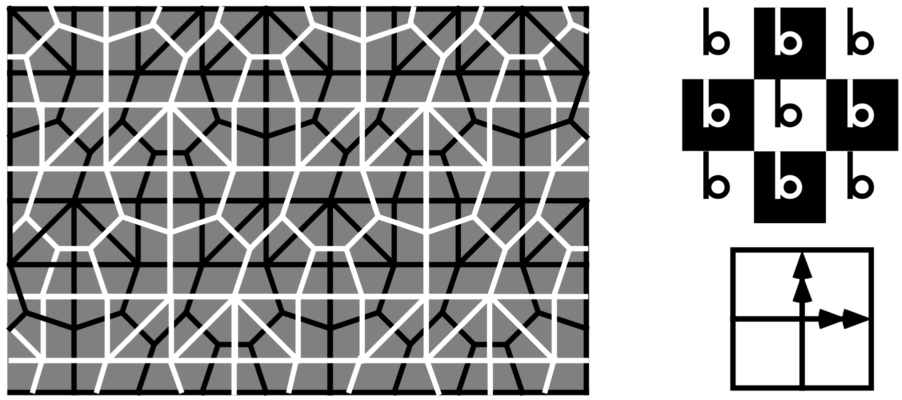

FIGURE 8. The pairing (p1, p1)

generating translations in the diagonal directions.)

Ambient group $p2$**.** The group $p2$ is generated by 180° rotations about three non-collinear points. The fundamental region is a parallelogram, and the rotations are on the midpoints of three of the sides. The rotation on the fourth side is the product of the other three. If all three rotations are self-dualities, then all rotations are self-dualities and the translation subgroup is the automorphism group, (p2, pl), realized in Figure 9. If a proper subset of the three generating rotations consists

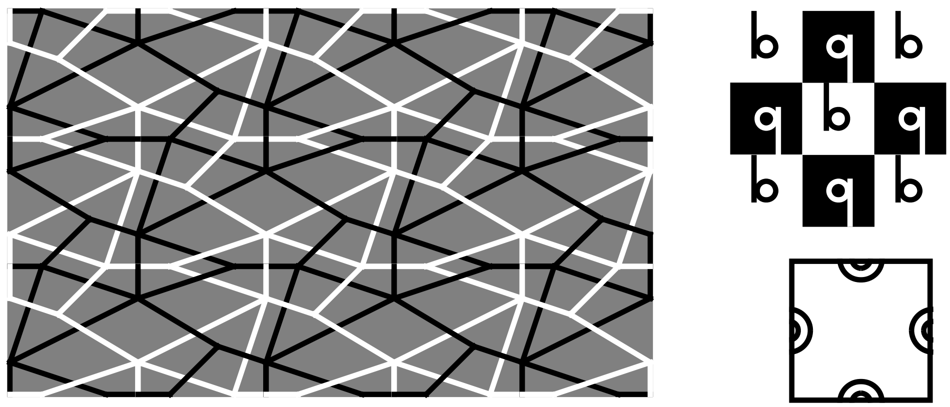

FIGURE 9. The pairing (p2, p1)

only of self-dualities, then, using the fact that we can replace any generator with the product of all three, we can choose the fundamental parallelogram so that the rotations on one pair of opposite sides are self-dualities. The pairing in this case is (p2, p2), see Figure 10.

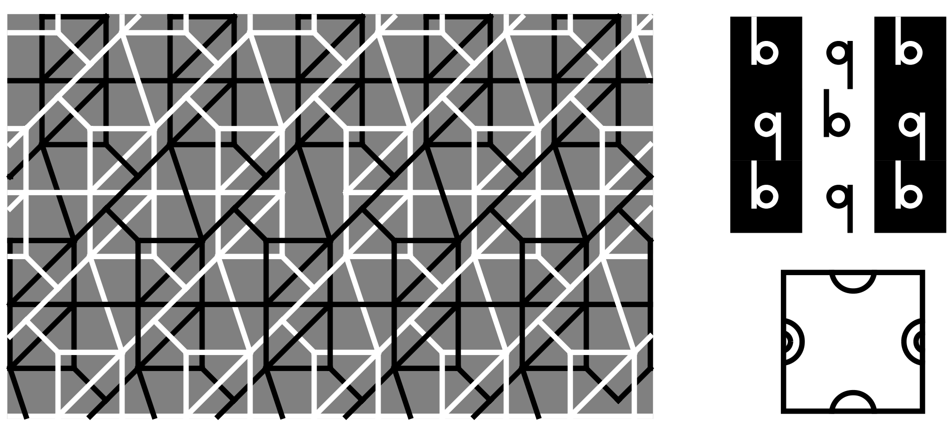

FIGURE 10. The pairing (p2, p2)

Ambient group ***pm*****.** The group *pm* is generated by two parallel reflections and a translation parallel to the lines of reflection. If neither reflection is a self-duality, then the translation must be, and the automorphism group is also *pm*, see Figure 11.

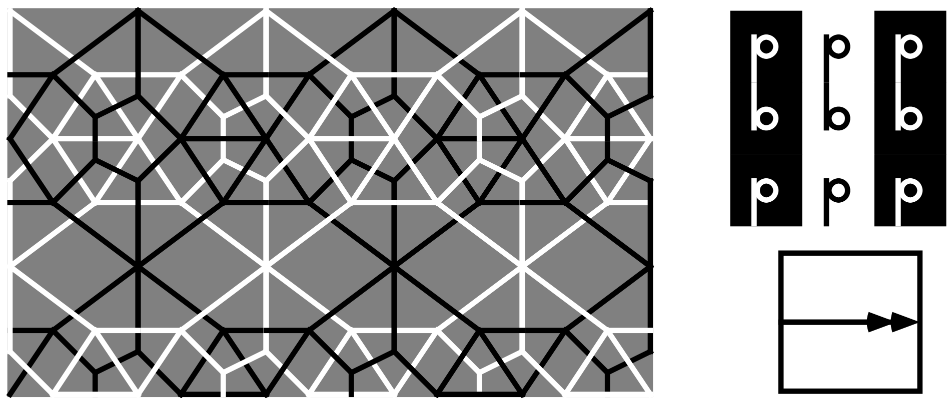

FIGURE 11. The pairing (pm, pm)(a)

This is also the case when one reflection is a self-duality and the translation is not, see Figure 12. These cases are geometrically distinguishable, and, following

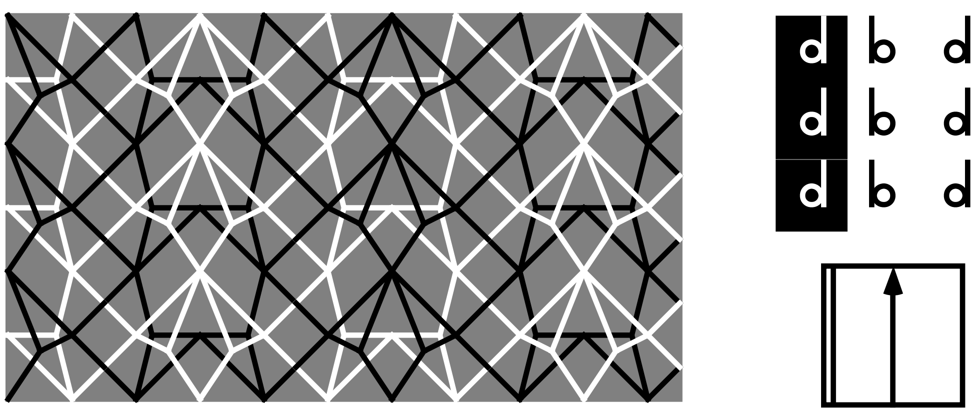

FIGURE 12. The pairing (pm, pm)(b)

Coxeter [5], we distinguish the pairings as (pm, pm)(a) and (pm, pm)(b). Of course, their marked diagrams are distinct.

If one reflection and the generating translation are dualities, then the pairing is (pm, cm), see Figure 13.

If both reflections are self-dualities, then the pairing is (pm, pg), see Figure 14, and if the generating translation is also a self-duality, and (pm, p1) if it is not, see Figure 15.

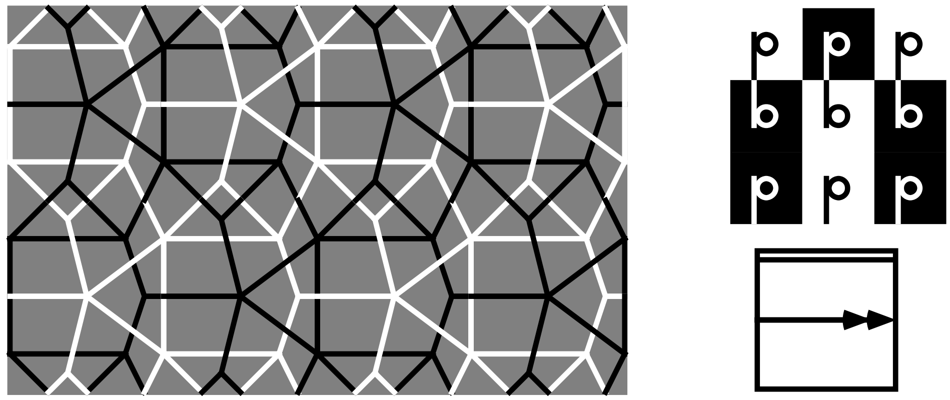

FIGURE 13. The pairing (pm, cm)

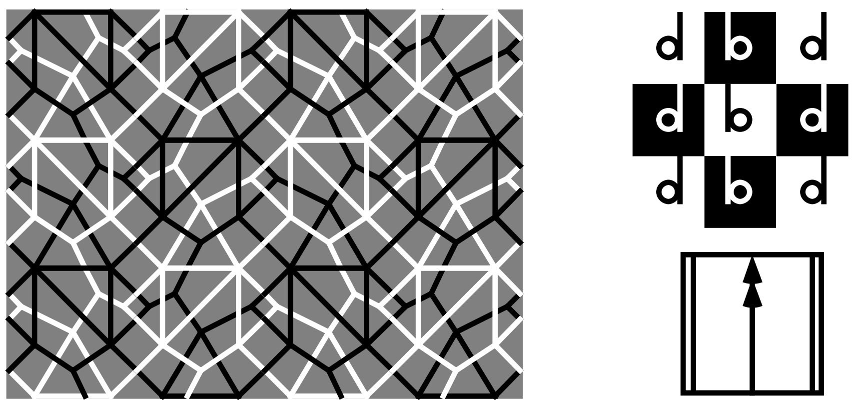

FIGURE 14. The pairing (pm, pg)

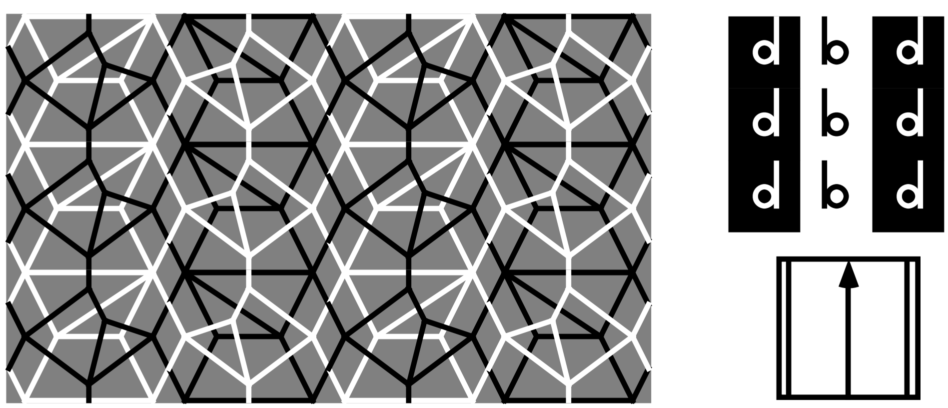

FIGURE 15. The pairing (pm, p1)

Ambient group *pmm***.** *pmm* is generated by the reflections in the sides of a rectangle. The pairings when one, three, or four of these reflections are dualities are (pmm, pmm), (pmm, pmg), and (pmm, p2) respectively, realized in Figures 16, 17, and 18.

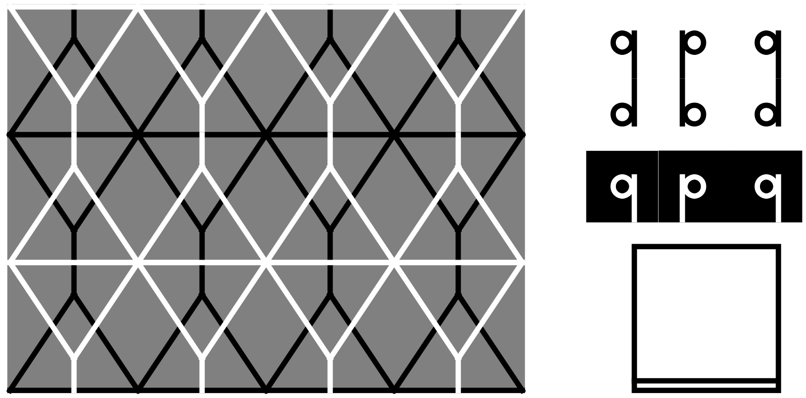

FIGURE 16. The pairing (pmm, pmm)

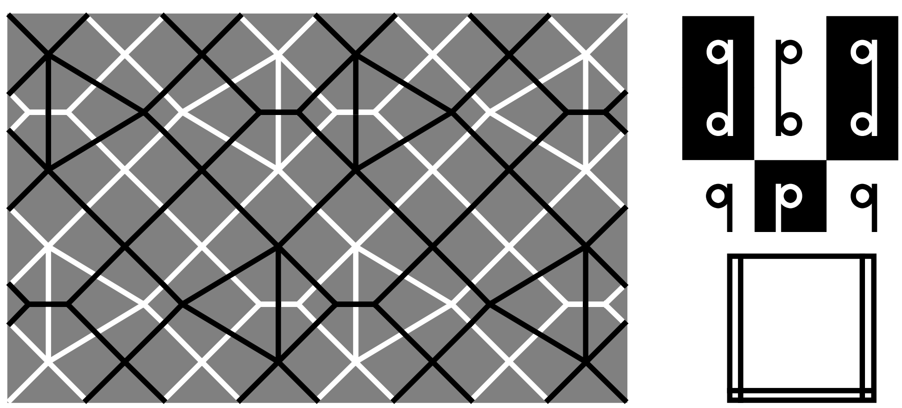

FIGURE 17. The pairing (pmm, pmg)

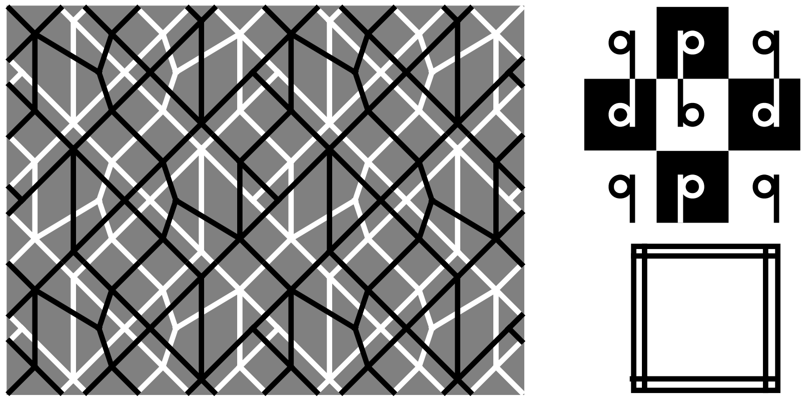

FIGURE 18. The pairing (pmm, p2)

If exactly two of these generating reflections are self-dualities, then either the self-dualities are reflections on opposite sides of the rectangle and we have (pmm, pm), see Figure 19, or they are on adjacent sides and the pairing is (pmm, cmm), see Figure 20.

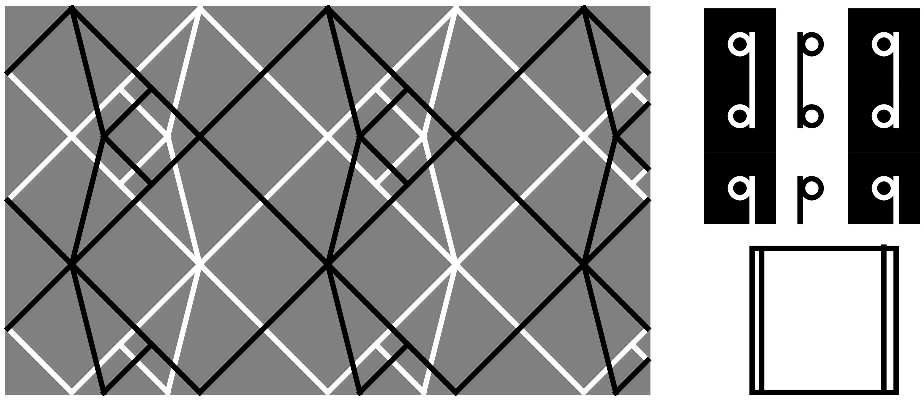

FIGURE 19. The pairing (pmm, pm)

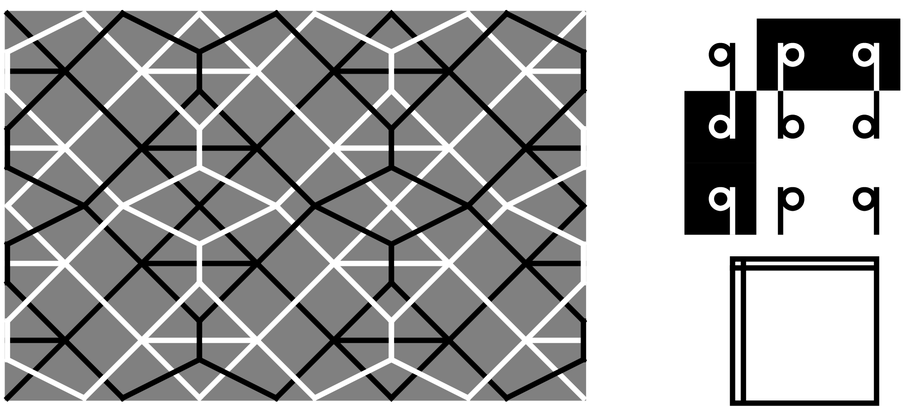

FIGURE 20. The pairing (pmm, cmm)

Ambient group ***pmg*****.** *pmg* is generated by two 180° rotations together with a reflection on a line parallel to the line between the poles. If neither rotation is a self-duality, the reflection must be a duality and the pairing is (pmg, p2), realized in Figure 21.

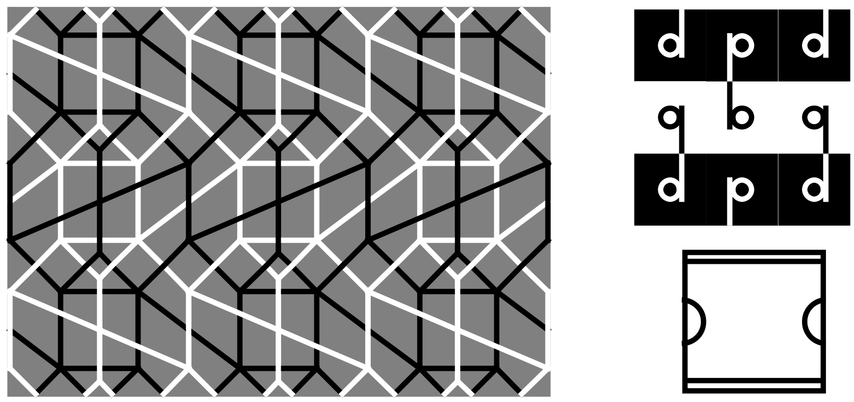

FIGURE 21. The pairing (pmg, p2)

If only one of the generating rotations is a self-duality, then the pairing is (pmg, pgg), see Figure 22, if the reflection is also a self-duality, and (pmg, pmg),

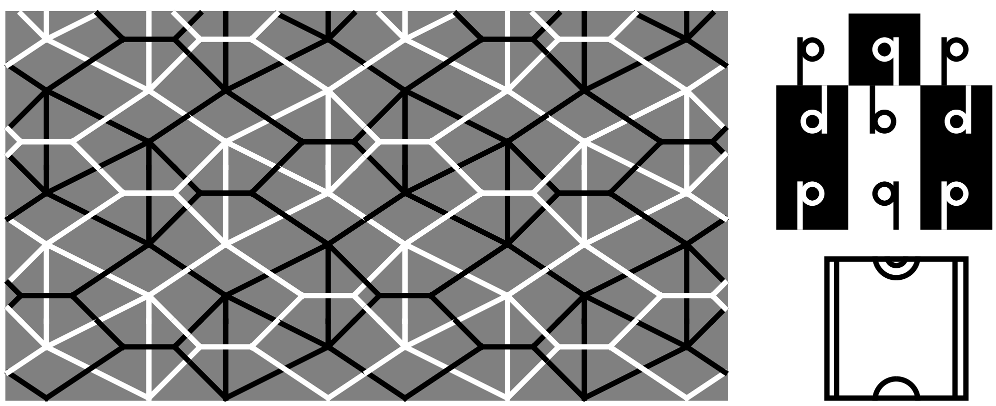

FIGURE 22. The pairing (pmg, pgg)

see Figure 23, if it is not.

If both generating rotations are self-dualities, then the pairing is (pmg, pg), see Figure 24, provided that the reflection is also a self-duality, and (pmg, pm), see Figure 25, if it is not.

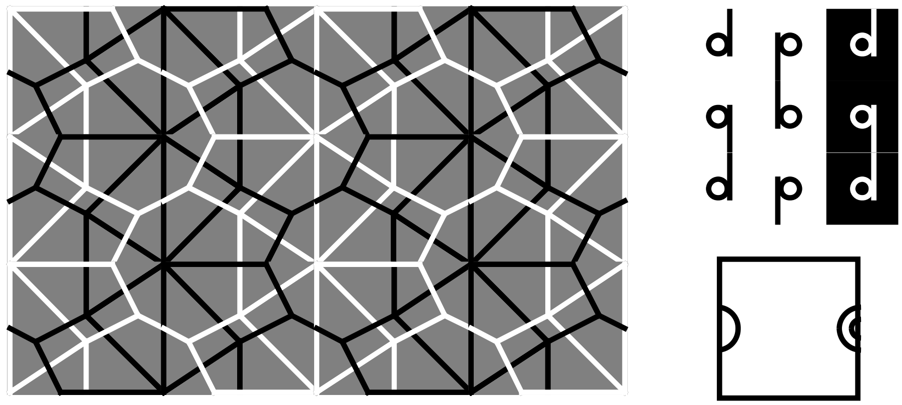

FIGURE 23. The pairing (pmg, pmg)

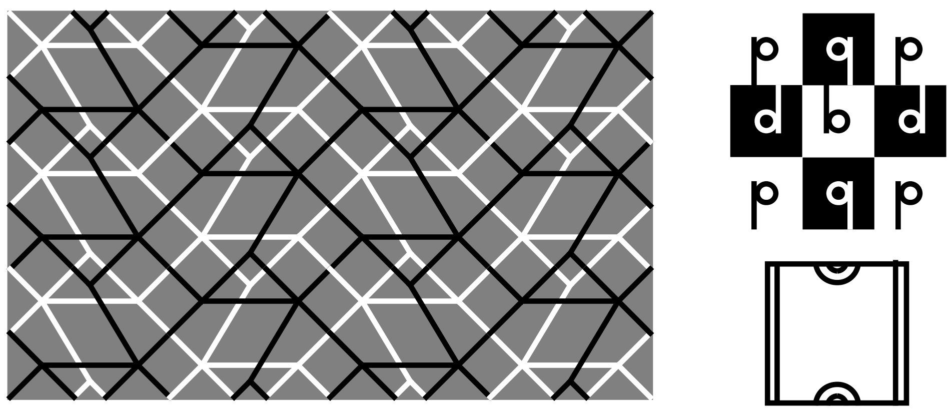

FIGURE 24. The pairing (pmg, pg)

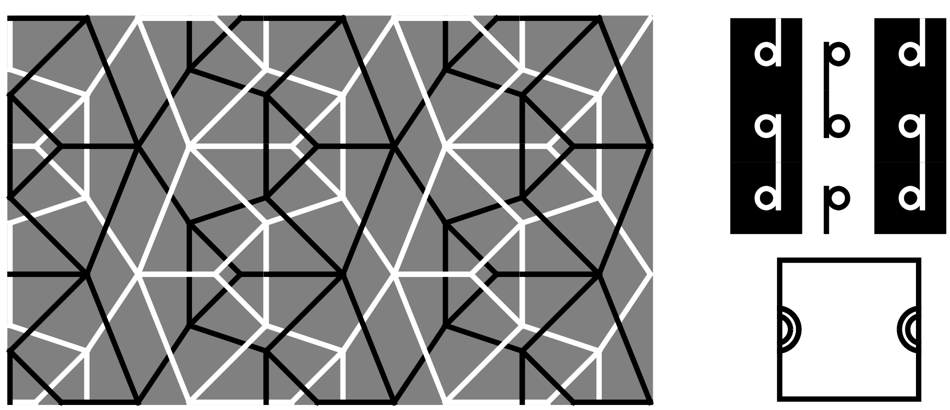

FIGURE 25. The pairing (pmg, pm)

Ambient group ***cm*****.** The group *cm* is generated by a reflection and a parallel glide reflection. The fundamental region may be taken to be a rectangle, two of whose opposite sides are on reflective lines. If the generating glide reflection is a self-duality and the reflection is not, then the pairing is (cm, pm), see Figure 26.

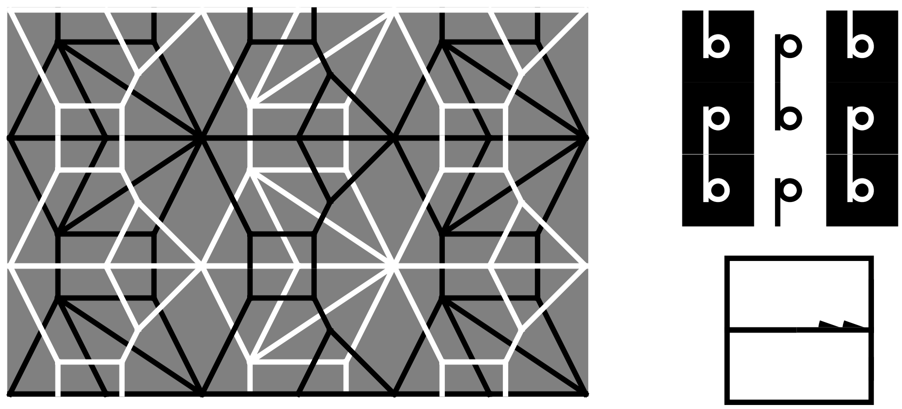

FIGURE 26. The pairing (cm, pm)

If, on the other hand, the generating glide reflection is not a self-duality, but the reflection is, then the pairing is (cm, pg), see Figure 27.

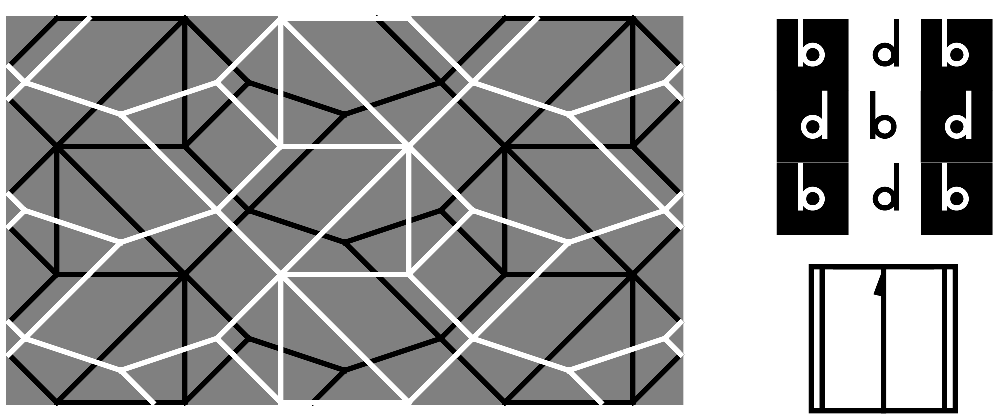

FIGURE 27. The pairing (cm, pg)

If both the generating glide reflection and the reflection are self-dualities, then the pairing is (cm, p1), see Figure 28.

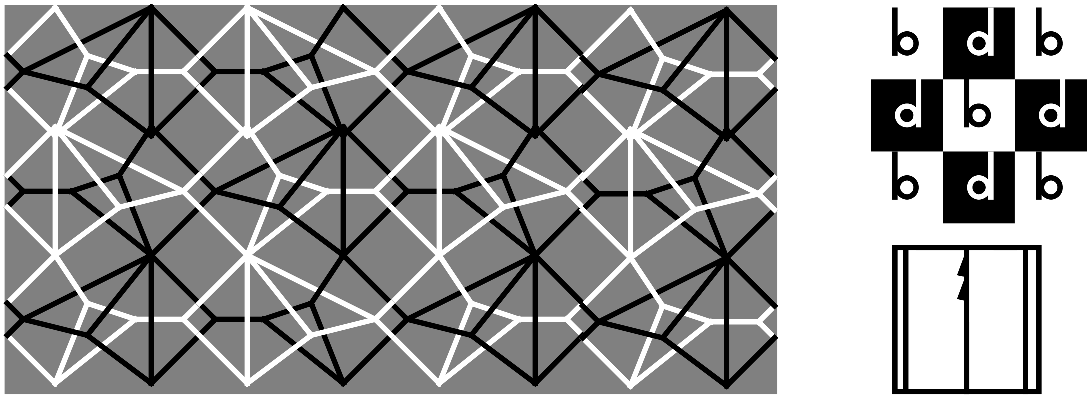

FIGURE 28. The pairing (cm, p1)

Ambient group *cmm***.** The group *cmm* is generated by reflections in two perpendicular lines, and a 180° rotation whose pole is not on either line.

If neither reflection is a self-duality, then the rotation must be, and the pairing is (cmm, pmm), see Figure 29.

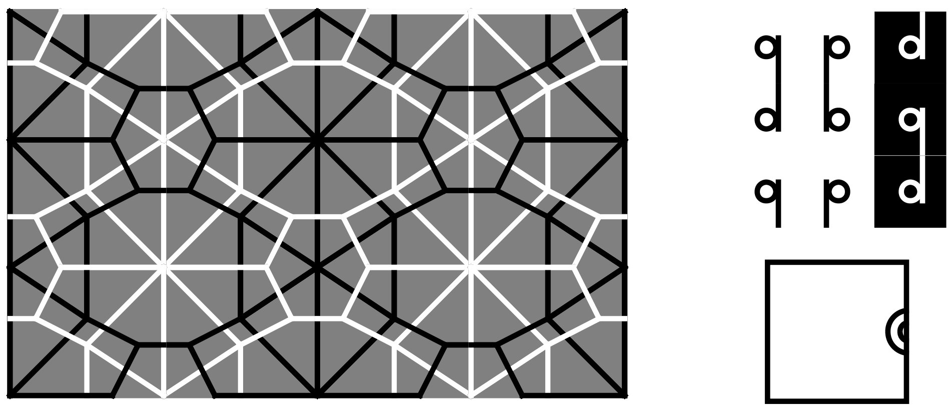

FIGURE 29. The pairing (cmm, pmm)

If one reflection is a self-duality, then if the rotation is a self-duality as well the pairing is (cmm, cm), see Figure 30, and if not then the pairing is (cmm, pmg), see

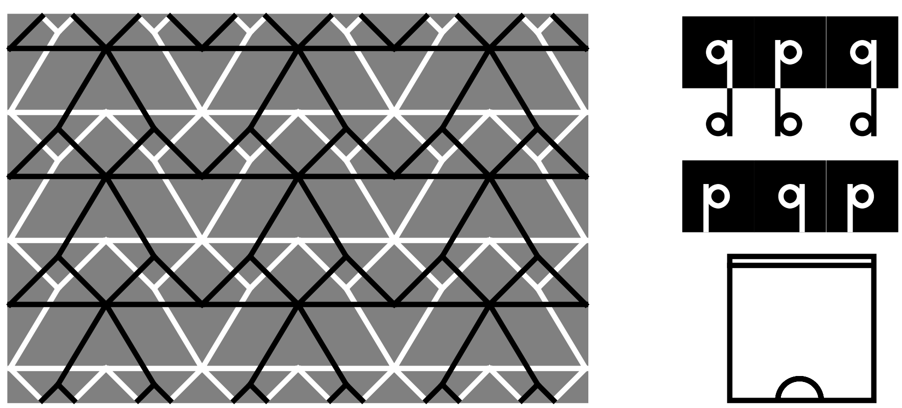

FIGURE 30. The pairing (cmm, cm)

Figure 31.

If both generating reflections are self-dualities, then the pairing is (cmm, pgg) if the rotation is also a self-duality, see Figure 32, and (cmm, p2) if it is not, see Figure 33.

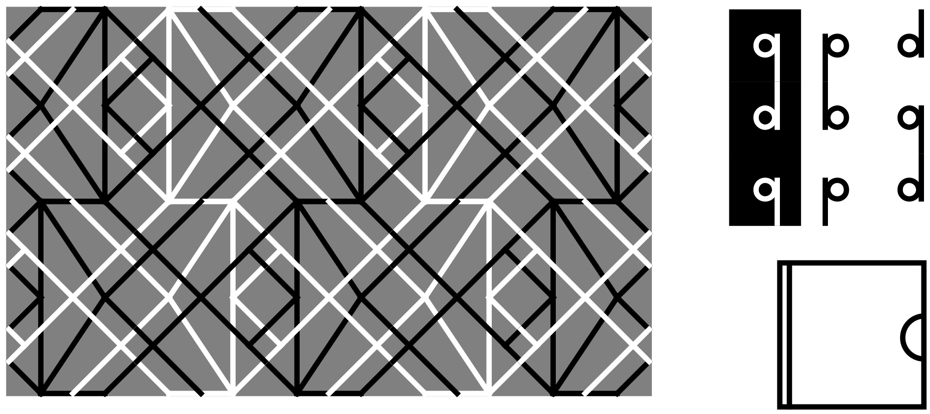

FIGURE 31. The pairing (cmm, pmg)

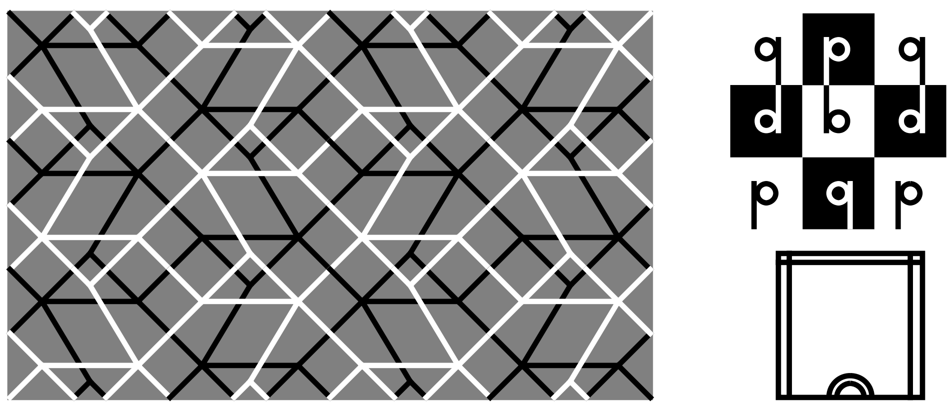

FIGURE 32. The pairing (cmm, pgg)

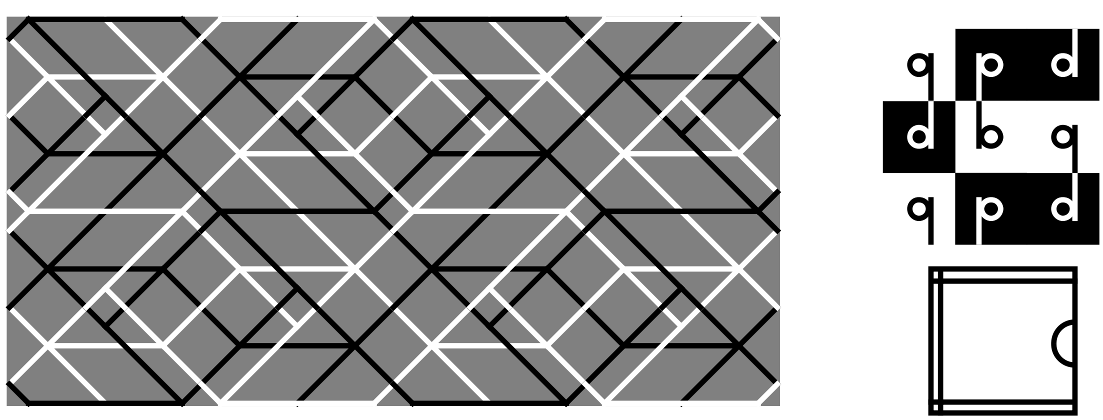

FIGURE 33. The pairing (cmm, p2)

Ambient group *pgg*. The group *pgg* is generated by two glide reflections along two perpendicular lines. If both are self-dualities, then the pairing is (pgg, p2), see Figure 34, while if only one is a self-duality, the pairing is (pgg, pg), see Figure 35.

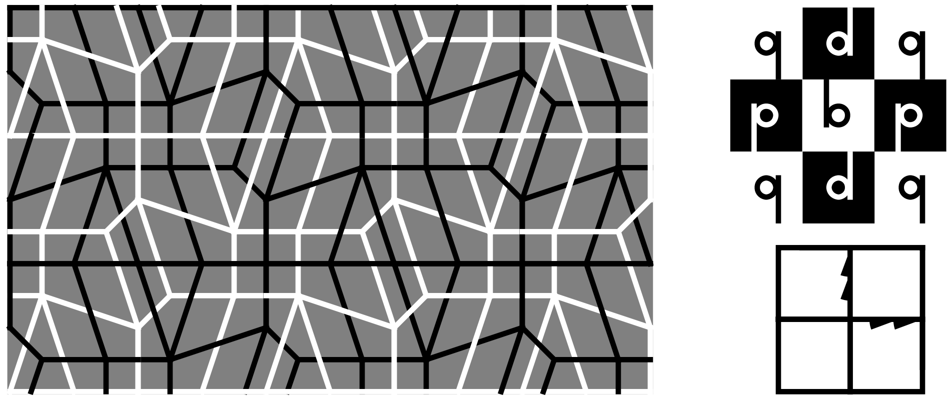

FIGURE 34. The pairing (pgg, p2)

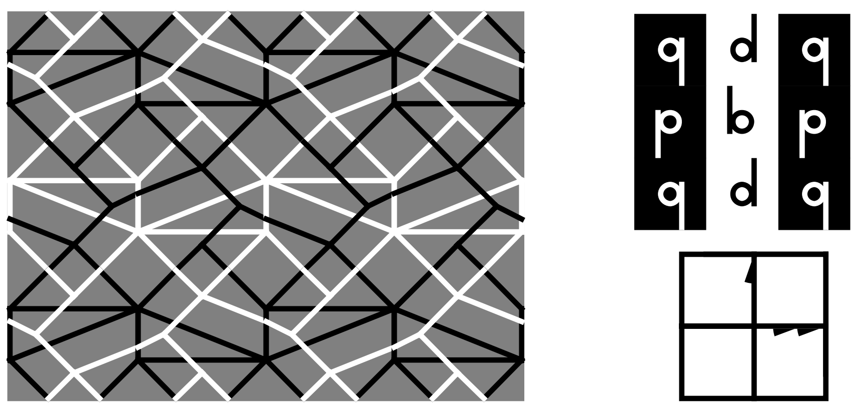

FIGURE 35. The pairing (pgg, pg)

Ambient group ***pg*****.** The group *pg* is generated by a glide reflection and a perpendicular translation. Equivalently, *pg* is generated by two parallel glide reflections, equal in length. If both glide reflections are self-dualities, then the pairing is (pg, p1), see Figure 36, while if only one is a self-duality, then the pairing is (pg, pg),

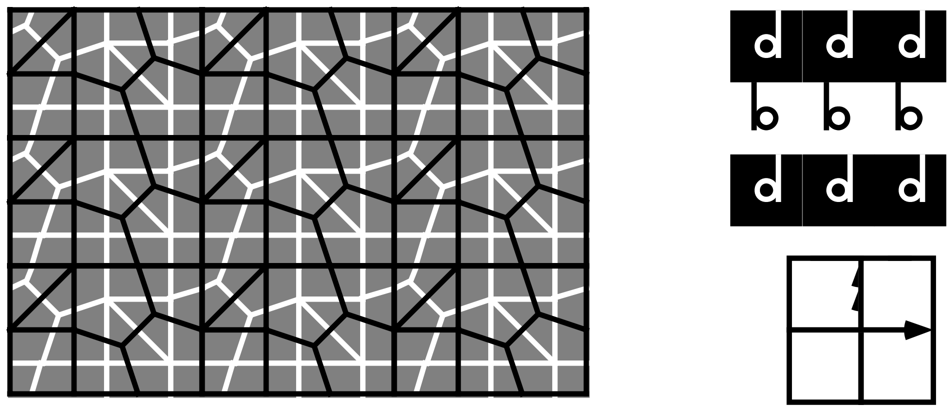

FIGURE 36. The pairing (pg, p1)

see Figure 37.

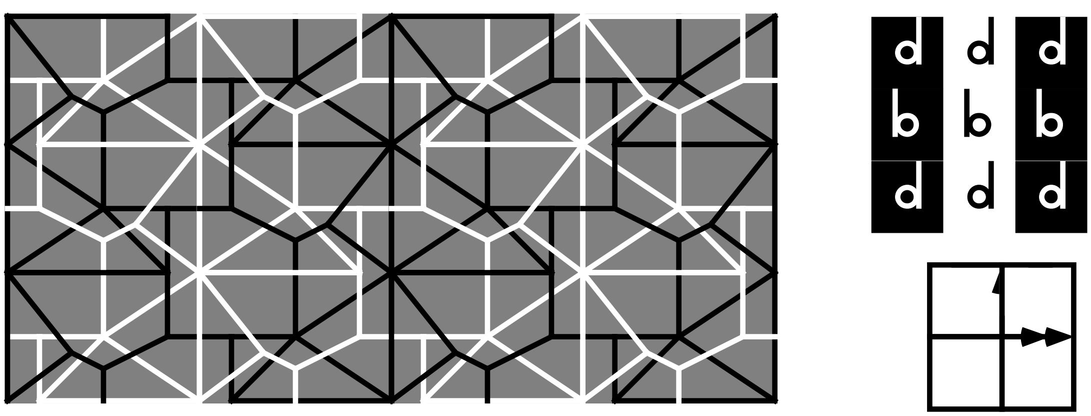

FIGURE 37. The pairing (pg, pg)

Ambient Group $p4$**.** The group $p4$ is generated by two rotations of order four, and the fundamental region may be taken to be a square with the poles of rotation at opposite corners. If both rotations are self-dualities, then the pairing is (p4, p2), see Figure 38.

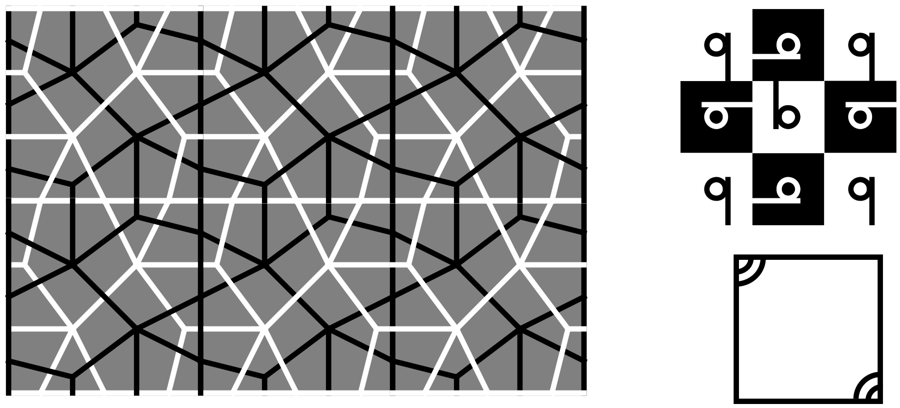

FIGURE 38. The pairing (p4, p2)

If only one of the order 4 rotations is a self-duality, then the pairing is (p4, p4), see Figure 39.

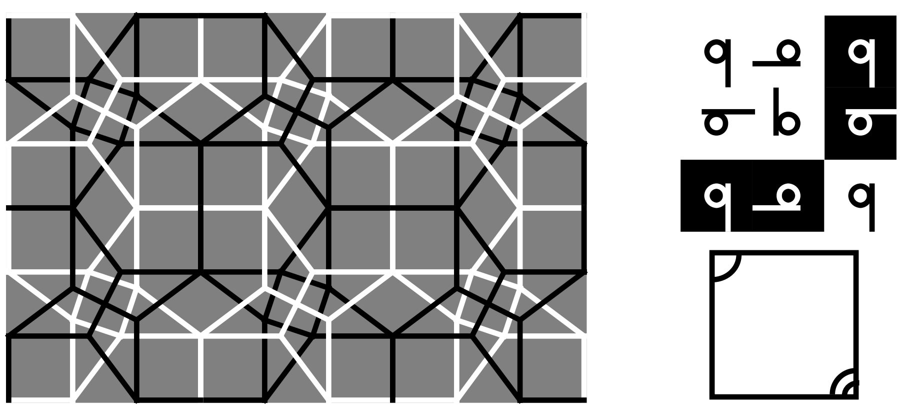

FIGURE 39. The pairing (p4, p4)

Ambient group ***p4g*****.** The group *p4g* is generated by an order 4 rotation and a reflection not passing through the pole of rotation. The fundamental region may be taken to be an isosceles right triangle with hypotenuse on the reflecting line and opposite vertex the pole of rotation. If both transformations are self-dualities, then the pairing is (p4g, pgg), see Figure 40.

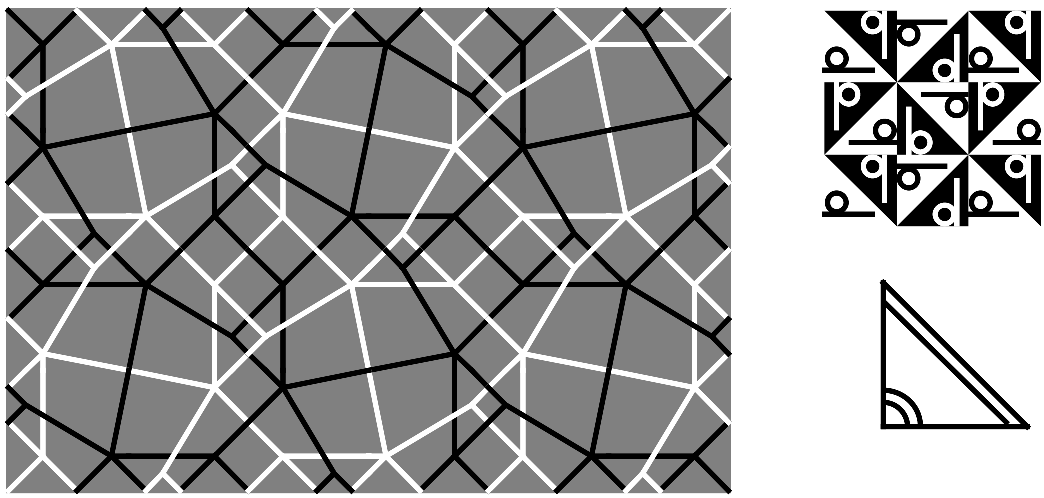

FIGURE 40. The pairing (p4g, pgg)

If only the reflection is a self-duality, then the pairing is (p4g, p4), see Figure 41, while if only the rotation is a self-duality the pairing is (p4g, cmm), see Figure 42.

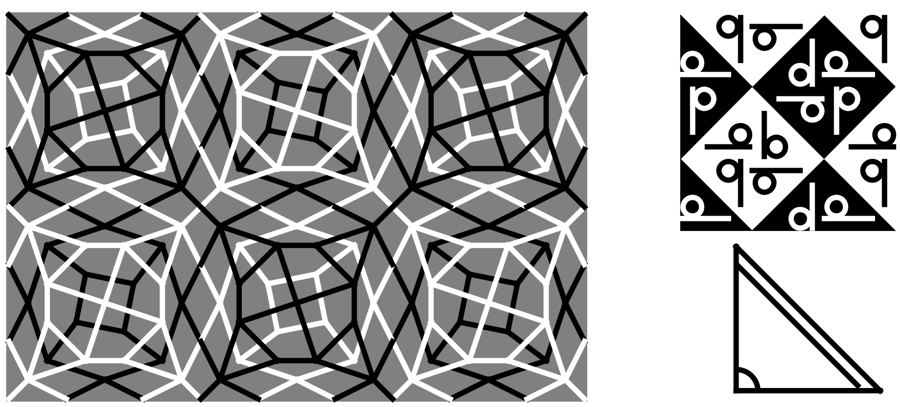

FIGURE 41. The pairing (p4g, p4)

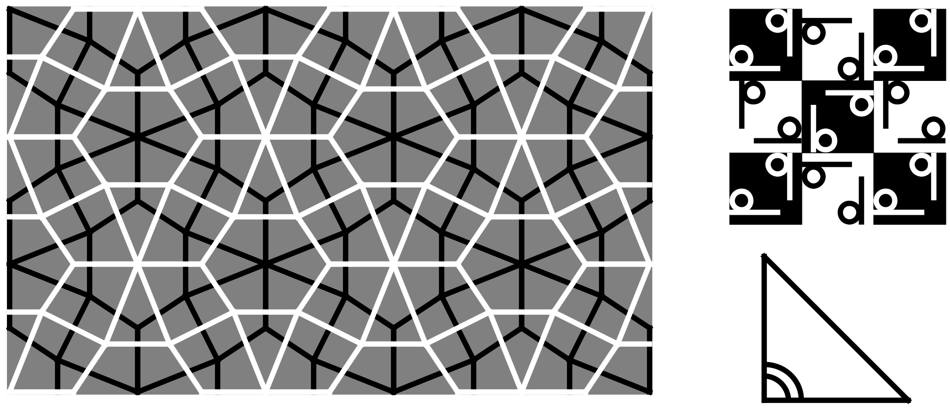

FIGURE 42. The pairing (p4g, cmm)

Ambient group $p4m$**.** The group $p4m$ is generated by the reflections in the sides on an isosceles right triangle. Since the fixed lines of two self-duality reflections are either parallel or perpendicular, reflections in the legs and hypotenuse of the right triangle cannot be both self-dualities. If the reflection on the hypotenuse is a self-duality, then the pairing is (p4m, pmm), illustrated in Figure 43.

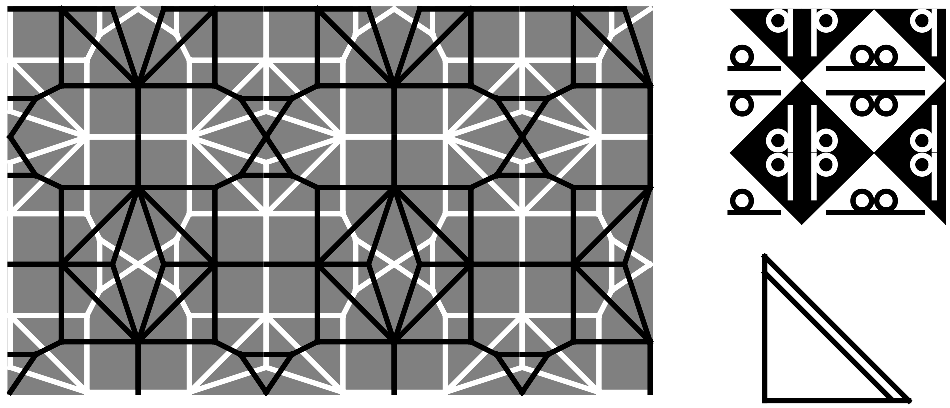

FIGURE 43. The pairing (p4m, pmm)

The pairing is (p4m, cmm), see Figure 44, if the reflections on both legs are

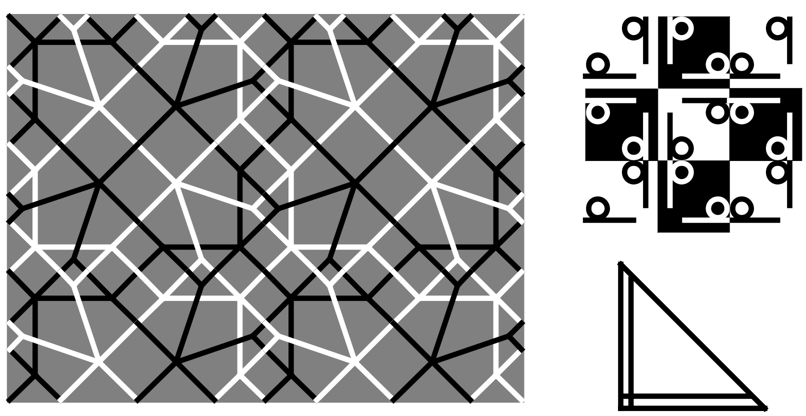

FIGURE 44. The pairing (p4m, cmm)

self-dualities, and (p4m, p4m), see Figure 45, if only one reflection is a self-duality.

REFERENCES

[1] D.S. Archdeacon, R.B. Richter, *The construction and classification of self-dual polyhedra*, JCTB, **54**, 1, pp 37-48, 1992.

[2] J. Ashley, B. Grünbaum, G. C. Shephard, W. Stromquist, *Self–duality groups and ranks of self–dualities* Applied Geometry and Discrete Mathematics, The Victor Klee Festschrift, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Volume 4, 1991, pp 11–50.

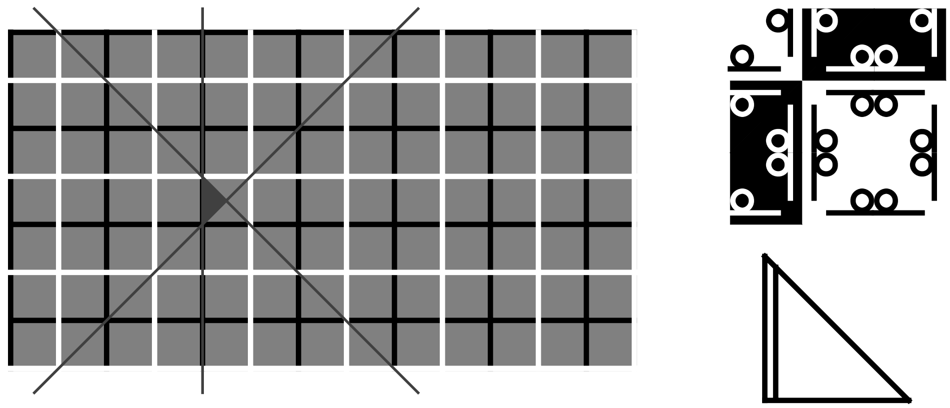

FIGURE 45. The pairing (p4m, p4m)

[3] D. W. Barnet and B. Grünbaum, *On Steinitz's Theorem concerning convex 3-polytopes and on some properties of planar graphs,* Many Facets of Graph Theory, Lecture Notes 110, Springer Verlag (1969) 27-40.

[4] H.S.M. Coxeter and W.O.J. Moser, *Generators and relations for discrete groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 14, Springer Verlag, 1972.

[5] H. S. M. Coxeter, *Coloured Symmetry,* in M. C. Escher: Art and Science, (H. S. M. Coxeter et. al., eds.) Elsevier, Amsterdam, 1986, pp. 15–33.

[6] A. Dean and B. Richter, *When is an algebraic duality a geometric duality?,* in Graph Theory, Combinatorics, and Applications, Y. Alavi and A.J. Schwenk (eds.), Wiley (1995), 991-997.

[7] C. Droms, B. Servatius, H. Servatius, The structure of locally finite 2-connected graphs, Electronic Journal of Combinatorics **2** (1995), #R17.

[8] B. Grünbaum and G.C. Shephard, *Is selfduality involutory?*, Amer. Math. Monthly 95(1988), 729-733.

[9] B. Grünbaum and G.C. Shephard, *Tilings and Patterns,* Freeman, New York, 1987.

[10] S. Jendroľ, *On symmetry groups of self-dual convex polyhedra*, Annals of Discrete Math. 51 (1992), 129-135.

[11] B. Servatius and H. Servatius, *The 24 symmetry pairings of self–dual maps on the sphere*, Discrete Math. 140, 167–183, (1995).

[12] B. Servatius and H. Servatius, Self-dual graphs, Discrete Math. **149**, 223–232, (1996).

[13] H. Whitney, 2-*isomorphic graphs,* Amer. J. Math. **55**, 1933, 245–254.

DEPARTMENT OF MATHEMATICS, WORCESTER POLYTECHNIC INSTITUTE, WORCESTER, MA 01609-2280