A NOTE ON THE RANK OF SELF-DUAL POLYHEDRA.

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Abstract. We examine how the symmetry of a self-dual polyhedron affects its rank, answering some questions in [4]

A polyhedron P is said to be *self-dual* if there is an isomorphism $\delta : P \to P^*$, where P^* denotes the dual of P. We may regard δ as a permutation of the elements of P which sends vertices to faces and vice versa, preserving incidence. For example, the regular tetrahedron and its dual are isomorphic, and the self-dual permutation may be taken to correspond to the antipodal map.

The character of the permutation δ has only recently been considered. In [3] an example of a self-dual polyhedron is given for which no self-dual permutation has order 2. Given a self-dual polyhedron P, the least order of any self-duality is called the *rank* of P, $r(P)$. It is easy to see that $r(P)$ must be a positive power of 2.

The possible symmetries of a self-dual polyhedron were enumerated in [4], and the following result is stated which indicates how the symmetry class can affect the rank.

Theorem 1. *If a self-dual polyhedron P has a central symmetry, then $r(P)$ is either 2 or 4.*

The symmetry does not completely determine the rank, as the following example illustrates. Figure 1 shows Schlegel diagrams of four self-dual polyhedra, each with symmetry group $[4]^+$. All have rank 2 except Figure 1b, which has rank 8.

In [5] it is shown that every self-dual polyhedron P corresponds to a bi-colored map M on the sphere obtained by embedding the graph of P (one color) together with the graph of P^* (second color), such that the automorphism group of the map M, $\mathrm{Aut}(M)$, is one of the isometry groups of the sphere, and $[\mathrm{Aut}(M), \mathrm{Aut}(P)] = 2$. In this setting the self-dualities correspond to the elements in $\mathrm{Aut}(M) - \mathrm{Aut}(P)$. We call $\mathrm{Aut}(M) \rhd \mathrm{Aut}(P)$ the *self-dual pairing* of P. For example, the pairing corresponding to the regular tetrahedron is $[3,4] \rhd [3.3]$, which reflects the usual embedding of the pair of dual tetrahedra in the cube, (see [2] for the notation of the isometry groups of the sphere).

The self-dual pairings were catalogued in [6], and the pairing does determine the rank.

Theorem 2. *If $Aut(P) = [2]^+$ or $Aut(P) = [2, 2^+]$, then $r(P)$ may be either 2 or 4.*

If $Aut(P) = [q]^+$, $q > 2$, then $r(P)$ may be either 2 or q/s, where s is the largest odd divisor of q.

If $Aut(P) \in \{[q], [2,2], [2,2]^+, [2^+,2^+], [2^+,4^+], [2^+,4], [3,3], [3,3]^+\}$, then $r(P) = 2$.

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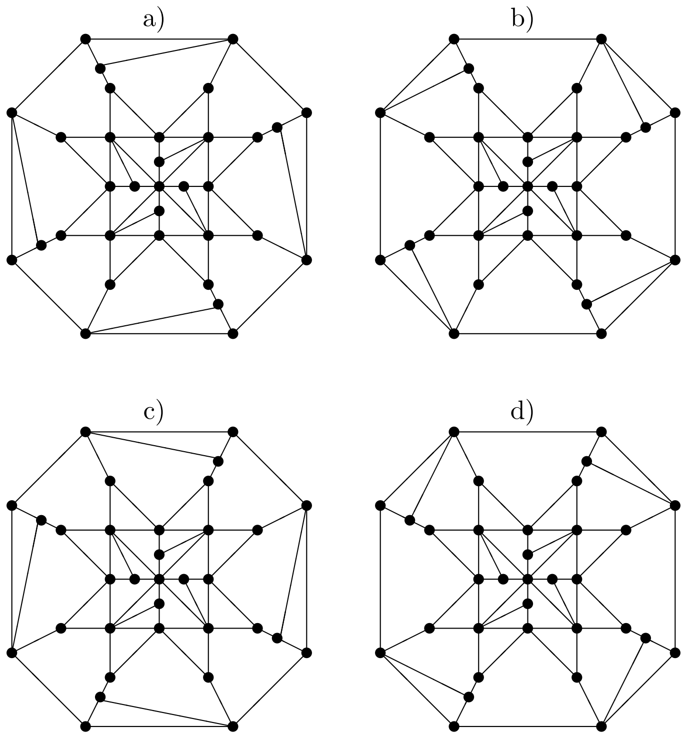

FIGURE 1. Self-dual polyhedra with fourfold rotational symmetry.

Proof. If $\mathrm{Aut}(P) = [2]^+$, then its pairing is either $[2,2]^+ \triangleright [2]^+$, $[2,2^+] \triangleright [2]^+$, in which case $r(P) = 2$, or $[4]^+ \triangleright [2]^+$ for which $r(P) = 4$.

If $\mathrm{Aut}(P) = [2,2^+]$, then the pairing of P is either $[2,2] \triangleright [2,2^+]$, so $r(P) = 2$, or $[2,4^+] \triangleright [2.2^+]$, in which case $r(P) = 4$.

If $\mathrm{Aut}(P) = [q]^+$, $q > 2$, then the pairing of P is $[2,q]^+ \triangleright [q]^+$, (for $q = 4$ see Figure 1a and d), $[2,q^+] \triangleright [q]^+$ (see Figure 1c), in which case $r(P) = 2$, or $[2,2q^+] \triangleright [q]^+$ (See Figure 1b), in which case the rank is q/s.

□

In particular, if P has any symmetry excepting rotational symmetry, then $r(P)$ is 2 or 4.

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