SELF-DUAL GRAPHS

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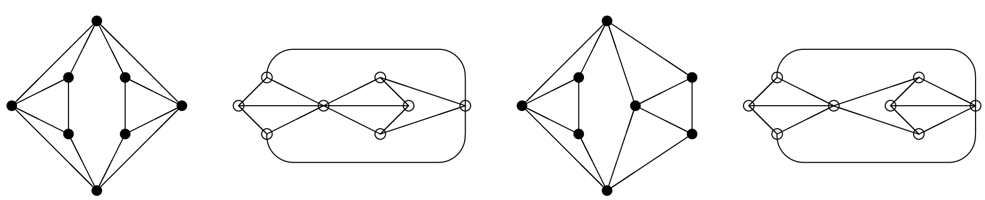
Abstract. We consider the three forms of self-duality that can be exhibited by a planar graph G, map self-duality, graph self-duality and matroid self-duality. We show how these concepts are related with each other and with the connectivity of G. We use the geometry of self-dual polyhedra together with the structure of the cycle matroid to construct all self-dual graphs.

1. Self-Duality of Graphs

1.1. **Forms of Self-duality.** Given a planar graph $G = (V, E)$, any regular embedding of the topological realization of G into the sphere partitions the sphere into regions called the *faces* of the embedding, and we write the embedded graph, called a *map*, as $M = (V, E, F)$. G may have loops and parallel edges. Given a map M, we form the *dual map*, M^* by placing a vertex f^* in the center of each face f, and for each edge e of M bounding two faces f_1 and f_2, we draw a dual edge e^* connecting the vertices f_1^* and f_2^* and crossing e once transversely. Each vertex v of M will then correspond to a face v^* of M^* and we write $M^* = (F^*, E^*, V^*)$. If the graph G has distinguishable embeddings, then G may have more than one dual graph, see Figure 1. In this example a portion of the map (V, E, F) is flipped over on a



$(V, E, F) \xrightarrow{*} (F^*, E^*, V^*)$ $(V, E, F') \xrightarrow{*} (F'^*, E^*, V^*)$

Figure 1.

separating set of two vertices to form (V, E, F'). Such a move is called a *Whitney flip*, and the duals of (V, E, F) and (V, E, F') are said to differ by a *Whitney twist.* If the graph (V, E) is 3-connected, then there is a unique embedding in the plane and so the dual is determined by the graph alone.

In general, an object is said to be *self-dual* if it is isomorphic to its dual, the most famous example being the regular tetrahedron, and self-duality has been studied in various contexts, see for example [1], [2], [6], and [8]. Given a map $X = (V, E, F)$ and its dual $X^* = (F^*, E^*, V^*)$, there are three natural notions of *self-duality.* The strongest, *map self-duality*, requires that X and X^* are isomorphic as maps, that is, there is an isomorphism $\delta : (V, E, F) \to (F^*, E^*, V^*)$ preserving incidences. A weaker notion requires only a graph isomorphism $\delta : (V, E) \to (F^*, E^*)$, in which case we say that the map (V, E, F) is *graph self-dual*, and we say that $G = (V, E)$ is a *self-dual graph.* More generally, we say that (V, E, F) is *matroid self-dual* if the cycle matroids of (V, E) and (F^*, E^*) are isomorphic, so there is a bijection between E and E^* preserving the cycle structure, or, equivalently, there is a permutation $E \to E$ which sends cycles to cocycles and vice versa. We note that, since a graphic matroid $M(G)$ is cographic if and only if G is planar, only planar graphs can be self-dual in any sense.

1.2. **The Self-dual Permutation.** Suppose that the map $X = (V, E, F)$ is self-dual, so that there is a bijection $\delta : (V, E, F) \to (F^*, E^*, V^*)$. Following δ with the correspondence $*$ gives a permutation Δ on $V \cup E \cup F$ which preserves incidence but which reverses dimension. The collection of all such permutations, or *self-dualities*, generate a group $\mathrm{Dual}(X)$ in which the map automorphisms $\mathrm{Aut}(X)$ of (V, E, F) are contained as a subgroup of index 2.

In the case of a self-dual graph $G = (V, E)$, following the self-duality $\delta : (V, E) \to (F^*, E^*)$, by $*$ does not define a permutation on $V \cup E$, however we can define a self-dual permutation on the edges of G alone, which will be a permutation on the edges E sending cycles to cocycles. In general, given a matroid M with a bijection $\delta : M \to M$ sending cycles to cocycles and vice versa, the group generated by all such permutations $\mathrm{Dual}(M(G))$ is called the *self-duality group of M*, and contains $\mathrm{Aut}(M)$ as a subgroup of index 2.

We call $\mathrm{Dual}(X) \rhd \mathrm{Aut}(X)$ and $\mathrm{Dual}(M) \rhd \mathrm{Aut}(M)$ the *self-dual pairing* of the map X and the cycle matroid M respectively. The possible symmetry groups of self-dual polyhedra were enumerated in [9] and in [12] the self-dual pairings of self-dual maps were enumerated and used to classify all self-dual maps. Briefly, given any self-dual map X, there is a drawing of the map and the dual map on the sphere so that $\mathrm{Dual}(X)$ is realized as a group of spherical isometries. In the notation of [3] the possible pairings are among the infinite classes $[2, q] \rhd [q]$, $[2, q]^+ \rhd [q]^+$, $[2^+, 2q] \rhd [2q]$, $[2, q^+] \rhd [q]^+$, and $[2^+, 2q^+] \rhd [2q]^+$; or are among the special pairings $[2] \rhd [1]$, $[2] \rhd [2]^+$, $[4] \rhd [2]$, $[2]^+ \rhd [1]^+$, $[4]^+ \rhd [2]^+$, $[2, 2] \rhd [2, 2]^+$, $[2, 4] \rhd [2^+, 4]$, $[2, 2] \rhd [2, 2^+]$, $[2, 4] \rhd [2, 2]$, $[2, 4]^+ \rhd [2, 2]^+$, $[2^+, 4] \rhd [2, 2]^+$, $[2^+, 4] \rhd [2^+, 4^+]$, $[2, 4^+] \rhd [2^+, 4^+]$, $[2, 2^+] \rhd [2^+, 2^+]$, $[2, 4^+] \rhd [2, 2^+]$, $[2, 2^+] \rhd [1]$, $[3, 4] \rhd [3, 3]$, $[3, 4]^+ \rhd [3, 3]^+$, and $[3^+, 4] \rhd [3, 3]^+$.

Given a pairing $D \rhd A$ on this list, a the self-dual map realizing this pairing can be constructed by drawing any partial map and dual map in a fundamental region for D, observing the natural boundary conditions, and then using the action of D to complete the drawing to the whole of the sphere.

2. Comparing forms of self-duality

It is clear that for a map (V, E, F) we have

(1) Map Self-duality $\Rightarrow$ Graph Self-duality $\Rightarrow$ Matroid Self-duality.

We are concerned to what extent these implications can be reversed. The next two theorems assert that, in the most general sense, they cannot.

Theorem 1. *There exists a map (V, E, F) such that $(V, E) \cong (E^*, V^*)$, but $(V, E, F) \ncong (F^*, E^*, V^*)$.*

Theorem 2. *There exists a map (V, E, F) such that $M(E) \cong M(E)^*$, but $(V, E) \ncong (F^*, E^*)$.*

In Figure 2 the map (V, E, F) in a) is map self-dual, as shown in b), however c)

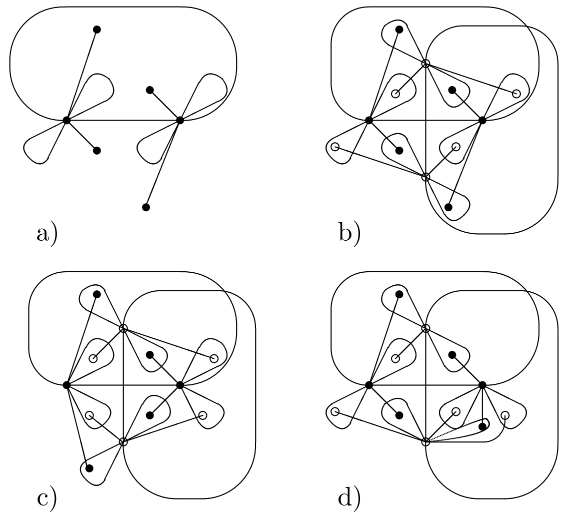


FIGURE 2.

illustrates embedding (V, E, F'), whose dual is isomorphic to (V, E, F') as a graph, but not as a map, and d) shows a map embedding (V, E, F'') whose whose dual is not even isomorphic to (V, E) as a graph.

2.1. **Self-dual maps and self-dual graphs.** In the previous examples the graphs were of low connectivity. We shall use n-connectivity for graphs and matroids in the Tutte sense, see [13, 14, 10], because Tutte n-connectivity is invariant under dualization. Note that the usual concept of 3-connectivity coincides with tutte 3-connectivity for simple graphs, and similarly for 3-connectivity and loopless graphs.

By Steinitz's Theorem, a planar 3-connected simple graph has a unique embedding on the sphere, in the sense that if p an q are embeddings, then there is a homeomorphism h of the sphere so that $p = hq$. By [15], any isomorphism between the cycle matroids of a 3-connected graph is carried by a graph isomorphism. Thus, for a 3-connected graph

$$\text{Map Self-duality} \Leftarrow \text{Graph Self-duality} \Leftarrow \text{Matroid Self-duality},$$

so self-dual 3-connected graphs, as well as self-dual 3-connected graphic matroids, reduce to the case of self-dual maps.

Since the examples in Figure 2 are only 1-connected, we must consider the 2-connected case. In Figure 3 we see an example of a graphically self-dual map whose graph is 2-connected which is not map self-dual. One might hope that, as was the case in Figure 2, that such bad examples can be corrected by re-embedding or rearranging, however we have the following stronger result.

THEOREM 3. *There exists a* 2*-connected map* (V, E, F) *which is graphically self-dual, so that* $(V, E) \cong (F^*, V^*)$*, but for which every map* (V', E', F') *such that* $M(E) \cong M(E')$ *is not map self-dual.*

Proof. Consider the map in Figure 3, which is drawn on an unfolded cube. The graph is obtained by gluing two 3-connected self-dual maps together along an edge

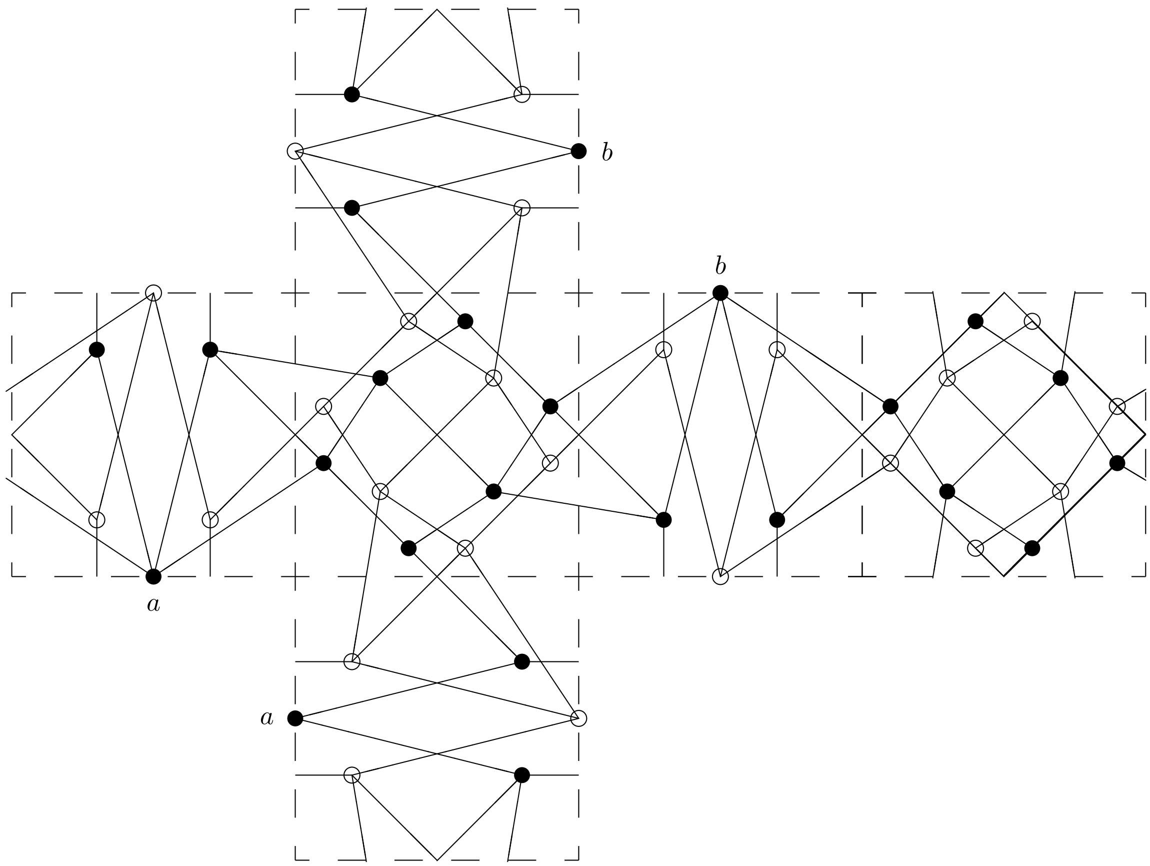


FIGURE 3.

(a, b) and erasing the common edge. One map has only two reflections as self-dualities, both fixing the glued edge, the other has only two rotations of order four as dualities, again fixing the glued edge. The graph self-duality is therefore a combination of both, an order 4 rotation followed by a Whitney twist of the reflective hemisphere. It is easy to see that all the embeddings of this graph, as well as the graph obtained after the Whitney flip, have the same property. □

We also have the following.

THEOREM 4. *There is a graphically self-dual map* (V, E, F) *with* (V, E) 1*-connected and having only* 3*-connected blocks, but for which every map* (V', E', F') *such that* $M(E) \cong M(E')$ *is not map self-dual.*

Proof. Consider the 3-connected self-dual maps in Figure 4. X_1 has only self-dualities of order 4, two rotations and two flip rotations, while X_2 has only a left-right reflection and a 180° rotation as a self-duality. Form a new map X by gluing two copies of X_2 to X_1 in the quadrilaterals marked with q's, with the gluing at the vertices marked v and v^*. X is graphically self-dual, as can easily be checked, but no gluing of two copies of X_2 can give map self-duality since every quadrilateral in X_1 has order 4 under any self-duality. □

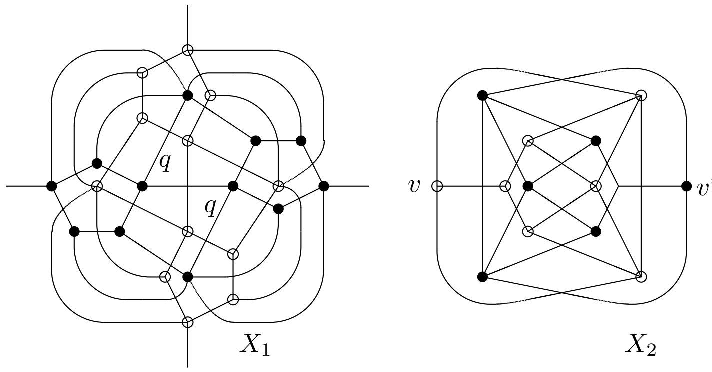


FIGURE 4.

In particular, self-dual graphs of connectivity less than 3 cannot in general be re-embedded as self-dual maps.

2.2. **Self-dual graphs and matroids.** If G is 1-connected, then its cycle matroid has a unique decomposition as the direct sum of connected graphic matroids, $M(G) = M_1 \oplus \cdots \oplus M_k$, and if G^* is a planar dual of G, then $M(G^*) = M(G)^* = M_1^* \oplus \cdots \oplus M_k^*$. If G is graph self-dual, then there is a bijection $\delta : M(G) \to M(G^*)$ sending cycles to cycles, and so there is a partition π of $\{1, \ldots, k\}$ such that $\delta : M_i \to M^*_{\pi(i)}$, and we see that $M(G)$ is the direct sum of self-dual connected matroids, together with some pairs of terms consisting of a connected matroid and its dual.

In the next theorem we see that not every self-dual graphic matroid arises from a self-dual graph.

THEOREM 5. *There exists a self-dual graphic matroid M such that for any graph $G = (V, E)$ with $M(G) = M$, and any embedding (V, E, F) of G, $(V, E) \not\cong (F^*, E^*)$.*

Proof. Consider M_1 and M_2, the cycle matroids of two distinct 3-connected self-dual maps X_1 and X_2 whose only self-dualities are the antipodal map. The matroid $M_1 \oplus M_2$ is self-dual, but its only map realizations are as the 1- vertex union of X_1 and X_2, which cannot be self-dual since the cut vertex cannot simultaneously be sent to both "antipodal" faces. □

So for 1-connected graphs, the three notions of self-duality are all distinct. For more details about 1-separable self-dual graphs see [5]. For 2-connected graphs, however, we have the following.

THEOREM 6. *If $G = (V, E)$ is a planar 2-connected graph such that $M(E) \cong M(E)^*$, then G has an embedding (V, E, F) such that $(V, E) \cong (F^*, E^*)$.*

Proof. Let (V, E, F) be any embedding of G. Then G is 2-isomorphic, in the sense of Whitney [15], to (F^*, E^*), and thus there is a sequence of Whitney flips which transform (F^*, E^*, V^*) into an isomorphic copy of G, and act as re-embeddings of G. Thus the result is a new embedding (V, E, F') of G such that $(V, E, F) \cong (F'^*, E^*, V^*)$. □

Thus, to describe 2-connected self-dual graphs it is enough, up to embedding, to describe self-dual 2-connected graphic matroids.

3. Automorphisms of 3-block Trees

3.1. **The 3-block Tree.** Any graph is the disjoint union of its connected components. If a graph is connected, then its block-cutpoint tree, see [7], shows how the graph may be constructed from 2-connected graphs and singleton edges by gluing them together at the cut vertices. If a graph is 2-connected, then there is a similar construction called the 3-*block tree* due to Tutte, [13], which was generalized to matroids by Cunningham and Edmonds, [4, 10].

Let M_i be a matroid on a set E_i, $i = 1, 2$. The 2-*sum of* M_1 *and* M_2 *along* e_1 *and* e_2, denoted by $M_1 \overset{(e_1,e_2)}{\oplus} M_2$, is defined on the set $(E_1 \cup E_2) - e_1 - e_2$ by taking the cycles in $M_1 \overset{(e_1,e_2)}{\oplus} M_2$ to consist of those cycles in M_i not containing e_i, as well as the sets $(C_1 - e_1) \cup (C_2 - e_2)$ where C_i is a cycle of M_i containing e_i. We also write that the edges e_1 and e_2 have been *amalgamated* in $M_1 \overset{(e_1,e_2)}{\oplus} M_2$.

The 2-sum of graphs is defined similarly, being careful to note the orientation of the amalgamated edge. Let G_i be graphs, $i = 1, 2$, with $e_i = (x_i, y_i)$ an edge of G_i. Define the 2-*sum* of G_1 and G_2 along e_1 and e_2 to be the graph obtained from the disjoint union of $G_1 - e_1$ and $G_2 - e_2$ when x_1 and x_2 are identified, as well as y_1 and y_2, see Figure 5. Clearly $M(G_1) \overset{(e,f)}{\oplus} M(G_2) = M(G_1 \overset{(e,f)}{\oplus} G_2)$.

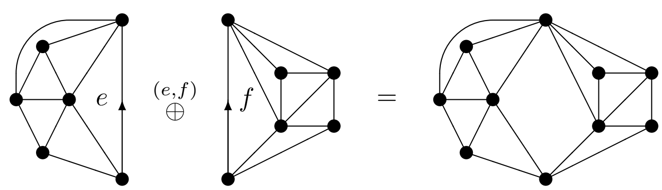


Figure 5.

A map is determined from a planar graph by choosing the faces, i.e., by choosing a dual graph. The 2-sum of maps is therefore obtained by simultaneously taking the 2-sum of G_1 and G_2 along e_1 and e_2, with the 2-sum of G_1^* and G_2^* along e_1^* and e_2^*, in other words, an orientation on e_i and e_i^* must be specified.

A matroid is called a 3-block if it is either 3-connected, or has at least 3 elements and consists of either one cycle or one cocycle.

A 3-*block tree* is a tree $\mathcal{T}$ such that each node α is labeled with a 3-block M_α and each link $\eta = \{\alpha, \beta\}$ is labeled to indicate which edge in M_α is to be amalgamated with which edge of M_β, and the labels satisfy:

(1) For each node α the labels on the links (α, β) from M_α are distinct, and
(2) for each link (α, β) the matroids M_α and M_β are not both cycles nor both cocycles.

The matroid $M(\mathcal{T})$ determined by $\mathcal{T}$ is obtained by taking the 2-sum of the matroids $\{M_\alpha\}$ along the elements determined by the labels on the links of $\mathcal{T}$. Every 2-connected matroid is encoded by a unique 3-block tree.

Let $\mathcal{T}$ and $\mathcal{T}'$ be 3-block trees. A *isomorphism of* 3-*block trees* is a triple $(f, \{f_\alpha\})$ where $f : \mathcal{T} \to \mathcal{T}'$, is a graph isomorphism and $f_\alpha : M_\alpha \to M_{f(\alpha)}$ is a matroid isomorphism such that if (α, β) is an edge of $\mathcal{T}$ amalgamating e_α with e_β, then $f(\eta)$ amalgamates $f_\alpha(e_\alpha)$ with $f_\beta(e_\beta)$. Since the 3-block tree decomposition is unique, every matroid isomorphism $F : M \to M'$ corresponds a unique isomorphism $(f, f_\alpha) : \mathcal{T}(M) \to \mathcal{T}(M')$.

4. Self-dual Matroids

As noted earlier, 3-connected self-dual graphic matroids are classified via self-dual polyhedra. On the other hand, 1-connected self-dual matroids are easily understood via the direct sum. In this section we show how a 2-connected self-dual matroid M with self-duality δ arises via 3-connected graphic matroids by recursively constructing its 3-block tree $\mathcal{T}(M)$ by adding orbits of pendant nodes. The following theorem shows that this construction is sufficient to obtain all 2-connected self-dual matroids.

Theorem 7. *Let M be a self-dual connected matroid with 3-block tree $\mathcal{T}$. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ be deleting all the pendant nodes, and let M' be the 2-connected matroid determined by $\mathcal{T}'$. Then M' is also self-dual.*

Proof. Let M be a self-dual connected matroid on a set E, so there is a matroid isomorphism $\Delta : M \to M^*$, so δ is a permutation of E sending cycles to cocycles. The 3-block tree of M^* is obtained from that of M by replacing every label with the dual label, so Δ corresponds to a bijection $(\delta, \{\delta_\alpha\})$ of $\mathcal{T}$ onto itself, such that for each node α of $\mathcal{T}$, $\delta_\alpha : M_\alpha \to M_{f(\alpha)}$ sends cycles of M_α to cocycles of $M_{f(\alpha)}$. The restriction of $(\delta, \{\delta_\alpha\})$ to $\mathcal{T}'$ has the same property and so corresponds to a self-dual permutation of M'. □

To examine the base case we note that every finite tree has a well defined central vertex or central edge which is fixed under every automorphism of the tree. If $\mathcal{T}$ has a central vertex α, then M_α must be self-dual, hence, since it cannot be a cycle or cocycle, M_α is a 3-connected self-dual matroid. If $\mathcal{T}$ has a central edge, (α, β), then $M_\alpha \overset{e}{\oplus} M_\beta$ must be self-dual and the self-dual permutation satisfies $\delta(\alpha) = \alpha$ and $\delta(\beta) = \beta$ or $\delta(\alpha) = \beta$ and $\delta(\beta) = \alpha$.

If $\delta(\alpha) = \alpha$ and $\delta(\beta) = \beta$ then both M_α and M_β are 3-connected self-dual matroids with self-dualities δ_α and δ_β both of which fix the edge e.

If $\delta(\alpha) = \beta$ and $\delta(\beta) = \alpha$ then $M_\beta = M_\alpha^*$, and $\delta_\beta\delta_\alpha$ is a matroid automorphism of M_α which fixes e.

We have the following.

Theorem 8. *Suppose M is a self-dual 2-connected matroid with self-dual permutation δ and let $e_1 \in M$. Let $\{e_1, \ldots, e_k\}$ be the orbit of e_1 under δ. Suppose one of the following:*

1. *k is even and M_0 is a 3-connected matroid or a cycle and δ_0 is a matroid automorphism of M_0 fixing an edge e_0.*
2. *k is odd and M_0 is a 3-connected self-dual matroid with self-dual permutation δ_0 fixing an edge e_0.*

For $i = 1, \ldots, k$ set $M_{2i+1} = M_0$ and $M_{2i} = M_0^*$. Let M' be the matroid obtained from M by 2-sums with the matroids M_i, amalgamating e_0 or e_0^* in M_i with e_i.

Let δ' be defined by $\delta'(e) = \delta(e)$ for $e \in M - \{e_1, \ldots, e_k\}$, $\delta' : M_i - e_0 \to M_{i+1} - e_0$ is induced by * for $i = 1, \ldots, k$ and $\delta' = \delta_0 : M_k \to M_1$. Then M' is a 2-connected self-dual matroid with self-dual permutation δ'.

Moreover, every 2-connected self-dual matroid and its self-duality is obtained in this manner.

Proof. The fact that this construction gives a 2connected self-dual matroid follows at once, since to check if δ' is a self-duality, it suffices to check that $(\delta')_\alpha$ sends cycles to cocycles on each 3-block. The fact that M_0 must be self-dual if k is odd follows by considering that δ'^k is a self-duality and maps $M_0 = M_1$ onto itself.

To see that all self-dualities arise this way, let $\delta' : M' \to M'$ be a self-duality, let α be a pendant node of $\mathcal{T}$, and set $M_0 = M_\alpha$. Let M be the self-dual matroid that results from removing removing from $\mathcal{T}(M')$ the k nodes corresponding to the orbit of the node α. δ' induces $\delta : M \to M$. Then the desired δ_0 is $(\delta^k)_\alpha$. □

5. The structure of self-dual graphs

Given the results of the previous section, we may construct all 2-connected self-dual graphs; start with any self-dual 2-connected graphic matroid M and chose any realization of M as a cycle matroid of a graph G. Theorem 6 asserts that G has an embedding as a self-dual graph. Alternatively, we may carry out a recursive construction in the spirit of Theorem 8 at the graph level, paying careful attention to the orientations in the 2-sums. The following theorem gives a more geometric construction.

Theorem 9. *Every 2-connected self-dual graph is 2-isomorphic to a graph which may be decomposed via 2-sums into self-dual maps such that the 2-sum on any two of the self-dual maps is along two edges, one of which is the pole of a rotation of order 4 and the other an edge fixed by a reflection.*

Proof. In case 1 of Theorem 8 we can always choose δ_0 to be the identity, and simply glue in the copies of the maps corresponding to M_0 and M_0^* compatibly to make a larger self-dual map.

In case 2 we must have that M_0 is a self-dual 3-block containing a self-duality fixing e_0, hence it corresponds to a self-dual map and δ_0 must be a reflection or an order 4 rotation fixing e_0, see [11], and likewise the 3-block to which it is attached must be such an edge. If both are of the same kind, then the 3-blocks may be 2-summed into a self-dual map. This leaves only the mismatched pairs. □

To see that 2-isomorphism is necessary in the above, consider the self-dual graph in Figure 6. The map cannot be re-embedded as a self-dual map, nor does it have a 2-sum decomposition described as above, but the graph is 2-isomorphic to a self-dual map.

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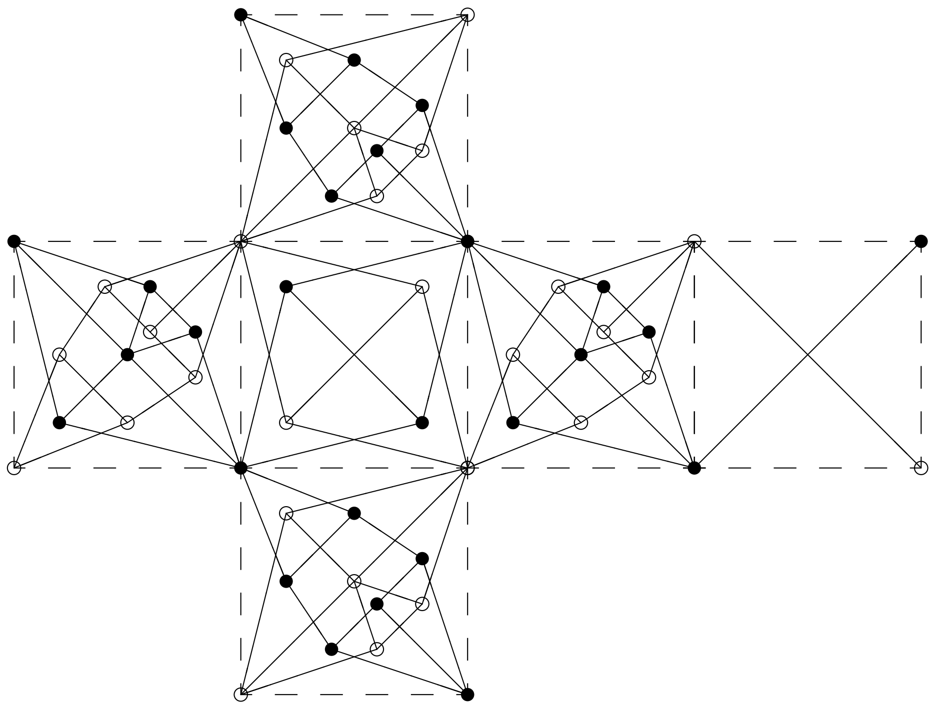

FIGURE 6.

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