SELF–DUAL MAPS ON THE SPHERE

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Abstract. We show how to recursively construct all self–dual maps on the sphere together with their self–dualities, and classify them according to their edge–permutations.

Although several well known classes of self–dual graphs, e.g., the wheels, have been known since the last century, [7], the general characteristics of self–dual graphs have only recently begun to be explored. In [10] two constructions are given to produce examples of large minimally self–dual graphs. In [2] self–dual polyhedra are constructed and classified.

Given a self–dual object, Grünbaum and Shephard [5] considered the self–dual correspondence as a permutation on the elements of the object itself, and asked if every self–dual object admitted a self–duality permuation of order 2. The question was answered negatively for polyhedra by Jendroľ [6] and by McCanna [8] and prompted a re–examination of self–dual polyhedra, [3]. In this article, we examine the more general setting of self–dual maps on the sphere, making no assumptions of higher connectivity on the underlying graphs, allowing a clear and unified approach.

1. Automorphisms of maps on the sphere

Let $\Gamma = (V, E)$ be a finite connected planar graph, so there exists a tame embedding ρ of Γ into the sphere, S^2. We regard two such embeddings, ρ and ρ', as equivalent if there is a homeomorphism f of S^2 such that $\rho' = f\rho$. The graph Γ may have parallel edges and loops, in which case there will be several inequivalent ways to place Γ in S^2. On the other hand, if Γ is 3–connected, then all embeddings of Γ are equivalent up to orientation. Unless there is danger of confusion, we will hereafter suppress mention of ρ. $S^2 - \Gamma$ consists of a disjoint union of open cells whose closures in S^2 are the faces of a realization of S^2 as a finite CW–complex, G, called a *map on the sphere*, or more briefly, just a map. An isomorphism of maps will be understood to be an isomorphism of cell complexes and we note that the CW–complex arising from an embedded graph will not in general be regular. By straightforward subdivision arguments one can show the following two propositions.

Proposition 1. *Every non-trivial orientation preserving map automorphism σ has exactly two fixed cells. Moreover, the map can be drawn so that σ is a rotation of S^2.*

Proposition 2. *Suppose σ is an orientation reversing map automorphism. If σ^2 is the identity, and some cell is sent into itself by σ, then the map can be drawn so that σ is a reflection of S^2 about an equator. If σ^2 is the identity and σ fixes no cell, then the map may be drawn such that σ is the antipodal map. If σ^2 is not the identity, then the map may be drawn so that σ is a rotatory reflection.*

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Note that when a map is drawn, as in the above propositions, to reflect the geometry of some map automorphism, the edges cannot in general be chosen to be geodesics.

Any map G determines a *dual* map, G^*, obtained by placing a vertex f^* in the interior of each face f and, if two faces f and f' meet along an edge e, then an edge e^* is drawn connecting f^* and f'^* such that e^* intersects G only once transversely in the interior of the edge e. Each vertex v will then lie in the interior of a face v^* of G^*. A map G is said to be *self–dual* if G and G^* are map isomorphic. A planar graph Γ is said to be *self–dual* if there is a map G of Γ in S^2 such that the 2–skeleton of G^* is isomorphic to the graph Γ. The example in Figure 1 shows that not all self–dual graphs arise in this manner. The graph and its dual are pictured. There is no

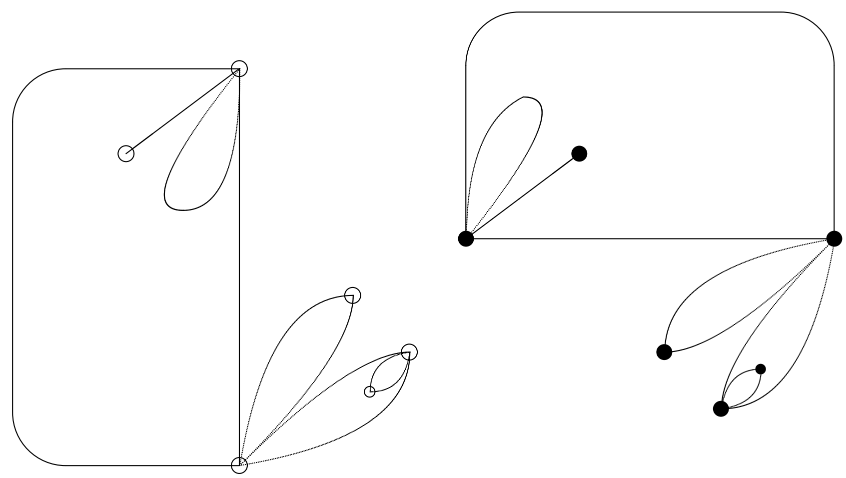


FIGURE 1. Self–dual graphs admitting no self–dual map

map isomorphism between them since the order of the objects attached to the large 2–cycle in the graph is incompatible with their order in the dual. Furthermore, it is easy to check that every other embedding of this graph is either not self–dual, or exhibits the same problem. Of course, such examples exist only among graphs of connectivity less than three.

Given a map $G = (V, E, F)$, we can perform the dual construction and regard the superposition of the dual map with the original map as single map, G_2, whose vertex set consists of the vertices V of G, the vertices F^* of G^*, and those points where the edges of G and G^* cross, denoted by $(E \cap E^*)$, so the edges of G_2 are the "half–edges" of G and G^*, and every face of G_2 is a quadrilateral. We will color the half–edges in G_2 originating from G and G^* differently, say red and blue respectively. The following is clear.

PROPOSITION 3. *Every map isomorphism δ from G to G^* induces a unique color reversing map automorphism δ_2 of G_2 and conversely.*

We call a color reversing map automorphism δ_2 of G_2 a *self–duality of G*, and define its *edge permuation*, $\Delta : E \to E$, by $\Delta(e) = \delta(e)^*$. Equivalently, we can consider Δ to be the permutation induced by δ_2 on those vertices of G_2 which are incident to edges of both colors, from which it is clear that $\Delta^2(e) = \delta_2^2(e)$. We note that the self–duality permutation considered in [5] corresponds to the permutation of all the vertices of G^2 induced by δ_2.

THEOREM 1. *Let G be a self–dual map, $\delta_2 : G_2 \to G_2$ a self–duality of G. Then δ_2 is realized by one of the following:*

(1) *a rotation of order 4, the poles being two elements in $E \cap E^*$.*
(2) *a rotation of order 2, the poles lying in the interiors of two quadrilaterals.*
(3) *the antipodal map.*
(4) *a simple reflection with equator intersecting the graph of G_2 only at vertices in $E \cap E^*$.*
(5) *a rotatory reflection of order 4 with poles at two vertices in $E \cap E^*$.*
(6) *a rotatory reflection of order $2k > 2$ which has one pole in V and one pole in F^* and for which δ_2^k is the antipodal map, a rotation, or a reflection.*

Proof. If δ_2 is orientation preserving then the map G_2 can be drawn so that δ_2 is a rotation. Since δ_2 is color reversing, the pole cannot be a vertex of G_2 of the form V or F^* nor the interior of a half edge, so a pole of the rotation is either a vertex in $E \cap E^*$ or in the interior of a quadrilateral. If it is the interior of a quadrilateral, then the rotation must be of order 2, otherwise it must be of order 4.

Suppose δ_2 is orientation reversing then the map can be drawn so that δ_2 is either the antipodal map, a reflection, or a rotatory reflection of order $2k$, $k > 1$. If δ_2 is a color reversing reflection, the objects along the equator can only be vertices of the form $E \cap E^*$ or interior points of quadrilaterals, see Figure 2. If δ_2 is a rotatory

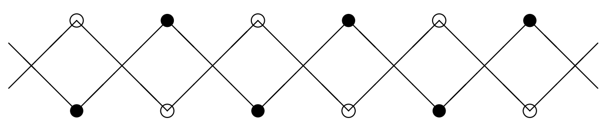


FIGURE 2. The equator of a reflection

reflection of order $2k$, $k > 1$, then δ_2^2 is a color preserving rotation, and hence the pole must be a vertex of G_2. Since the poles are exchanged by δ_2, both vertices will be either in $V \cup F^*$, or both of the form $e \cap e^*$. If the vertex is of the form $e \cap e^*$, then the rotation is of order 2, so δ_2 has order 4. □

We note that these six transformations correspond to the six contructions described in [2].

To show that all these self–dualities are possible, we will exhibit some self–dual maps that will be of use later. The two smallest non–trivial self–dual maps both have two vertices and two edges, and are illustrated in Figure 3. In these figures, the map and the dual map are superimposed, as in G_2, with the vertices of G and G^* distinguished by solid and hollow circles, and the vertices of $E \cap E^*$ indicated by simple crossings. The 1–thorned rose map exhibits self–dualities of types 2 and 3, the map of the single dipole exhibits self–dualities of type 1, 2 and 4. For types 5 and 6 we turn to the wheels and the asteras respectively, see Figures 4a and 4b for examples with a rotatory reflections of order 12 and 6, respectively. In particular, all asteras and wheels are self–dual. Figure 4e is called a 3–thorned rose. We will see later that Figure 4b together with its duality is reducible to that of and Figure 4d or e. Figures 4c and f are examples of *dipole trees*, that is a tree with doubled edges, called *dipoles*. These maps also have a self–duality of type 6. It might be

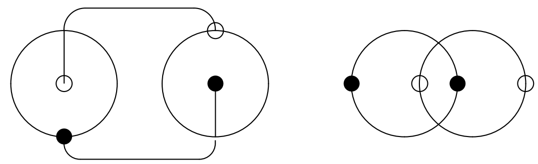
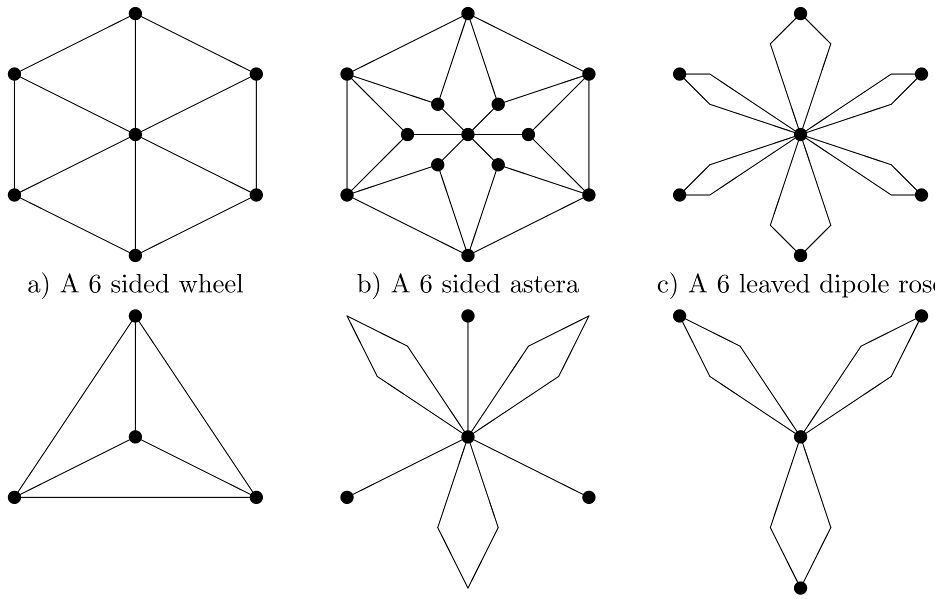


FIGURE 3. The 1-thorned rose and the dipole



a) A 6 sided wheel b) A 6 sided astera c) A 6 leaved dipole rose

d) A 3 sided wheel e) A 3 thorned rose f) A 3 leaved dipole rose

FIGURE 4. Some self–dual maps

less obvious that every dipole tree has a self–dual map and admits a particularly simple self–duality permutation.

THEOREM 2. *A graph Γ has a map which admits a reflective self–dualtity δ_2 such that every edge is fixed under Δ if and only if Γ is a dipole tree.*

Proof. Let $\{e_1, e_2\}$ be a pendant dipole at v and assume by induction that $\Gamma - \{e_1, e_2\}$ has a map such that for each vertex v, v and $\delta_2(v)$ lie on the boundary of the same quadrilateral. This is trivially true for the single dipole. We may then add the dipole $\{e_1, e_2\}$ along the path from v to v^* as in Figure 5.

Conversely, if δ_2 is a reflection and Δ fixes every edge, then $\delta(e) = e^*$ for all e. Consider the star S of a vertex v. Since $\delta(S) = S^*$, and since δ is a map isomorphism, S is simultaniously the star of a vertex and the boundary of a face, and so must be pictured as in Figure 6. In particular, every vertex of valence greater than 2 is a cut vertex, and every cycle is of length 2, and the graph is a dipole tree. □

THEOREM 3. *Let G be a self–dual map with self–duality δ_2. Then the edge permuation Δ has one of the following shapes:*

(1) *two edges fixed, all other cylces of length 4,*

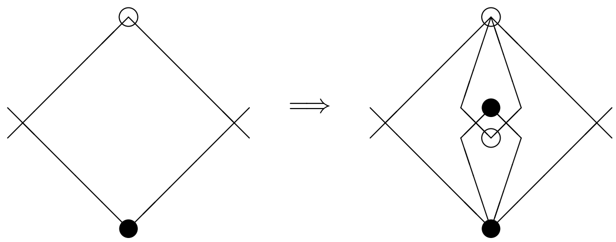


FIGURE 5. Adding a dipole

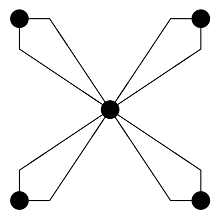


FIGURE 6. A face which is a star

(2) *all cycles of length 2,*
(3) *all non-trivial cycles of length 2, the edges of the trivial cycles comprising a closed path in G,*
(4) *all cycles of length 4 except for one cycle of length 2,*
(5) *all cycles of length $2k > 2$,*
(6) *all cycles of length $2k > 2$ except for a collection of cycles of length k, whose edges comprise a closed path in G.*

Hence we can distinguish the types of non-involutory self–dualities by the shapes of the cycles in the edge permutation Δ.

Proof. One merely needs to interpret the cases of Theorem 1. We note that if all the cycles of Δ are of length $2k$, $k > 1$, then δ_2^k is a rotation or the antipodal map if k is even or odd respectively. Lastly, if the cycles of Δ consist of one transposition and one cycle of length 4, δ_2 must be a rotatory reflection about two vertices of the form $e \cap e^*$, and not a rotatory reflection about two vertices $V \cup F^*$. □

2. Operations on self–dual maps

In this section we show how all self–dual maps may be recursively constructed by describing first how to reduce a given self–duality to a "canonical form" using *edge deletion* and *edge contraction*. These dual processes are natural from the point of view of matroid theory, and have been considered also in [1, 9].

If e is a non-separating edge of G, then let $G - e$, denote the map G with the edge e erased, so that the two faces of G which are incident to e become amalgamated in $G - e$. If e is not a loop, let $G \cdot e$ denote the dual operation, that is, $G \cdot e = (G^* - e^*)^*$. $G \cdot e$ can be described as amalgamating the two endpoints of e by letting e shrink to a point. See Figure 7.

LEMMA 1. *Let $\delta_2 : G_2 \to G_2$ be a self–duality of order $2k$, with the edge permutation Δ defined by $\Delta(e) = \delta(e)^*$. Suppose that $|\{\Delta^i(e)\}| = 2k$. The sets $\{\Delta^{2i}(e)\}$ and $\{\Delta^{2i+1}(e)\}$ do not both separate G.*

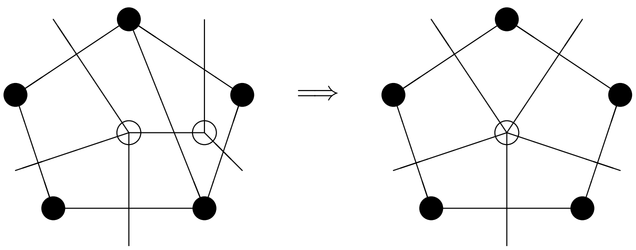


FIGURE 7. An edge deletion

Proof. If δ_2^2 is the identity, then $k = 1$ and if e is a separating edge it follows that e^* is a loop, hence so is $\delta(e)^*$, therefore $\delta(e)^*$ is not a separating edge.

If δ_2^2 is not the identity, then δ_2^2 is a rotation of order k. If $\{\delta_2^{2i}(e)\}$ is a separating set, then $\{\delta_2^{2i}(e)^*\}$ is a cycle C of length k permuted transitively by δ_2^2, and which separates S^2 into two components. If $\{\delta_2^{2i}(e)\}$ also formed a cycle of length k, then $k/2$ of its vertices would be on one side of C, and $k/2$ on the other, see Figure 8, but any rotation that preserves such a structure is at most of order $k/2$. So $\{\delta_2^{2i}(e)^*\}$,

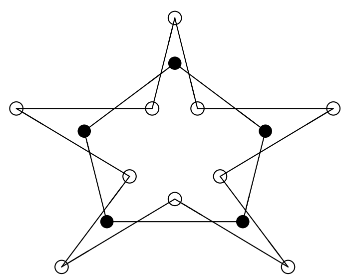


FIGURE 8. Dual separating cycles

and hence $\{\delta_2^{2i+1}(e)\}$ as well, is not a separating set. □

A *duality reduction* is defined as follows. Suppose G has an edge e whose Δ orbit has size $2k$. Using the lemma, suppose without loss of generality that $\{\delta_2^{2i}(e)\}$ is not separating. Then the reduction of G along $\{\delta_2^{2i}(e)\}$ is the map $(G - \{\Delta^{2i}(e)\}) \cdot \{\Delta^{2i+1}(e)\}$. The map isomorphism $\delta : G \to G^*$ induces a map isomorphism between $(G - \{\Delta^{2i}(e)\}) \cdot \{\Delta^{2i+1}(e)\}$ and its dual, since

$$\begin{aligned}
((G - \{\Delta^{2i}(e)\}) \cdot \{\Delta^{2i+1}(e)\})^* &= ((G - \{\delta_2^{2i}(e)\}) \cdot \{\delta_2^{2i+1}(e)^*\})^* \\
&= (G - \{\delta_2^{2i}(e)\})^* - \{\delta_2^{2i+1}(e)\}) \\
&= (G^* \cdot \{\delta_2^{2i}(e)^*\}) - \{\delta_2^{2i+1}(e)\}) \\
&= (G^* - \{\delta_2^{2i+1}(e)\}) \cdot \{\delta_2^{2i}(e)^*\}) \\
&= (\delta(G) - \{\delta(\Delta^{2i}(e))\}) \cdot \{\delta(\Delta^{2i+1}(e))\}).
\end{aligned}$$

The restriction of Δ to the edges of the reduced graph is the edge permutation of the reduced self–duality.

As an example of a duality reduction consider the astera of Figure 9 with δ_2 being a rotatory reflection of order 5 and angle $2\pi/6$, with the deletions and contractions indicated. A different sequence of deletions and contractions will yield the 3–sided

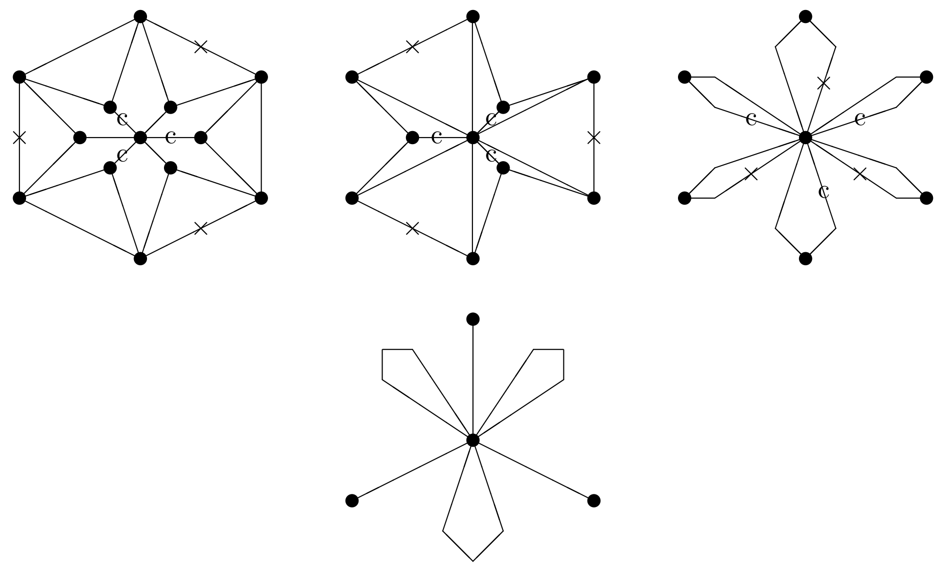


FIGURE 9. A sequence of duality reductions

wheel. If the self–duality is changed to the rotatory reflection of angle $2\pi/3$ then we shall see that the end result must be a dipole tree.

THEOREM 4. *Every self–dual map G with self–duality δ_2 of order $2k$ is reducible to to a self–dual map of the same geometric type, and which is either the map of a thorned rose, a wheel, or a dipole tree.*

Proof. If δ_2 is a rotation of order 4, every edge orbit is of order 4 except for the two fixed edges at the poles. When all the 4–cycles have been removed from the map, only the polar edges remain, and the map is the map of the dipole.

If δ_2 is a rotation of order 2, Δ has only transpositions. When all but one of them have been removed we are left with a graph on two edges, which may be either the dipole or the 1-thorned rose.

If δ_2 is a reflection, then when all the transpositions are removed from Δ the result is a dipole tree by Theorem 2. Similarly, if δ_2 is a rotatory reflection and δ_2^k is a reflection, so k is odd, then when the cycles of length $2k$ are removed from Δ that reflection will fix every edge and the reduced map is again a dipole tree.

If δ_2 is a rotatory reflection and δ_2^k is the antipodal map or a rotation, then every edge cycle in Δ is of order 2k, and if we reduce until there is only one edge cycle, we arrive at the map of either the k–wheel, or the k–thorned rose, since either half the edges must be incident to the vertex at one pole, and so half must be the boundary of the face at the other, or all the edges must be incident to the vertex at the one pole. □

We remark that if our duality reduction allows as well the removal of orbits of pairs of pendant dipoles, as in the proof of Theorem 2, then the dipole trees can be reduced to dipole roses of odd size.

We now show how to reverse the δ–reduction. For what follows it is necessary to fatten each vertex of G_2 in $V \cup F^*$ to a small disk. Let f be a face of G and let

v and w be two vertices, not necessarily distinct, on the boundary of f. Thus there is a path p in G_2 which joins the fat vertices v and w and which only intersects G_2 through the fat vertex f^*. If p can be drawn so that, for all i, $\delta_2^i(p) \cap p$ is either empty or identical with p, then we say G is *δ–expandable by p*, since we can then augment G_2 by adding the paths $\delta_2^i(p)$ as edges, and splitting the fat vertices crossed by $\delta_2^i(p)$. This process, called *δ–expansion*, is illustrated in Figure 10, where

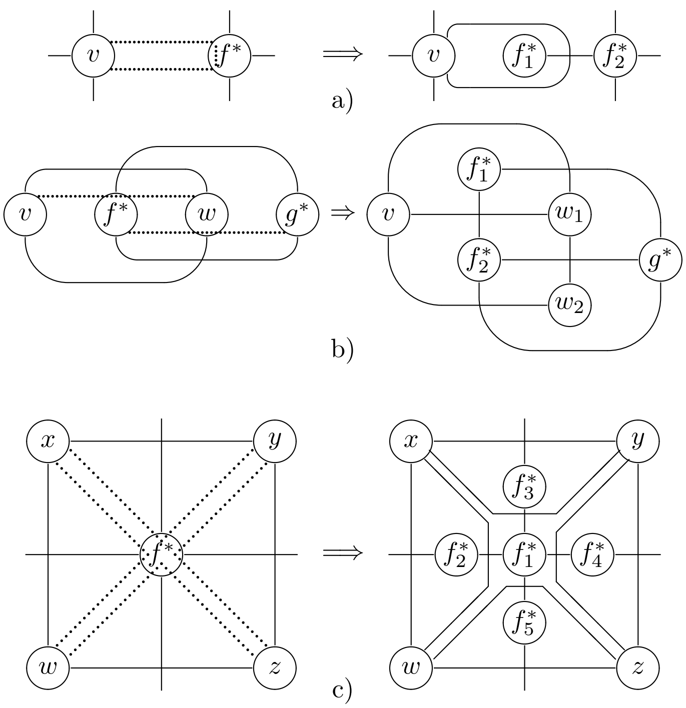


FIGURE 10. Some examples of expansion

a) shows the addition of a loop and pendant edge, b) shows the splitting of vertices near the pole of a rotation about a quadrilateral, and c) shows that the same vertex may be split several times, as at one pole of a rotatory reflection.

We note that not every p can be drawn to satisfy the condition that, for all i, $\delta_2^i(p) \cap p$ is either empty or identical with p, e.g, if p passes over the equator of a reflection.

Theorem 4 can now be reworded to say

THEOREM 5. *Every self–dual map G with self–duality δ_2 of order $2k$ is obtainable by a sequence of δ–expansions of either the map of the dipole, the 1-thorned rose, a wheel, or a dipole tree, depending on the geometric type of the self–duality.*

It is now easy to construct self–dual maps with specified duality properties. For example, Grünbaum [5] asked if there were any self–dual polyhedra which only had self–dualities of order 4. The question was answered by Jendrol̆ [6]. We can answer this by starting with the dipole map, with the self–dualities being the rotations and rotatory reflections of order four on an edge. We kill the extra-symmetry by expanding with some loops, see Figure 11. The ellipses in Figures 11 and 12 indicate

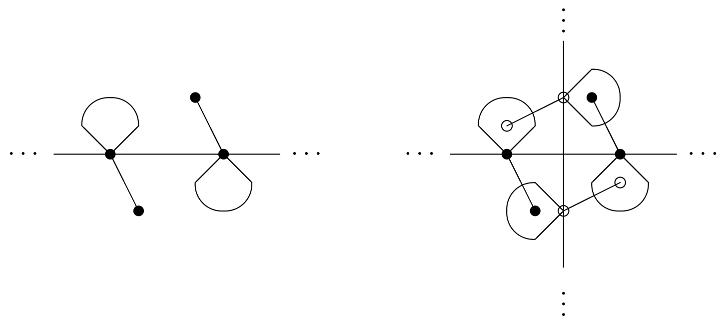


FIGURE 11. All self–dualities of order 4

edges that pass through infinity. If a 3–connected map with the same property is required, we can simply expand the loops into polygons, obtaining for instance the self–dual pair of Figure 12, which is smaller than the polyhedron obtained by

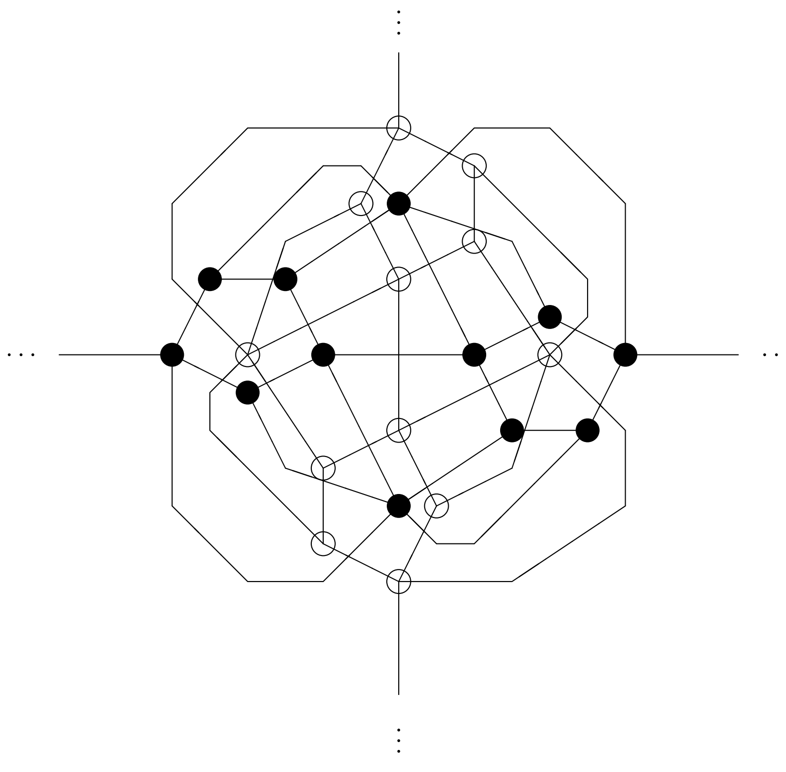


FIGURE 12. A polyhedron with all self–dualities of order 4

Jendroľ.

3. The self–duality groups

Given a self–dual map G, we define the *duality group of of* G, $\mathrm{Dual}(G)$, to be the group of all colored map automorphisms of G_2. If G is a self–dual map, the subgroup $\mathrm{Aut}(G)$ of all color preserving map automorphisms of G_2, which is equivalent to the group of map automorphisms of G, has index 2 in $\mathrm{Dual}(G)$, and the other coset comprises the set of self–dualities of G. Both Dual(G) and Aut(G) belong to the collection of finite groups of isometries of S^2, see [4]. In this view, we have constructed all maps with a given cyclic self–duality group. The possible combinations of Dual(G) and Aut(G) have been catalogued in [11].

Self–dual maps on surfaces of higher genus appear to belong, via covering spaces, to the realm of self–dual tilings, which have been examined extensively in [3].

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