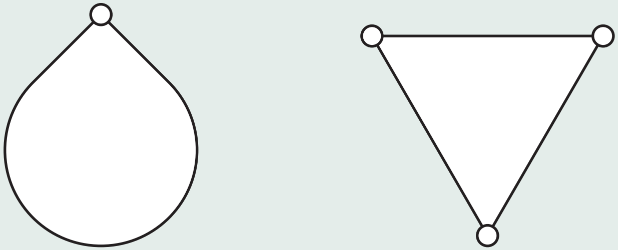
Geometric and Petrie (Self) Duality

Brigitte Servatius

1. Polygonal maps

Start with a set of polygons.



Polygonal maps
Forms of Duality
Self-Duality
Maps are not enough
The Block-Cutpoint Tree
The 3-Block Tree
The Program
Petrie Dual
Orientability
3D Realizability
Ribbon Embeddings
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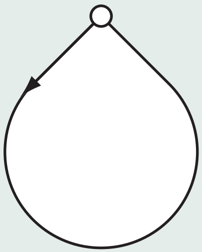
Consider them *oriented*

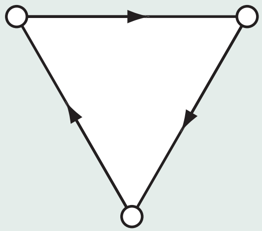
Assume: total # of edges is even.

$\exists$ a perfect matching of edges: m
For each matched edge pair specify

$+$ matched respecting edge orientation
$-$ matched reversing edge orientation

$(m, \pm)$ is a *map* provided the resulting complex is connected.





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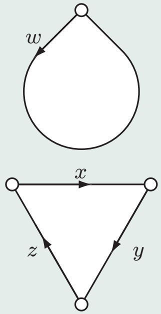
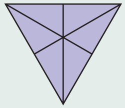
Example

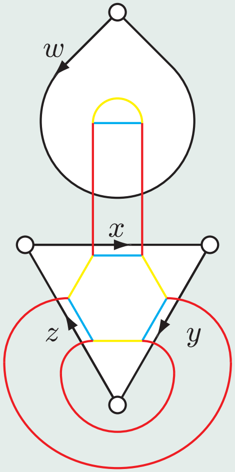


$$(z, y; -)$$
$$(x, w; +)$$

Idea
Barycentric Subdivision



This information is conveniently collected in the flag graph



Flag graph: Edges 3 colored.
01 cycles: faces
12 cycles: vertices
02 cycles: edges
of the resulting complex

$\mathfrak{M}(\tau_0, \tau_1, \tau_2)$

In [2] we find:
"The spirit of the present paper is probably best described by the desire to rid the theory of regular polyhedra of the psychologically motivated crutch of 'membranes' spanning the polygons used as building blocks."

2. Forms of Duality

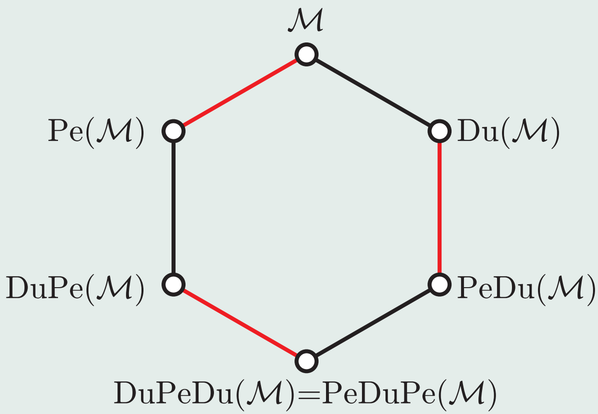
Map $\mathfrak{M} = (\tau_0, \tau_1, \tau_2)$

Dual $\mathrm{Du}(\mathfrak{M}) = (\tau_2, \tau_1, \tau_0))$

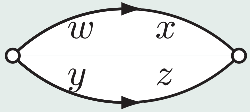
Antipodal Dual $\mathrm{A}(\mathfrak{M}) = (\tau_0, \tau_1, \tau_0\tau_2))$

Petrie Dual $\mathrm{Pe}(\mathfrak{M}) = (\tau_0\tau_2, \tau_1, \tau_0))$

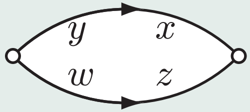
	Du		Pe		Du		Pe		Du		Pe	
τ_0	$\longleftrightarrow$	τ_2	$\longleftrightarrow$	$\tau_0\tau_2$	$\longleftrightarrow$	τ_0	$\longleftrightarrow$	τ_2	$\longleftrightarrow$	$\tau_0\tau_2$	$\longleftrightarrow$	τ_0
τ_1	$\longleftrightarrow$	τ_1	$\longleftrightarrow$	τ_1	$\longleftrightarrow$	τ_1	$\longleftrightarrow$	τ_1	$\longleftrightarrow$	τ_1	$\longleftrightarrow$	τ_1
τ_2	$\longleftrightarrow$	τ_0	$\longleftrightarrow$	τ_0	$\longleftrightarrow$	$\tau_0\tau_2$	$\longleftrightarrow$	$\tau_0\tau_2$	$\longleftrightarrow$	τ_2	$\longleftrightarrow$	τ_2
$\mathcal{V}$	$\longleftrightarrow$	$\mathcal{F}$	$\longleftrightarrow$	$\mathcal{F}$	$\longleftrightarrow$	$\mathcal{V}\tau_0$	$\longleftrightarrow$	$\mathcal{F}\tau_2$	$\longleftrightarrow$	$\mathcal{V}$	$\longleftrightarrow$	$\mathcal{V}$
$\mathcal{E}$	$\longleftrightarrow$	$\mathcal{E}$	$\longleftrightarrow$	$\mathcal{E}\tau_0$	$\longleftrightarrow$	$\mathcal{E}\tau_0$	$\longleftrightarrow$	$\mathcal{E}\tau_2$	$\longleftrightarrow$	$\mathcal{E}\tau_2$	$\longleftrightarrow$	$\mathcal{E}$
$\mathcal{F}$	$\longleftrightarrow$	$\mathcal{V}$	$\longleftrightarrow$	$\mathcal{V}\tau_0$	$\longleftrightarrow$	$\mathcal{F}$	$\longleftrightarrow$	$\mathcal{V}$	$\longleftrightarrow$	$\mathcal{F}\tau_2$	$\longleftrightarrow$	$\mathcal{F}$
G	$\longleftrightarrow$	G^*	$\longleftrightarrow$	G^*	$\longleftrightarrow$	G^p	$\longleftrightarrow$	G^p	$\longleftrightarrow$	G	$\longleftrightarrow$	G
G^*	$\longleftrightarrow$	G	$\longleftrightarrow$	G^p	$\longleftrightarrow$	G^*	$\longleftrightarrow$	G	$\longleftrightarrow$	G^p	$\longleftrightarrow$	G^p



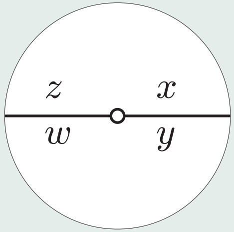
Example



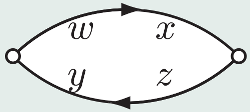
$\mathfrak{M}$ $\tau_0 = (xw)(yz)$ $\tau_1 = (xz)(yw)$ $\tau_2 = (xz)(yw)$



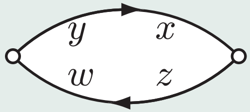
Pe$\mathfrak{M}$ $\tau_0 = (xy)(zw)$ $\tau_1 = (xz)(yw)$ $\tau_2 = (xz)(yw)$



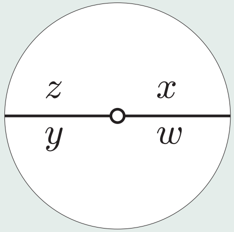
DuPe$\mathfrak{M}$ $\tau_0 = (xz)(yw)$ $\tau_1 = (xz)(yw)$ $\tau_2 = (xy)(zw)$



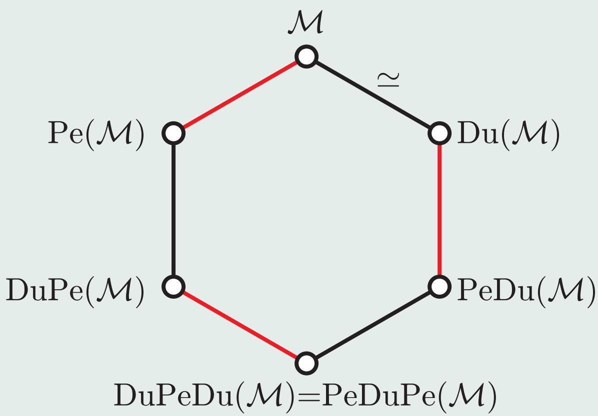
PeDuPe$\mathfrak{M}$ $\tau_0 = (xw)(yz)$ $\tau_1 = (xz)(yw)$ $\tau_2 = (xy)(zw)$

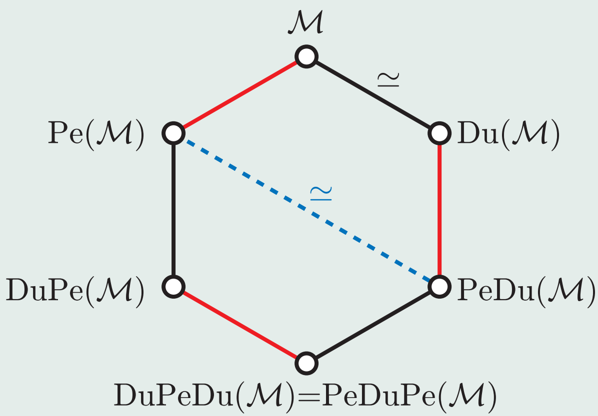


PeDu$\mathfrak{M}$ $\tau_0 = (xy)(zw)$ $\tau_1 = (xz)(yw)$ $\tau_2 = (xw)(yz)$



Du$\mathfrak{M}$ $\tau_0 = (xz)(yw)$ $\tau_1 = (xz)(yw)$ $\tau_2 = (xw)(yz)$


$\mathcal{M}$
$\simeq$
$\mathrm{Pe}(\mathcal{M})$
$\mathrm{Du}(\mathcal{M})$
$\mathrm{DuPe}(\mathcal{M})$
$\mathrm{PeDu}(\mathcal{M})$
$\mathrm{DuPeDu}(\mathcal{M})=\mathrm{PeDuPe}(\mathcal{M})$


$\mathcal{M}$
$\simeq$
Pe($\mathcal{M}$)
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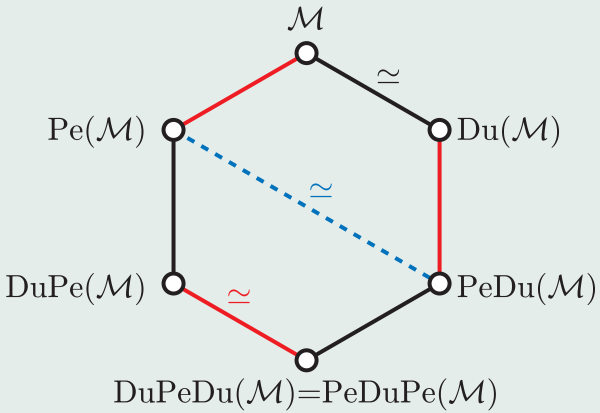
$\simeq$
DuPe($\mathcal{M}$)
PeDu($\mathcal{M}$)
DuPeDu($\mathcal{M}$)=PeDuPe($\mathcal{M}$)


$\mathcal{M}$
$\simeq$
Pe($\mathcal{M}$)
Du($\mathcal{M}$)
$\simeq$
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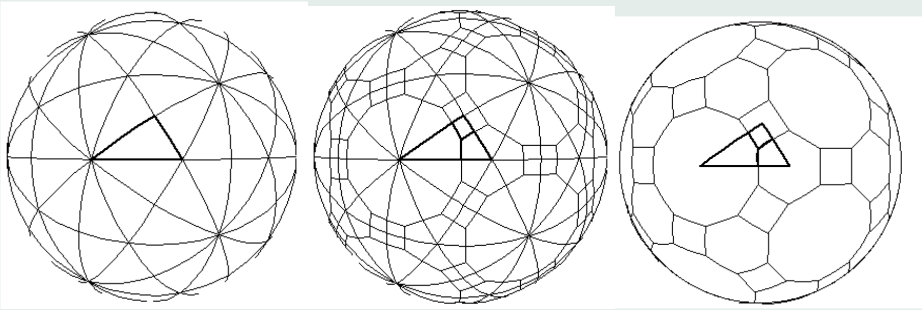
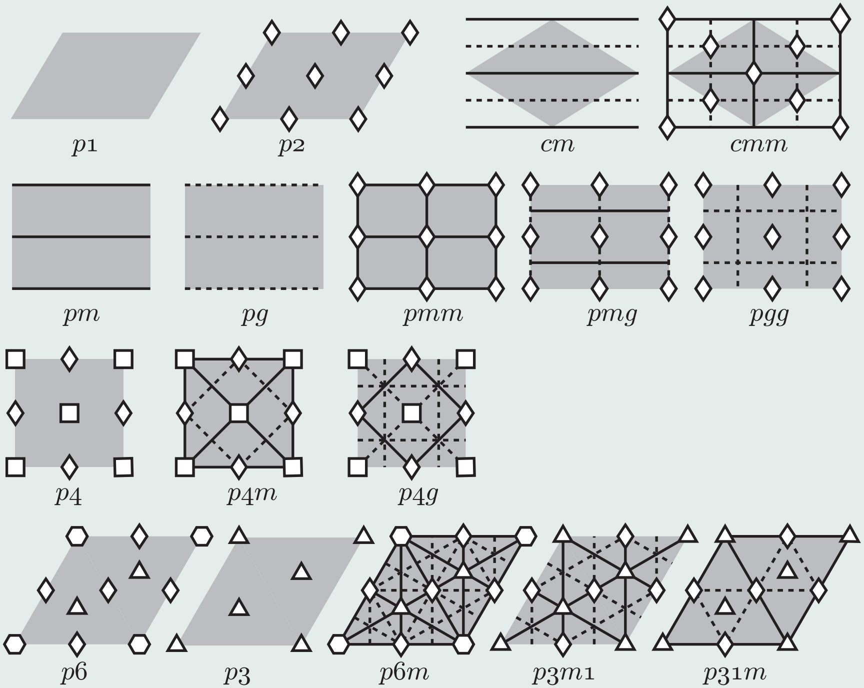
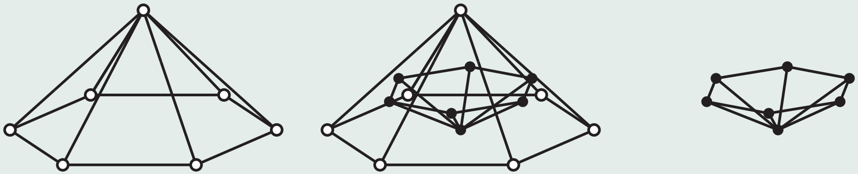
$\simeq$
PeDu($\mathcal{M}$)
DuPeDu($\mathcal{M}$)=PeDuPe($\mathcal{M}$)

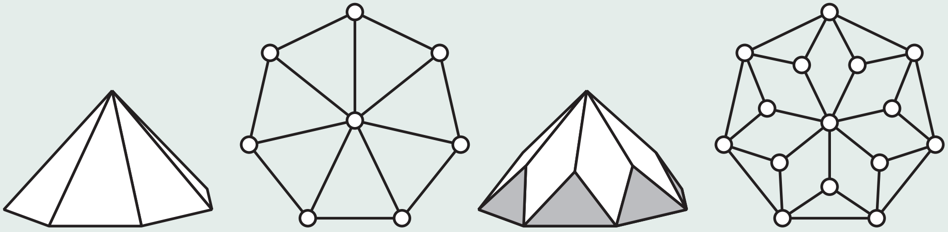
3. Self-Duality



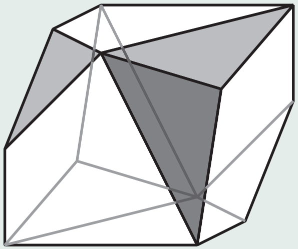

p1
p2
cm
cmm
pm
pg
pmm
pmg
pgg
p4
p4m
p4g
p6
p3
p6m
p3m1
p31m

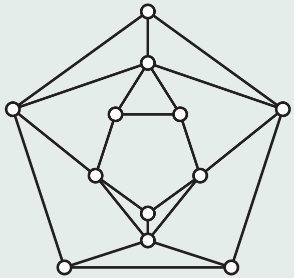
[4], [5]





[3]





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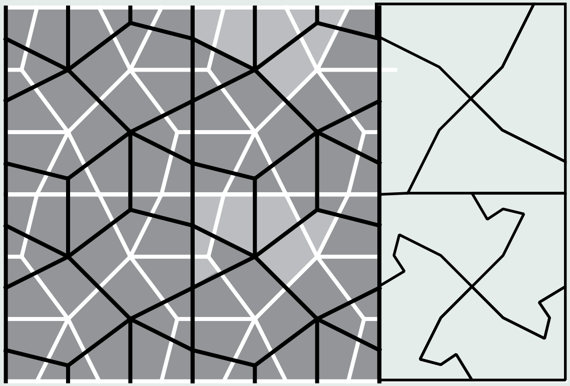
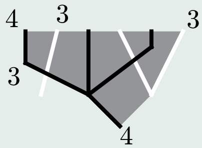
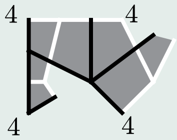
[7]







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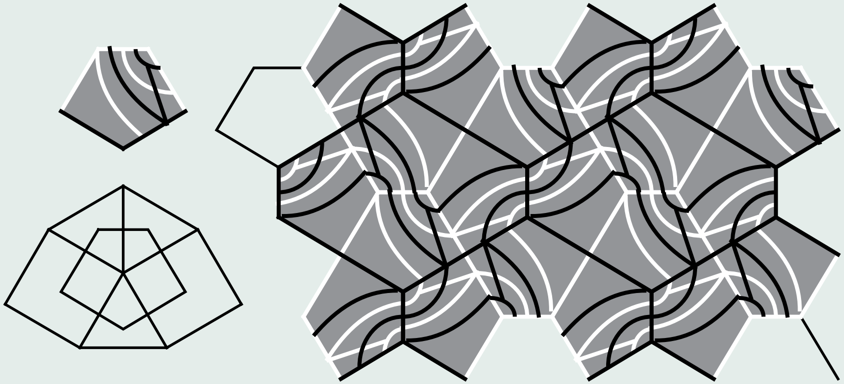
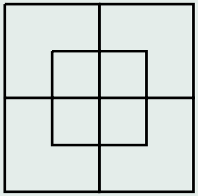
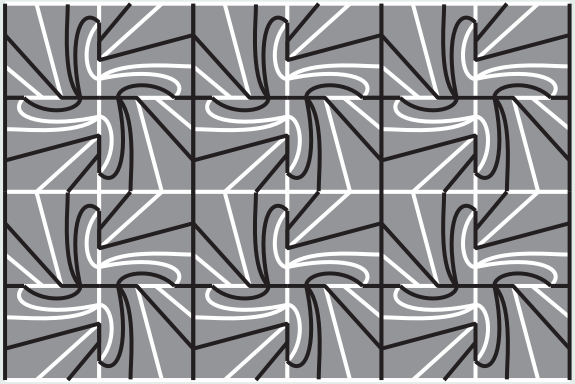
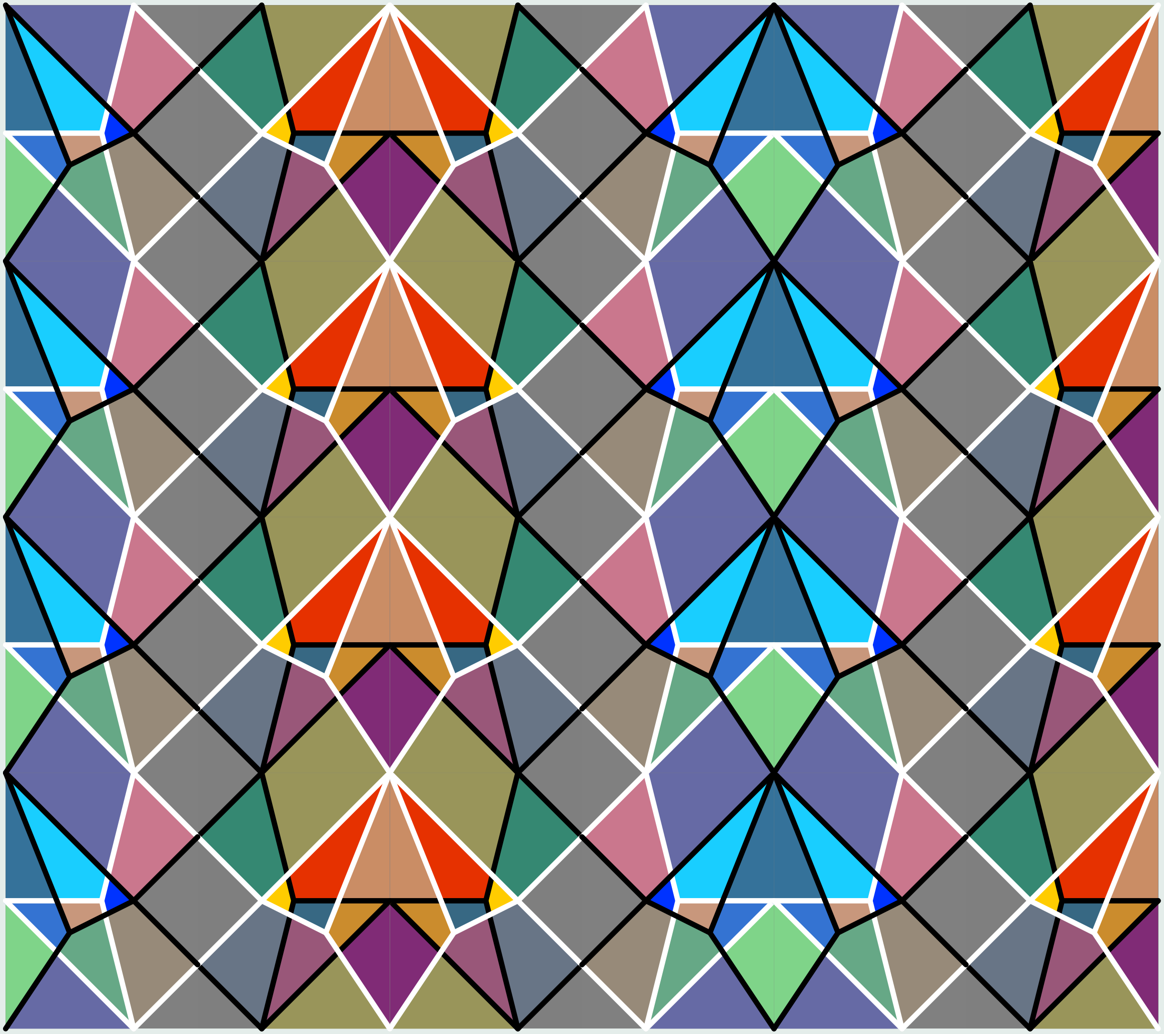


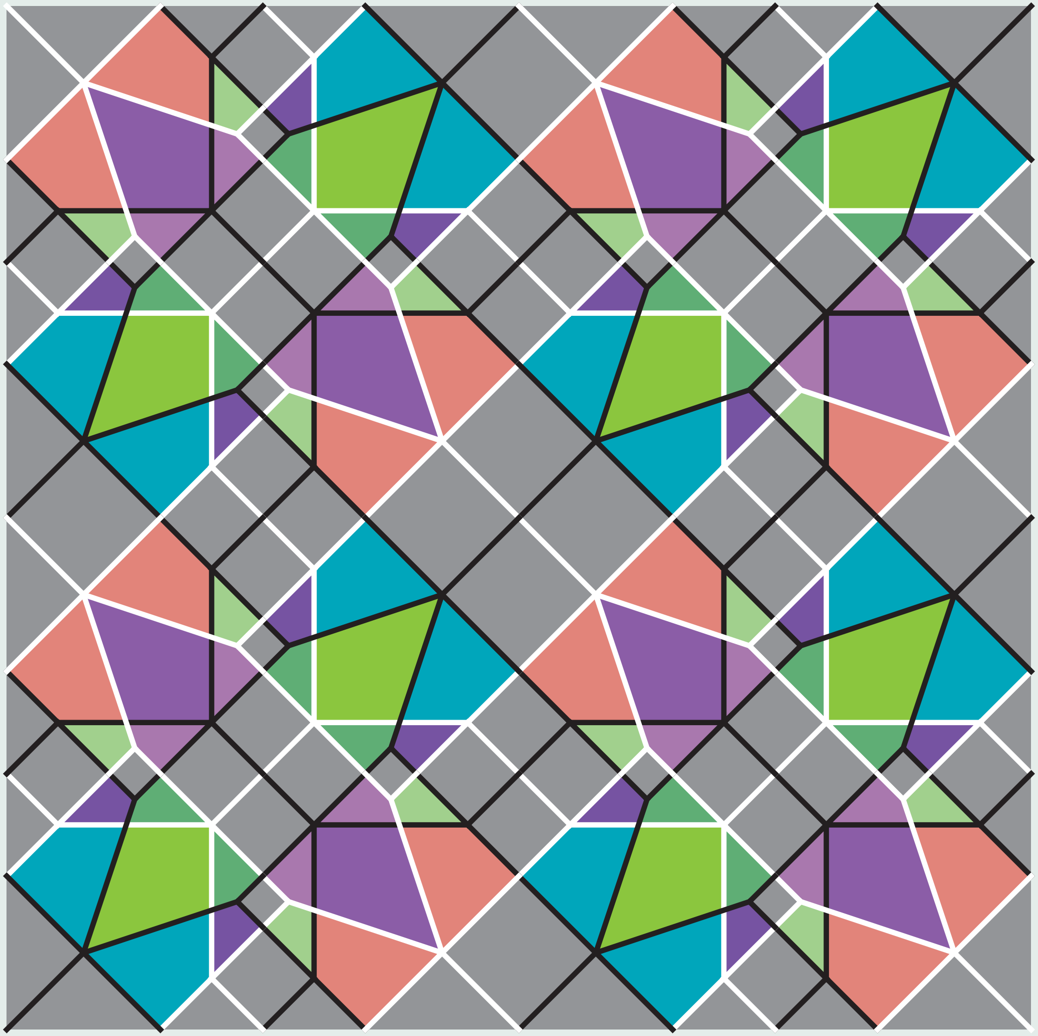
[7]





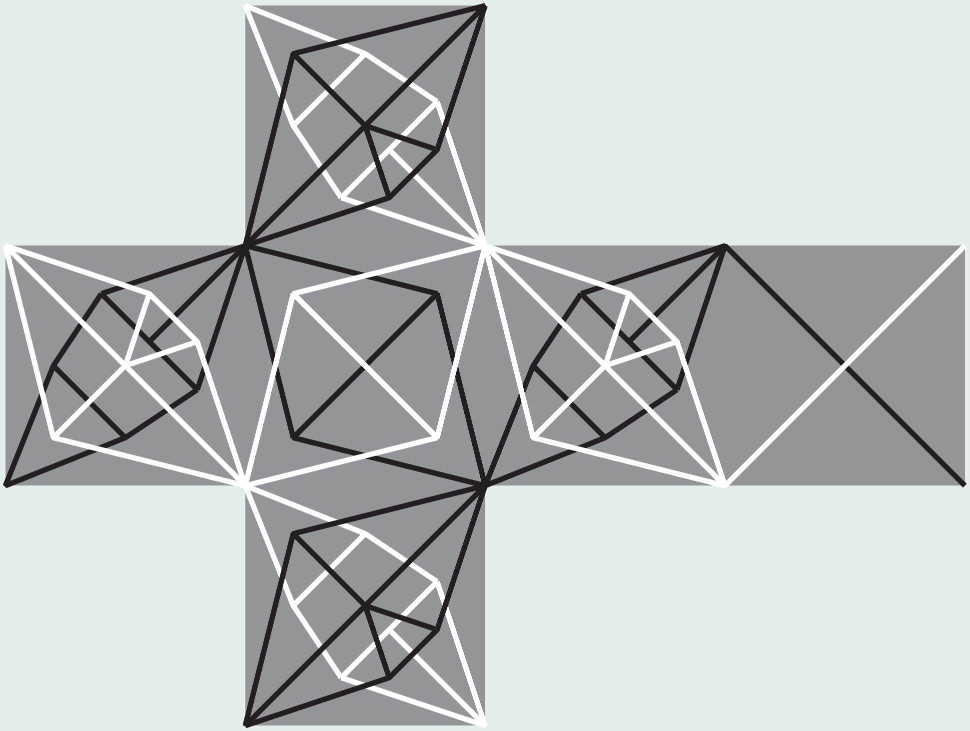


[3]



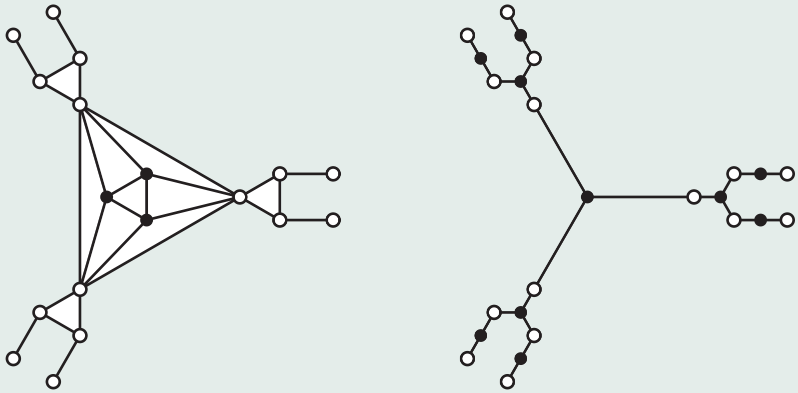
4. Maps are not enough

[6]



A self-dual graph with no corresponding self-dual map.

5. The Block-Cutpoint Tree



For any graph G, the cycle matroid of G is the direct sum over the cycle matroids of the blocks of G:

$$\mathfrak{M}(G) = \sum \mathfrak{M}(G_i)$$

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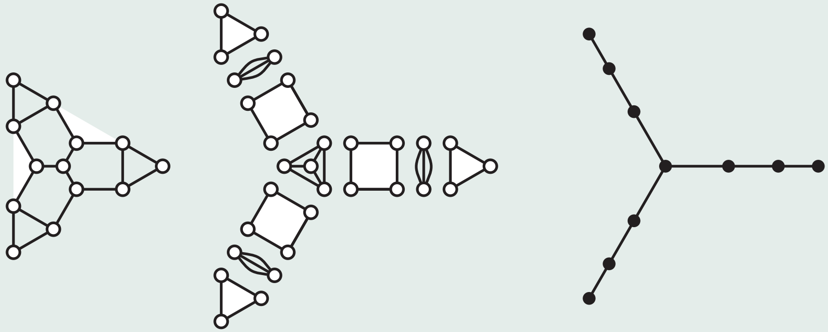
6. The 3-Block Tree



For a 2-connected graph G, the cycle matroid of G is the 2-sum over the cycle matroids of the 3-blocks of G:

$$\mathfrak{M}(G) = \mathfrak{M}(G_0) \bigoplus_{e_1} \mathfrak{M}(G_1) \dots \bigoplus_{e_k} \mathfrak{M}(G_k)$$

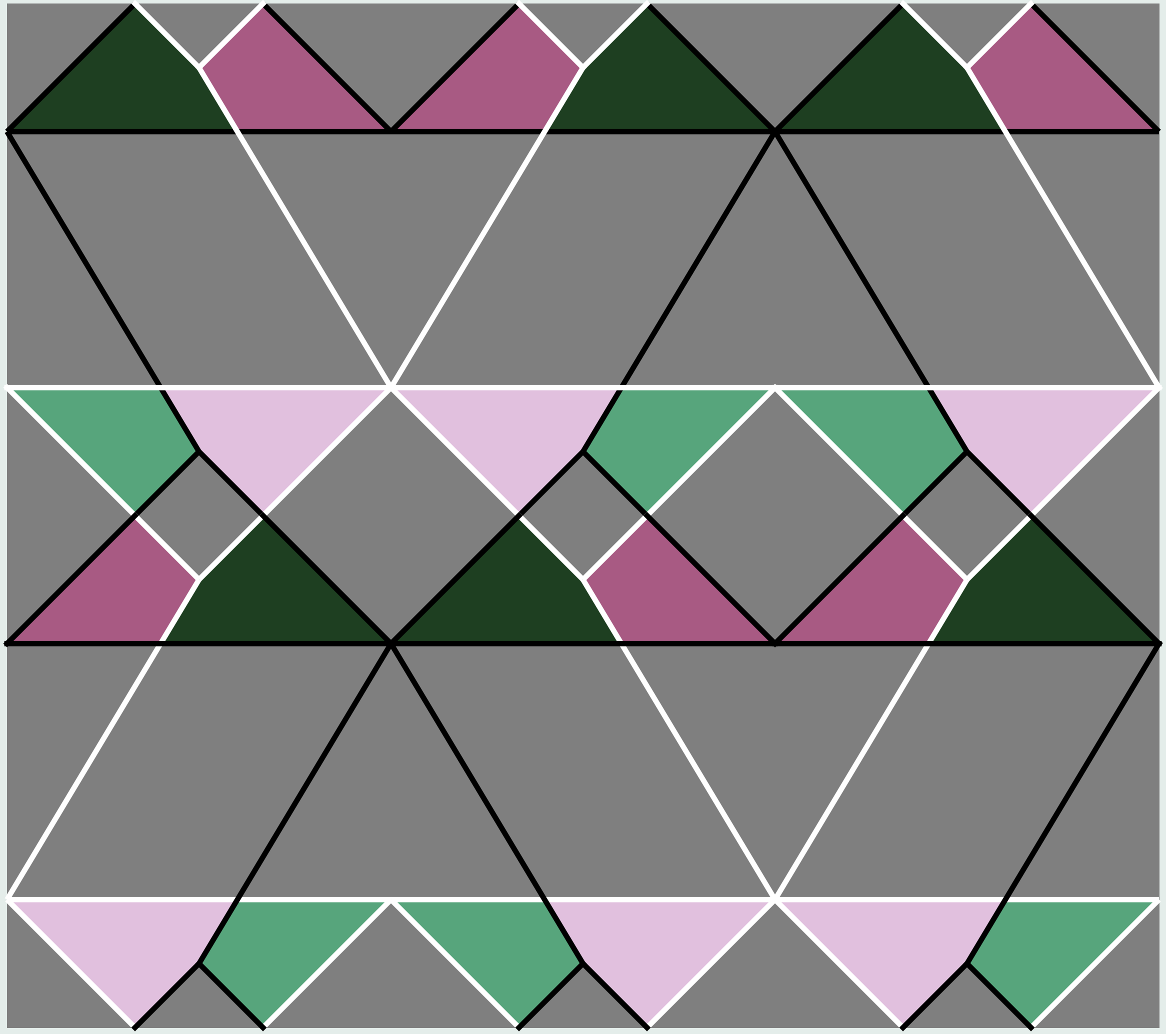
7. The Program

Program: Given a planar graph with whose automorphism group has finitely many vertex orbits.

1. If 3-connected – embed and straighten so that the automorphisms are represented by isometries (euclidean, spherical, or hyperbolic).

   Apply geometric methods.

2. Else if two-connected – form the 3–block tree and use the program on each block, and merge with data on the automorphisms of the tree.
3. Else if connected – from the block-cutpoint tree and apply the program to each block, merging with the tree automorphisms.
4. Else apply program to each connected component, and merge with permutations of isomorphic components.



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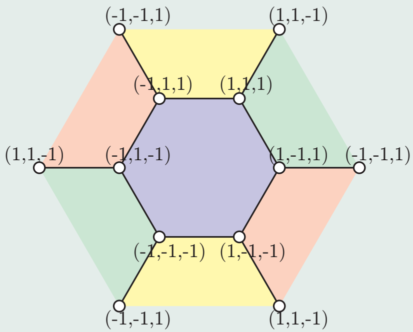
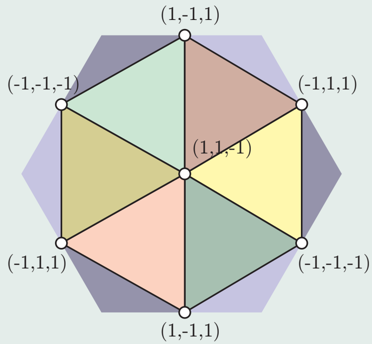
8. Petrie Dual

The Petrie dual of a cube and its geometric dual.
The opposite sides of the hexagon are identified.
Both maps are on the torus.





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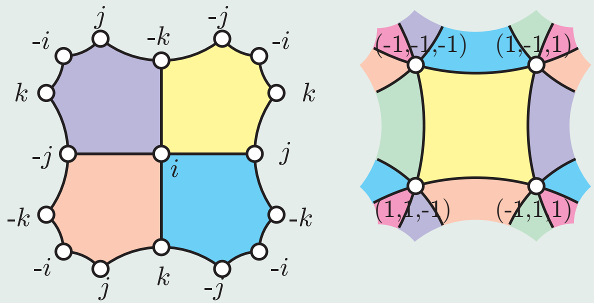

The Petrie dual of an octahedron and its dual.
Both maps are on the four crosscap surface.

9. Orientability

$\langle \tau_1\tau_2, \tau_2\tau_0, \tau_0\tau_1\rangle = \langle \mathcal{V}, \mathcal{E}, \mathcal{F}\rangle$ has
either one flag orbit $\longrightarrow$ non-orientable map
or two flag orbits $\longrightarrow$ orientable map

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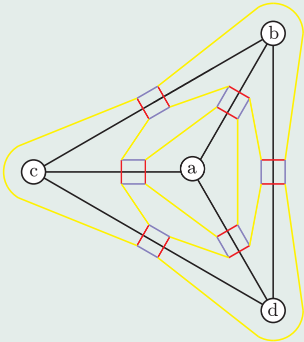
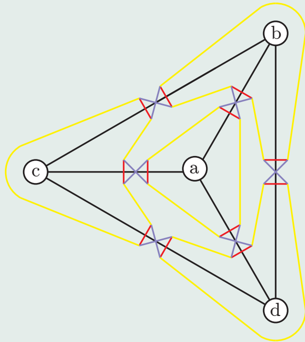
Examples of Petrie duals - Tetrahedron



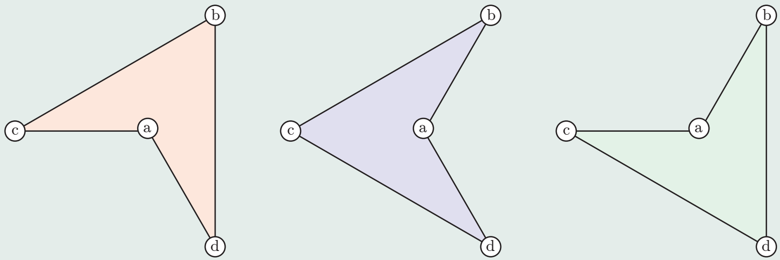
a)



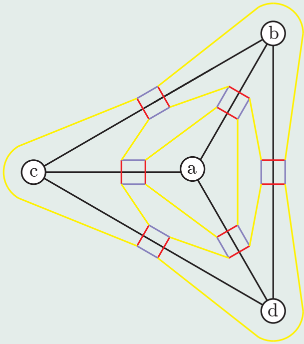
b)

a) The tetrahedron with superimposed flag graph, with the flag matchings τ_0 in blue, τ_1 in yellow and τ_2 in red. b) The flag graph for the Petrie Dual.

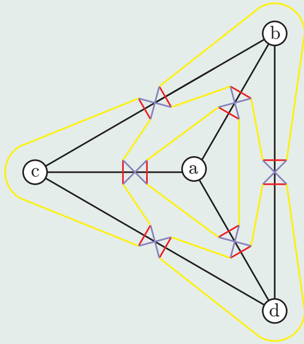
The three Petrie quadrilaterals,



Examples of Petrie duals - Tetrahedron



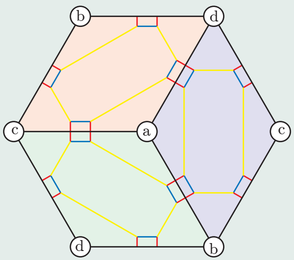
a)



b)

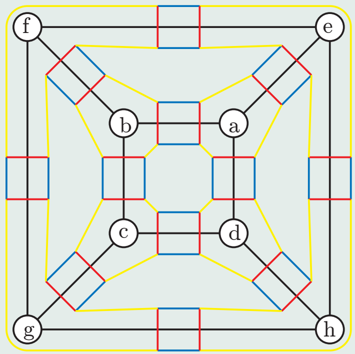
a) The tetrahedron with superimposed flag graph, with the flag matchings τ_0 in blue, τ_1 in yellow and τ_2 in red. b) The flag graph for the Petrie Dual.

A hexagon with opposite sides identified with a twist – non-orientable. (projective plane)

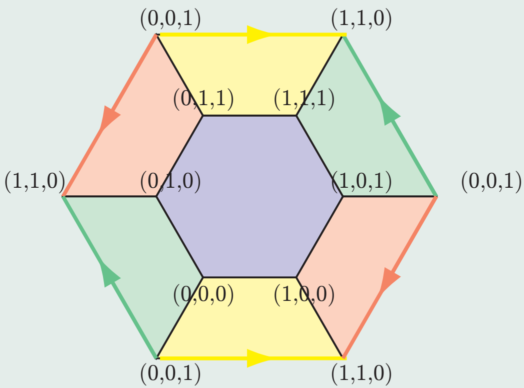


Examples of Petrie duals - Cube

The cube has four hexagonal Petrie cycles

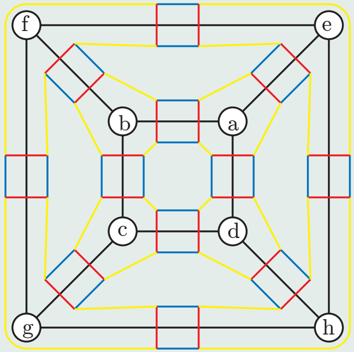


Four Petrie hexagons

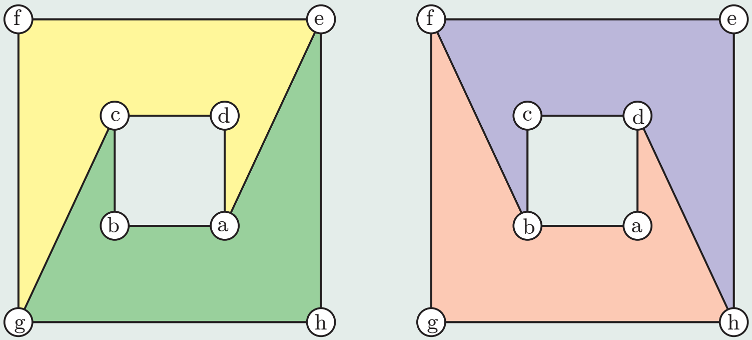


Examples of Petrie duals - Cube

The cube has four hexagonal Petrie cycles



Four Petrie hexagons arranged as the upper and lower half of a torus. which combine in pairs to form two annuli, which in turn join to form a torus – orientable.



Tetrahedron – Petrie dual not orientable

Cube – Petrie dual orientable

Theorem

Let $\mathfrak{M}(\tau_0, \tau_1, \tau_2)$ *be an orientable map.*
$\mathrm{Pe}(\mathfrak{M})$ *is orientable if and only if* $G(\mathcal{V}, \mathcal{E})$ *is bipartite.*

Theorem

Let $\mathfrak{M}(\tau_0, \tau_1, \tau_2)$ *be an non-orientable map.*
$\mathrm{Pe}(\mathfrak{M})$ *is non-orientable if* $G(\mathcal{V}, \mathcal{E})$ *is bipartite.*

The graph of a self-Petrie orientable map must be bipartite.

The graph of a self-Petrie non-orientable map need not be bipartite.

Recall Whitney's Theorem

[8] *A 3-connected planar graph has an essentially unique embedding on the sphere.*

9.1. Consequence

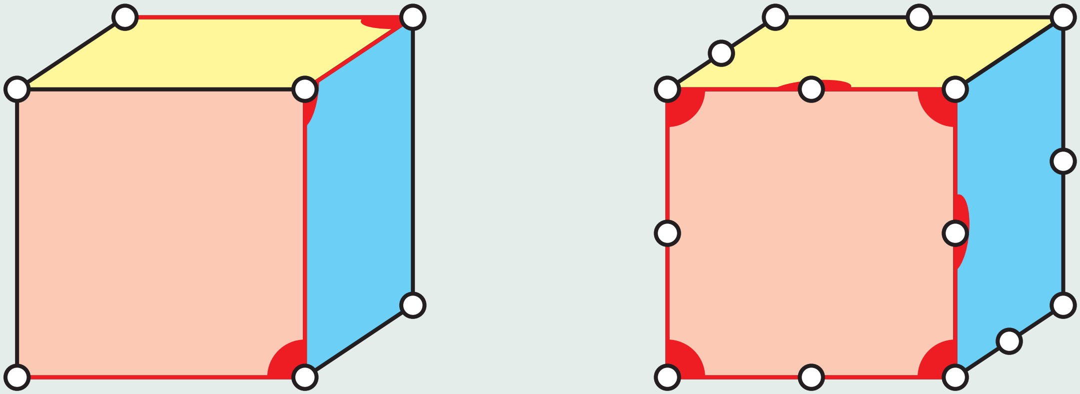
A self-Petrie graph on the sphere can be at most 2-connected.

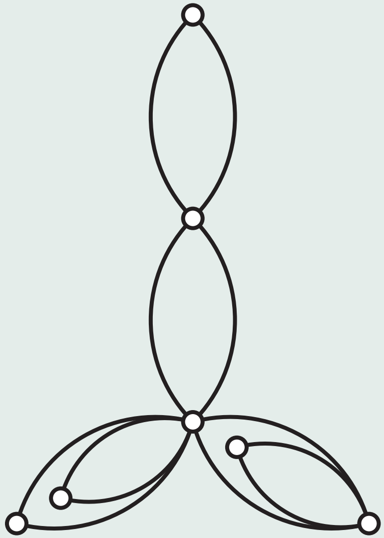
Question

Can there be a map $\mathfrak{M}$ on the sphere which is both is self-dual and self-Petrie ?

Theorem

Every three connected planar graph has a subdivision which is self-Petrie.



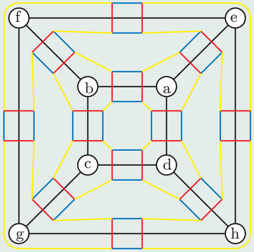


10. 3D Realizability

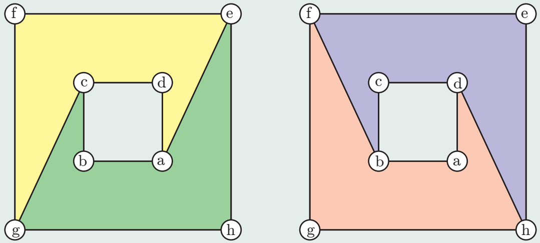
$\mathfrak{M}$ and Du($\mathfrak{M}$) can be drawn on the same surface in a nice way.
There is a 3D representation in the orientable case.

$\mathfrak{M}$ and Pe($\mathfrak{M}$) describe the same graph on two different surfaces.
In the orientable case, can we find two surfaces in 3D such that their intersection is their common graph?

Example: Cube graph



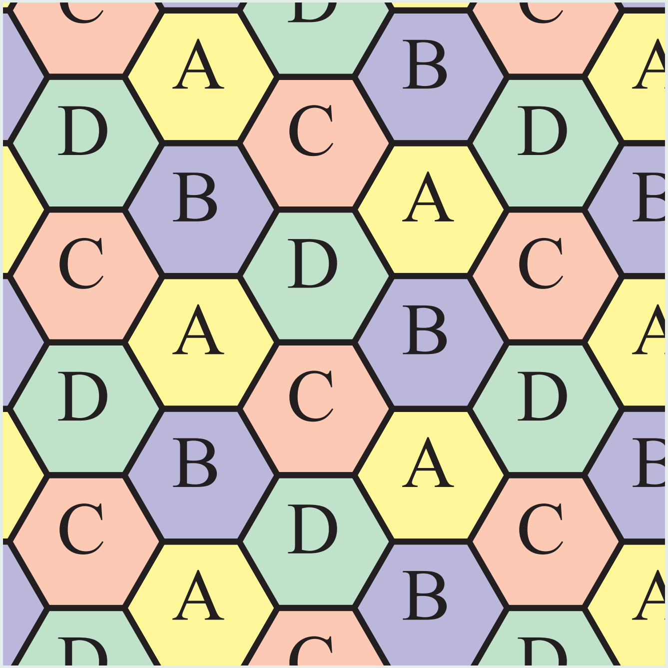
$\mathfrak{M}$
Sphere



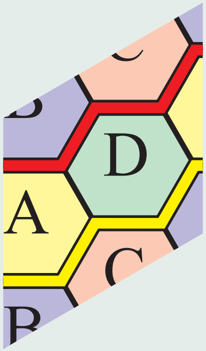
$\mathrm{Pe}(\mathfrak{M})$
Torus

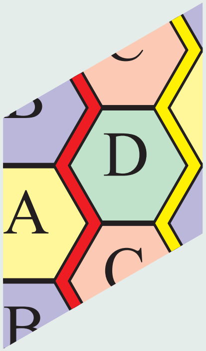
Fact: In any 3D representation of $\mathrm{Pe}(\mathfrak{M})$ at least one pair of quadrilaterals is linked.

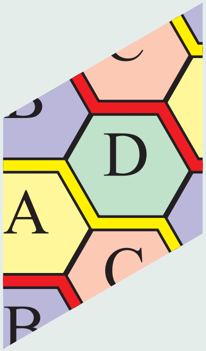
A flat torus in the plane with 4 hexagons.



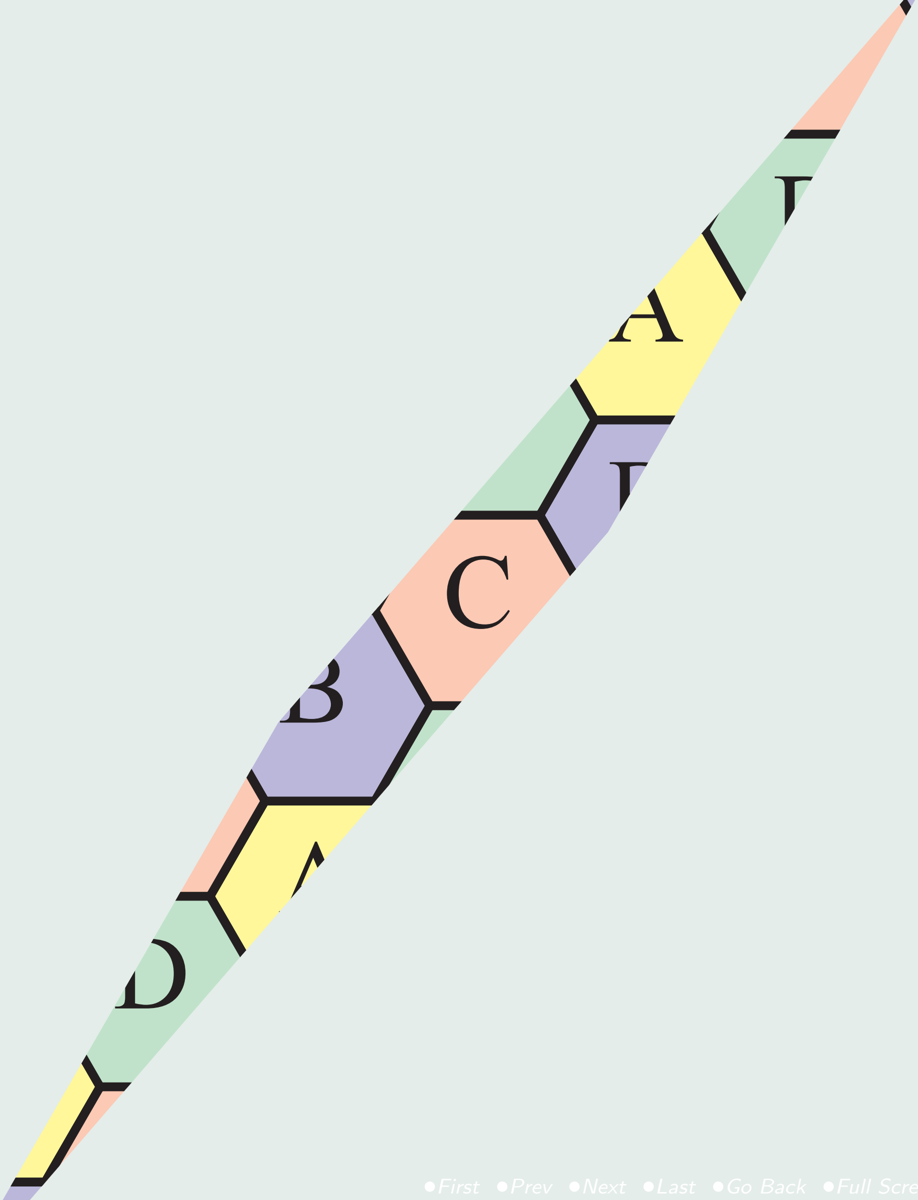
A framing of the torus in 3D

a) 

b) 

c) 

Other Framings


A
C
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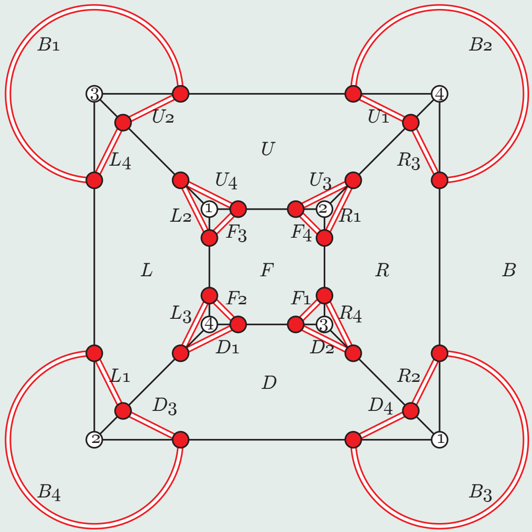
D

11. Ribbon Embeddings

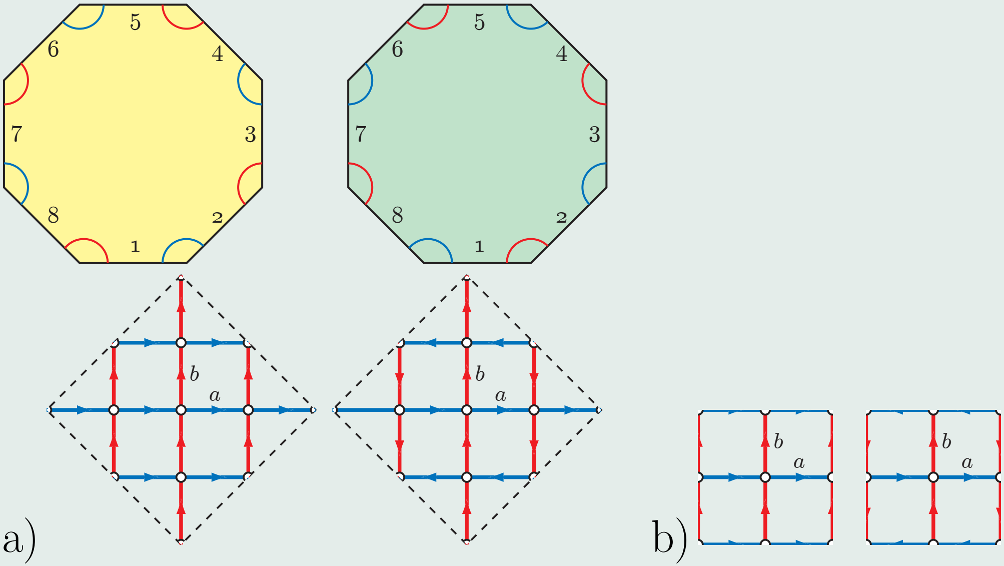
Example: Cube graph

Question: Is there at least a ribbon complex? The cube graph with ribbons sewn on for each four cycle and each Petrie six cycle?



Answer: No (by studying the labeled graph [1] associated to the ribbon complex.)

For self-Petrie-dual maps it is possible to have ribbon complexes in $\mathbb{R}^3$
in fact
These examples are realizable as intersections of 2 spheres or two tori respectively



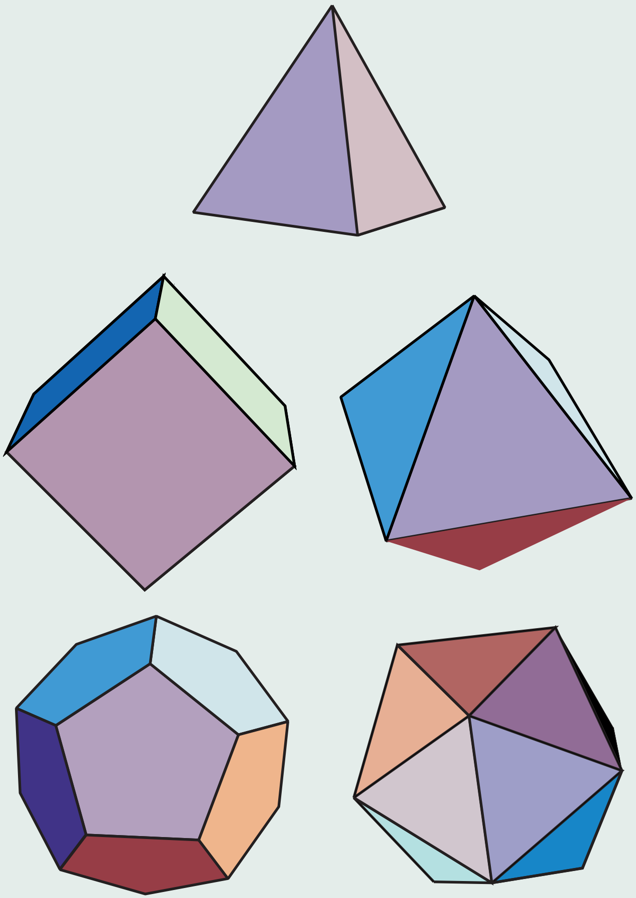
Non-orientable case: Example – Klein Bottle

Ribbon Complex? Yes!

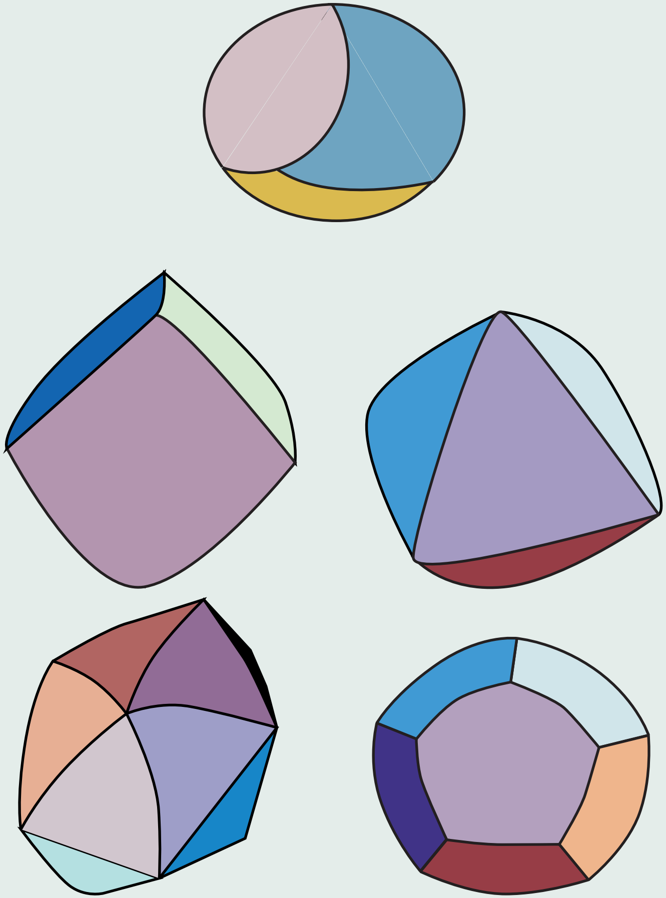
And the moral is:
If we have a 3D representation of a graph drawing on a surface, and then forget about the membranes in the Grünbaum spirit, we can still, from the existence of linked cycles exclude some surfaces by looking just at the skeleton.

12. Regular Objects

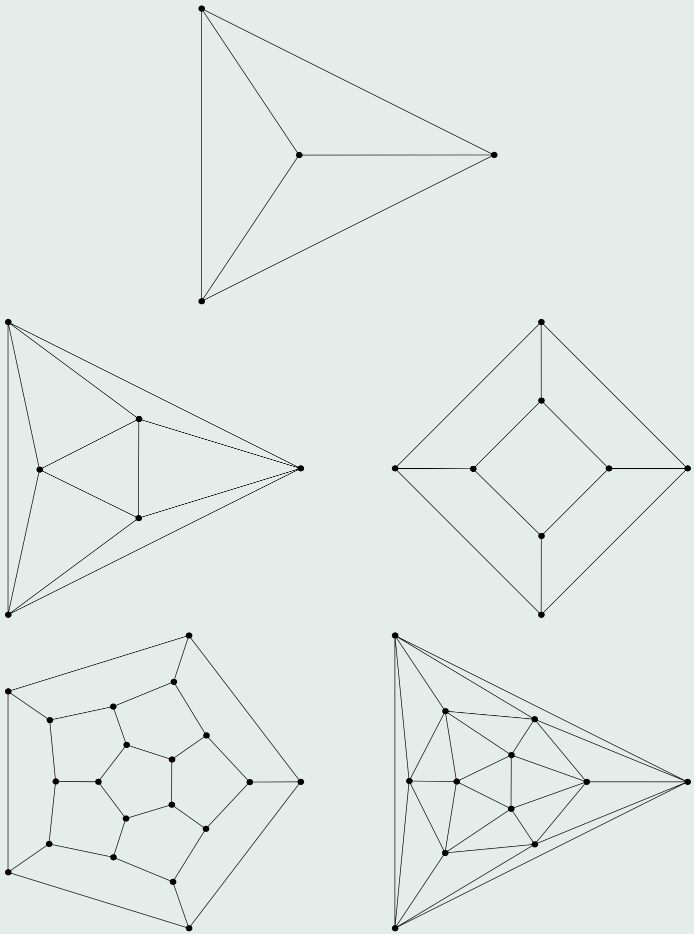
–To a geometer - Platonic Solids



–To a topologist - Platonic Maps



–To a graph theorist - Platonic Graphs



13. Combinatorics

$\mathfrak{M}$ is a regular map on the sphere:
vertices: m-valent
faces: n-valent.

$$2|E| = m|V| = n|F|$$

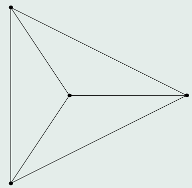
Euler characteristic: $\chi(\mathfrak{M}) = 2$
Thus

$$\frac{2|E|}{m} - |E| + \frac{2|E|}{n} = 2.$$

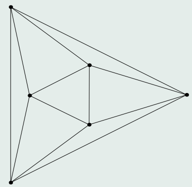
Integer solutions: $(m, n > 1)$

$$1/n + 1/m > 1/2,$$

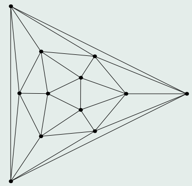
$m = 2$, $n \geq 2$, (an n-cycle separating the sphere into two n-gonal faces)



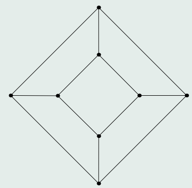
$m = 3$, $m = 3$ (tetrahedron)



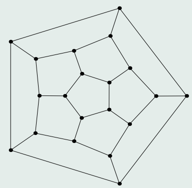
$m = 3$, $n = 4$ (octahedron),



$m = 3$ and $n = 5$ (icosahedron),

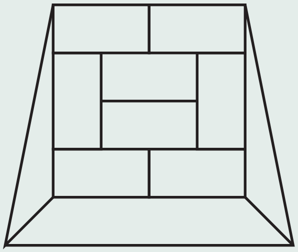


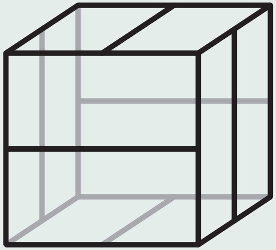
$m = 4$ and $n = 3$ (cube),

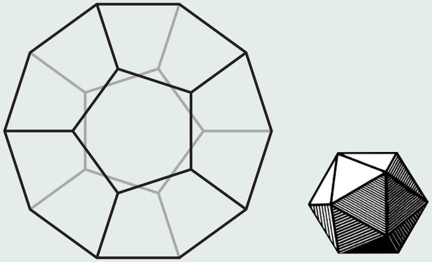


$m = 5$ and $n = 3$ (dodecahedron),
$m \geq 6$ and $n = 2$ (two vertices connected by n edges forming n 2-gons).

14. How many automorphisms?







15. Regular maps

map type

If $\mathfrak{M}$ is a regular map, then its underlying graph or multi-graph is vertex-transitive, edge-transitive and face-transitive. In particular, every face of must have the same number of edges (say k) and every vertex must have the same valency (say m), and every Petrie-cycle must have the same length (say p). In this case we say that $\mathfrak{M}$ has *type* $\{k, m\}_p$.

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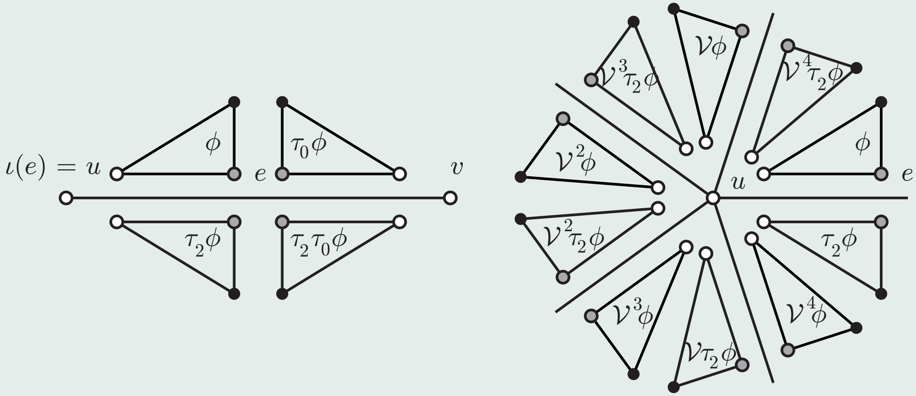


For a regular map $\mathfrak{M}$ of type $\{k, m\}$, the involutions τ_0, τ_1 and τ_2 generate the automorphism group $\mathrm{Aut}(\mathfrak{M})$ and satisfy the full $(2, k, m)$ triangle group relations

$$\tau_0^2 = \tau_1^2 = \tau^2 = (\tau_0\tau_2)^2 = (\tau_1\tau_2)^k = (\tau_0\tau_1)^m = 1.$$

(Standard) Wilson operators

Du and Pe are operators of order 2 on the family of all regular maps. Conder calls the composite operators $DuPe$ and $PeDu$ *triality* operators. They are of order 3.

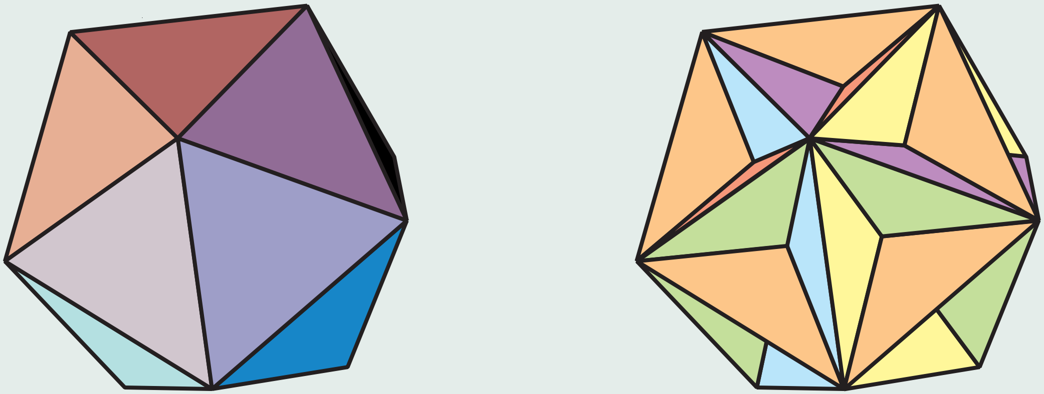
If $\mathfrak{M}$ has type $\{k, m\}_q$ then

- $Du(\mathfrak{M})$ has type $\{m, k\}_q$
- $Pe(\mathfrak{M})$ has type $\{q, m\}_k$
- $DuPe(\mathfrak{M})$ has type $\{q, k\}_m$
- $PeDu(\mathfrak{M})$ has type $\{m, q\}_k$
- $DuPeDu(\mathfrak{M})$ has type $\{k, q\}_m$

so the operators Du and Pe generate a group of order 6 and give all permutations of the three parameters k, m and q.

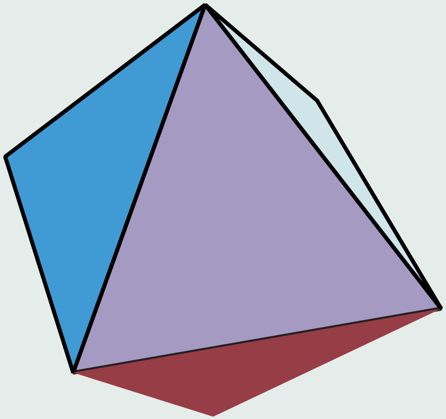
Wilson ‘hole’ operators

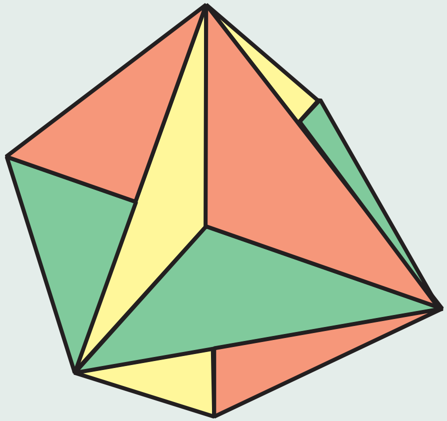
$\mathfrak{M}$ d-valent, $\gcd(e, d) = 1$
$\mathfrak{M}^e$: replacing $\mathcal{V}$ with $\mathcal{V}^e$.



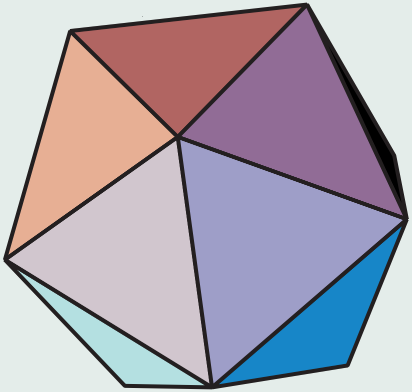
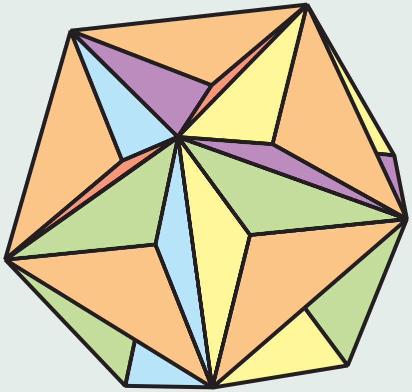
Icosahedron $\mathfrak{I}$ and $\mathfrak{I}^2$.

Coxeter (1937), Wilson (1979), Nedela and Škoviera (1997)





Octahedron $\mathfrak{O}$ and $\mathfrak{O}^2$
$\gcd(4,2) = 2$, $\mathfrak{O}^2$ is not a surface.

If $\mathfrak{M}$ is regular, then so is $\mathfrak{M}^e$, with the same underlying graph, and the same automorphism group. Taking $\mathfrak{M} \to \mathfrak{M}^e$ takes the vertex-stabilizing automorphism $\tau_2\tau_1$ to $(\tau_2\tau_1)^e$, and the canonical generating triple (τ_2, τ_0, τ_1) to $(\tau_2, \tau_0, \tau_2(\tau_2\tau_1)^e)$.

Kaleidoscopic Self-dual and Self-Petrie maps

If the map $\mathfrak{M}$ is self-dual and self-Petrie, then $\mathfrak{M}$ is invariant under all six of the 'standard' Wilson operators (in the group generated by D and P), and we say $\mathfrak{M}$ has trinity symmetry. If the map $\mathfrak{M}$ is isomorphic to all of its power maps $\mathfrak{M}^e$ (for e coprime to the valence), then $\mathfrak{M}$ is invariant under all of the Wilson 'hole' operators, and we say $\mathfrak{M}$ is kaleidoscopic. Wilson [9] conjectured the existence of d-valent kaleidoscopic regular maps which are self-dual and self-Petrie for all even d.
Maps that are regular, kaleidoscopic and have trinity symmetry are in a sense the most highly symmetric of all.

Questions:

Do such maps exist? and for what valences? How large is the group generated by all the map operators?

Examples

- A 2-cycle embedded on the sphere, with type $\{2,2\}_2$
- A regular map of type $\{4,4\}_4$ on the torus
- A regular map of type $\{6,6\}_6$ on a surface of genus 10

Theorem

[Archdeacon, Conder & Siráň (2010)] For every positive integer n, there exists a kaleidoscopic regular map of type $\{2n,2n\}_{2n}$ with trinity symmetry on an orientable surface of genus n^3-2n^2+1.
The existence of this family was conjectured by Steve Wilson as a PhD student in 1976, without the extra kaleidoscopic assumption. The theorem can be proved with the help of some combinatorial group theory.

Sketch proof(Conder).

In the abstract group G with presentation

$$\langle \tau_0, \tau_1, \tau_2 \mid \tau_2^2 = \tau_0^2 = \tau_1^2 = (\tau_2\tau_0)^2 = [(\tau_2\tau_1)^2, (\tau_0\tau_1)^2] = 1\rangle,$$

the elements $u = (\tau_2\tau_1)^2$, $v = (\tau_0\tau_1)^2$ and $w = (\tau_2\tau_0\tau_1)^2$ generate a normal subgroup L of index 8. Also Reidemeister-Schreier theory shows that L is free abelian (of rank 3). Hence for every n, the subgroup L_n generated by n'th powers of elements of L is normal in G, with quotient G/L_n being the automorphism group of a regular map $\mathfrak{M}_n$ of type $\{2n, 2n\}_{2n}$.

Moreover, the normal subgroup L_n is preserved by each of the operators $D : (\tau_2, \tau_0, \tau_1) \to (\tau_0, \tau_2, \tau_1)$, $P : (\tau_2, \tau_0, \tau_1) \to (\tau_2, \tau_2\tau_0, \tau_1)$ and $H_e : (\tau_2, \tau_0, \tau_1) \to (\tau_2, \tau_0, \tau_2(\tau_2\tau_1)^e)$ for e coprime to n, and so the map $\mathfrak{M}_n$ is kaleidoscopic with trinity symmetry. QED

New examples from old

[Archdeacon, Conder & Siráň (2010)] also have a construction that takes an orientable regular kaleidoscopic map of degree d with trinity symmetry, and produces from it a regular kaleidoscopic map of degree dn with trinity symmetry, for every positive integer n.

Question (Conder):

Is there an example of odd valence?

Answer (Conder):

Yes! There's an example of type $\{15, 15\}_1 5$ on a non-orientable surface (of large genus), with automorphism group $A_5 \times A5 \times A5$ [Conder, 2010]

15.1. Open question (Conder):

Are there any with odd prime valence?

How large can the operator group be?

For any regular map with trinity symmetry, the standard Wilson operators Du and Pe generate a group of order 6. For any kaleidoscopic regular map of valence k, the Wilson hole operators H_e generate a group of order $\phi(k)$, where ϕ is Euler's ϕ-function.

15.2. Theorem

[Conder, Kwon, Siráň 2011] Let $\omega(n)$ be the number of prime divisors of n. Then for the kaleidoscopic regular map $\mathfrak{M}_n$ of type $\{2n, 2n\}_{2n}$ with trinity symmetry, the set of all the Wilson operators generates a group of order $6(\phi(2k))^3/2^i$ where $i = \omega(n)$ if $n \not\equiv 0 \bmod 4$, $i = \omega(n)+1$ if $n \equiv 4 \bmod 8$, or $i = \omega(n) + 2$ if $n \equiv 0 \bmod 8$.

Question (Conder):

Is the order of the operator group bounded by the valence?

No!

15.3. Theorem

[Conder, Kwon, Siráň 2011] For all $n > 0$,
$\exists \mathfrak{M}$ of type $\{8;8\}_8$.
$\mathfrak{M}$ kaleidoscopic, regular, self-dual, and self-Petrie dual.
automorphism group of order $128n^{16}$
operator group of order divisible by $48n$.
(Even for fixed valence, the operator group can be arbitrarily large.)

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