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k-Plane Matroids and Whiteley's Flattening Conjectures

(cubocatahedron movie)

(pedestal movie)

Worcester Polytechnic Institute

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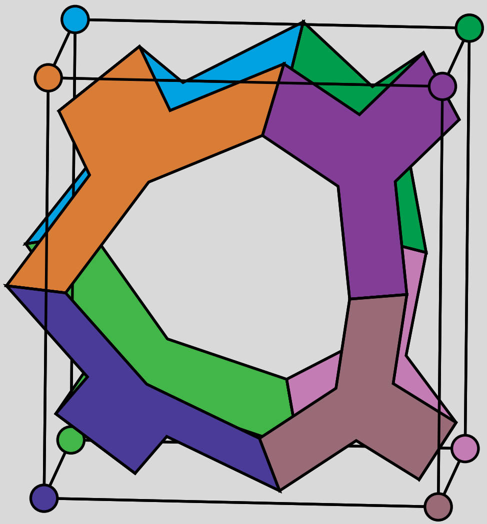

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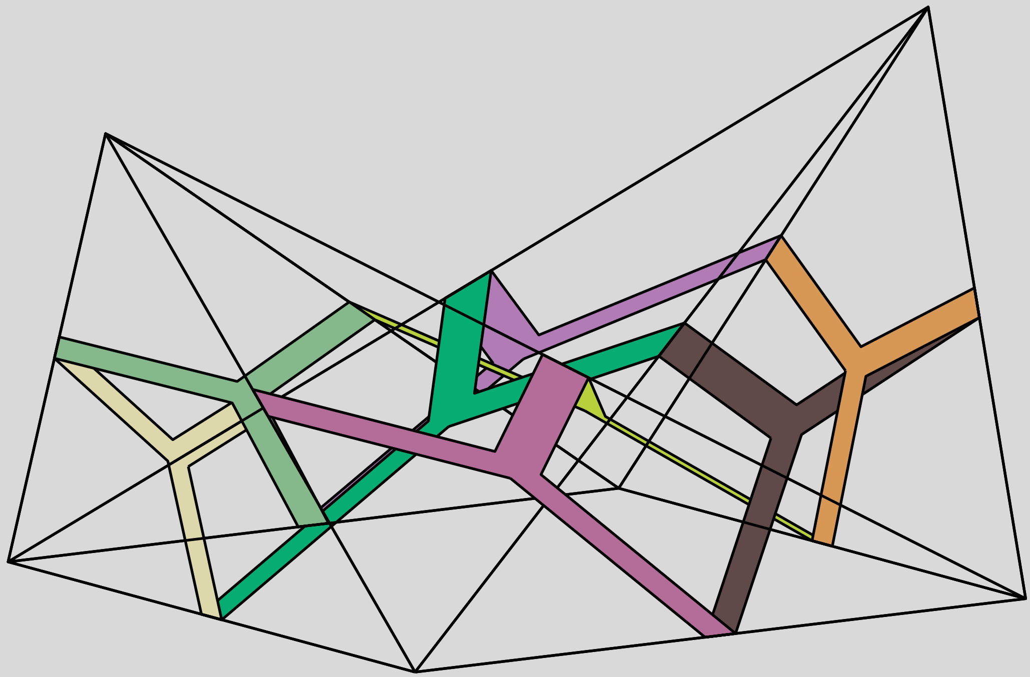

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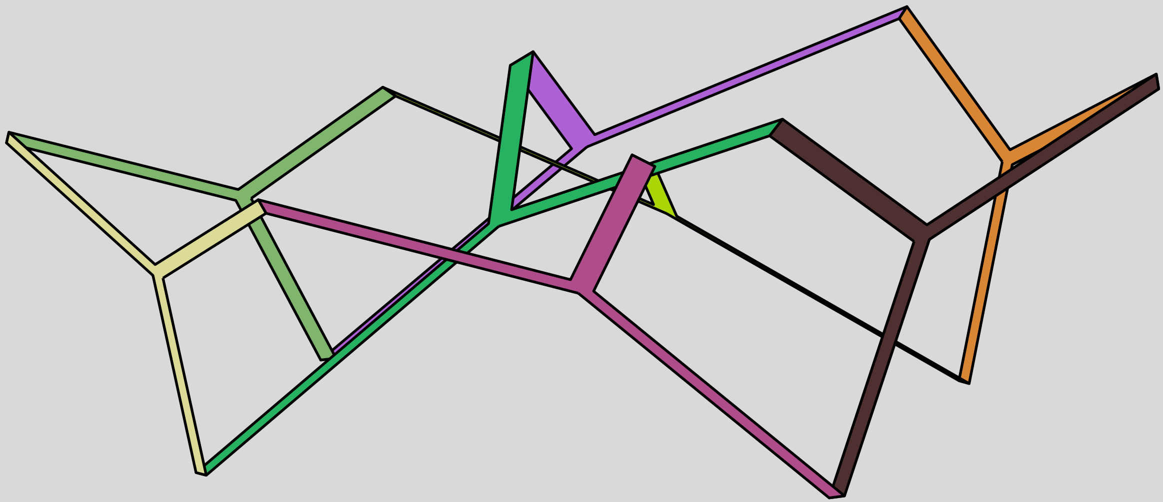

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(Bricard Ocatahedron Movie)

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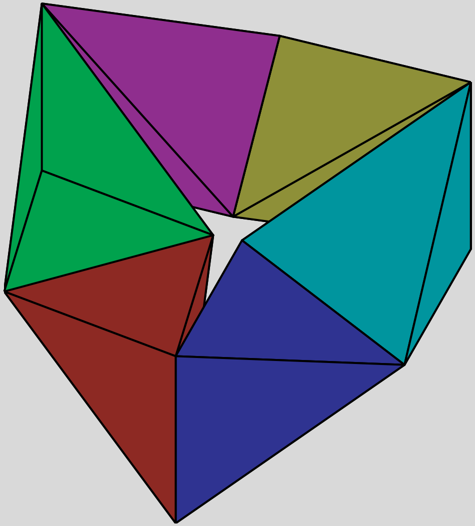

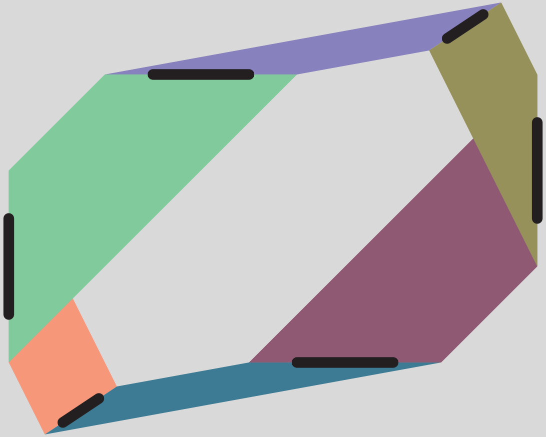

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Theorem 1 *A multigraph G can be realized as an infinitesimally rigid body and hinge framework in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)G$ has $\binom{d+1}{2}$ edge-disjoint spanning trees. (Tay and Whiteley, 1984)*

Recent Advances in the Generic Rigidity of Structures, Tiong-Seng Tay and Walter Whiteley Structural Topology # 9, 1984 Many body and hinge structures are built under additional constraints. For example in architecture flat panels may be used in which all hinges are coplanar. In molecular chemistry, we can model molecules by rigid atoms hinged along the bond lines so that all hinges to an atom are concurrent. This is the natural projective dual for the architectural condition.

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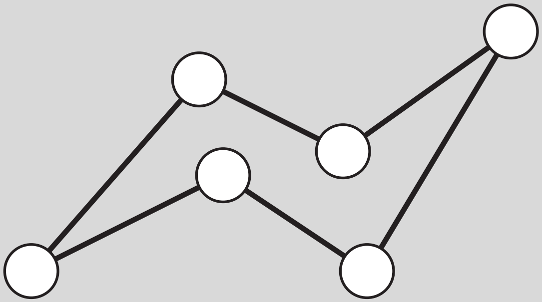
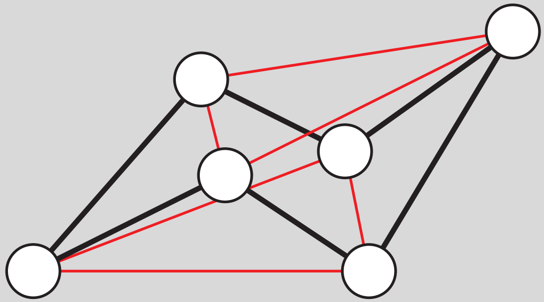
1. The Molecular Conjecture

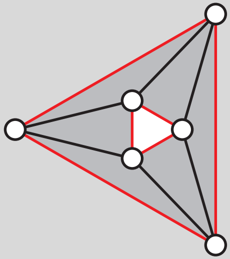
A multigraph is generically rigid for hinged structures in n-space if and only if it is generically rigid for hinged structures in n-space with all hinges of body v_i in a hyperplane H_i of the space.

Jackson-Jordán 2009

Conjecture 1 *Let $G(V, E)$ be a graph with minimum vertex degree at least two. Then $r(G^2) = 3|V| - 6 - def(G)$.*







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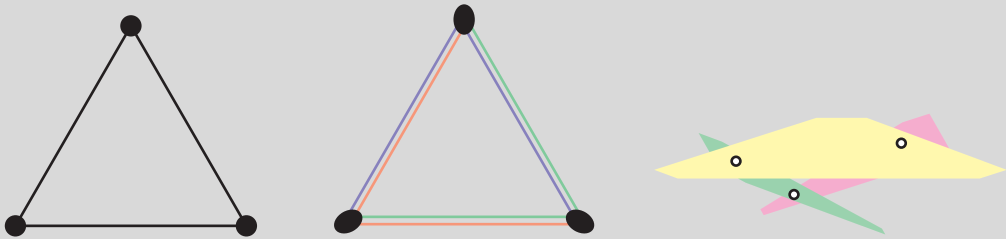
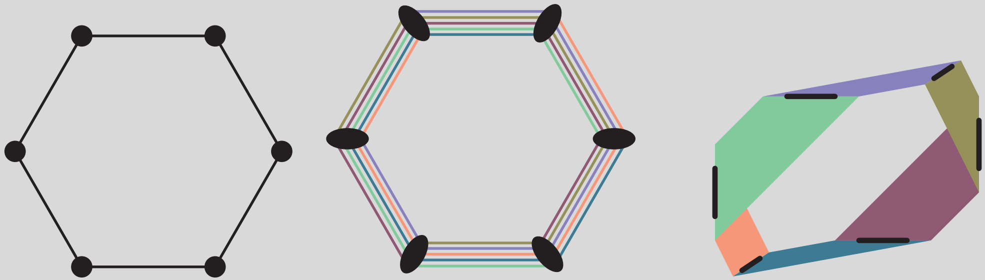
dim 2



dim 3



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Theorem 2 *Katoh-Tanigawa 2009: Let* $G = (V, E)$, $|V| \geq 2$, $def(G) = k$, $k \geq 0$. *Then there exists a (non-parallel if* G *is simple) panel and hinge realization* $(G, \boldsymbol{p})$ *in* $\mathbb{R}^d$ *satisfying*

$$rank R(G, \boldsymbol{p}) = \binom{d+1}{2}(|V| - 1) - k$$

.

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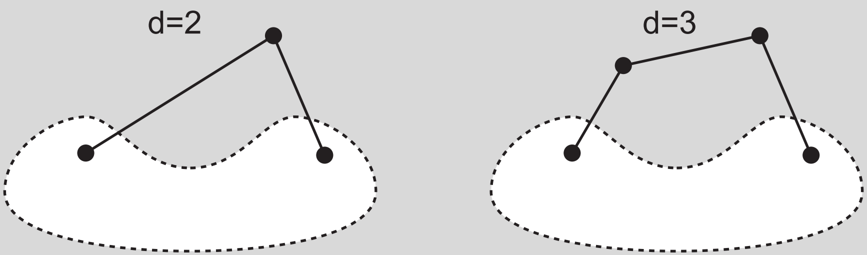
2. Katoh-Tanigawa Proof

2.1. Structure of edge minimal G with $def(G) = k$

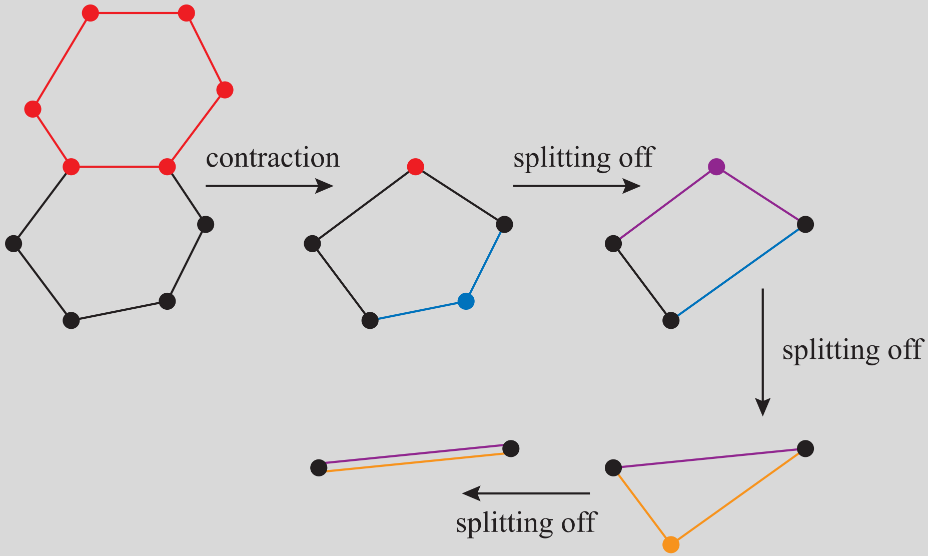
- Not edge 3-connected
- Subgraphs are minimal
- If edge 2-connected, then
  - either G contains a proper full-rank subgraph,
  - or, if not, then it is either a cycle of at most d vertices or contains a chain of length at least d.



2.2. Inductive construction of edge minimal G with $def(G) = 0$

There exists a sequence $G = G_1, G_2, \ldots, G_m$, G_i minimal w.r.t. $def(G_i) = 0$, such that

- G_m is a 2-gon
- G_{i+1} is obtained from G_i by
  - splitting off at a vertex of degree 2
  - contraction of a proper full rank subgraph



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3. The rigidity matrix for body and hinge structures

The set of infinitesimal motions of a body-and-hinge framework $(G, \mathbf{p})$ is the nullspace of the *rigidity matrix* $R(G, \mathbf{p})$. For $G = (V, E)$, and $\mathbf{p} : V \to \mathbb{R}^d$

R has $|E| \left(\binom{d+1}{2} - 1\right)$ rows and $|V|\binom{d+1}{2}$ columns

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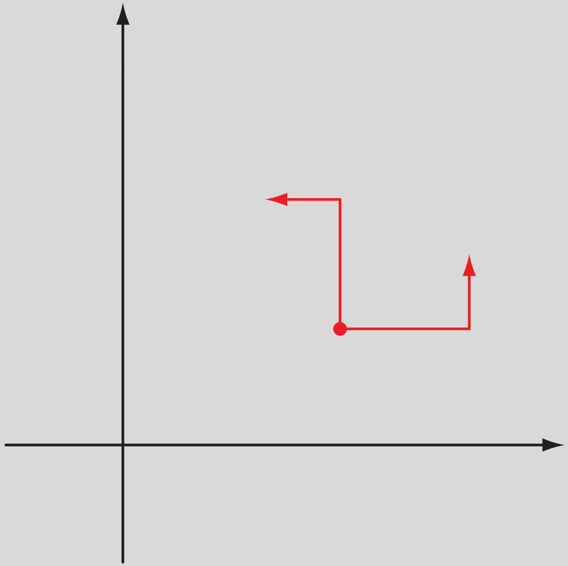
$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} = \begin{bmatrix} \omega b \\ -\omega a \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



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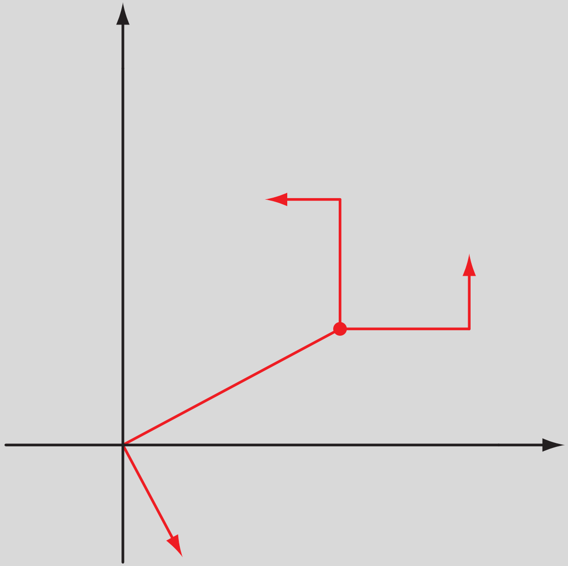
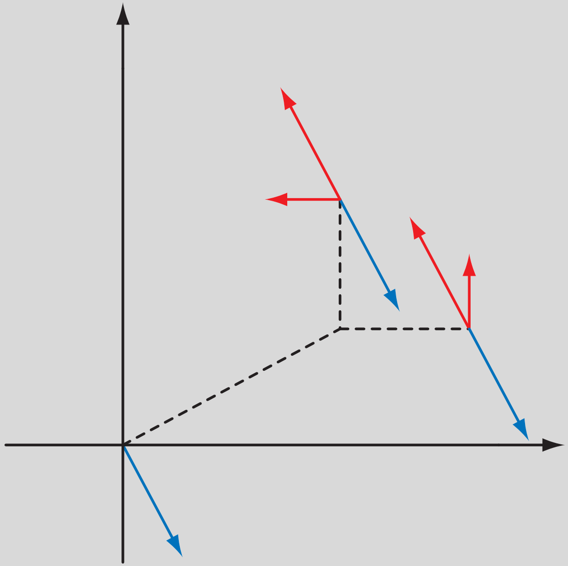
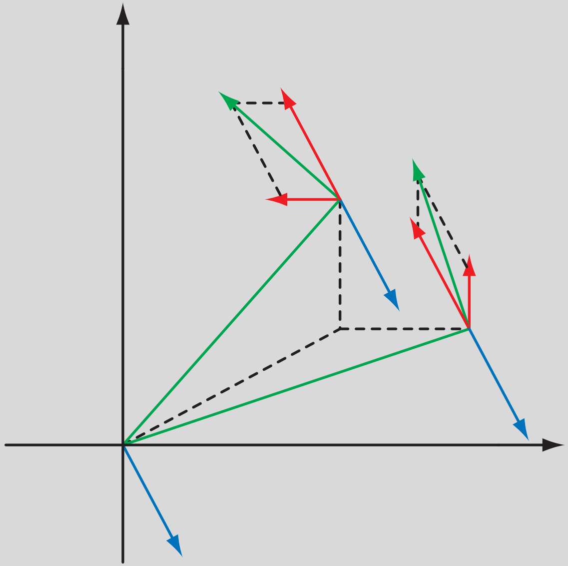


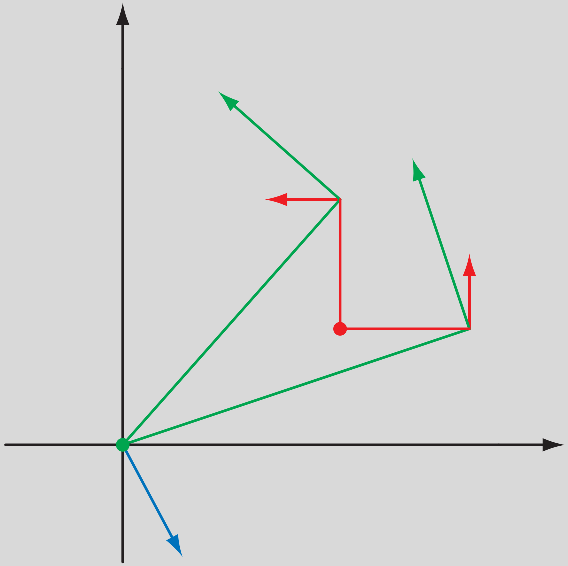
$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} = \begin{bmatrix} \omega b \\ -\omega a \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} = \begin{bmatrix} \omega b \\ -\omega a \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} = \begin{bmatrix} \omega b \\ -\omega a \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



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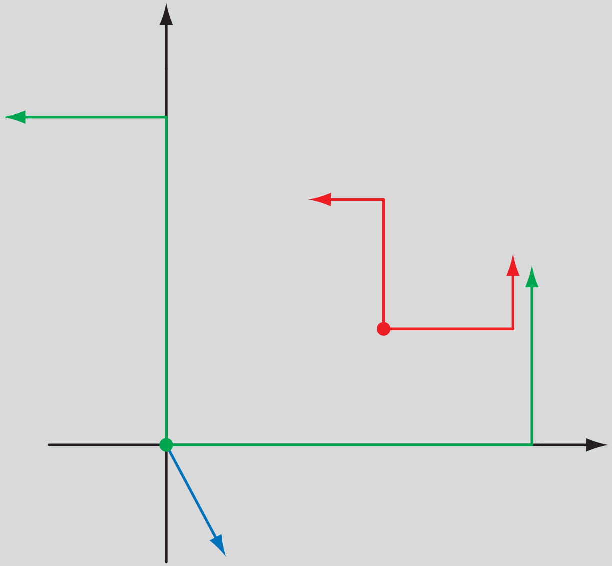
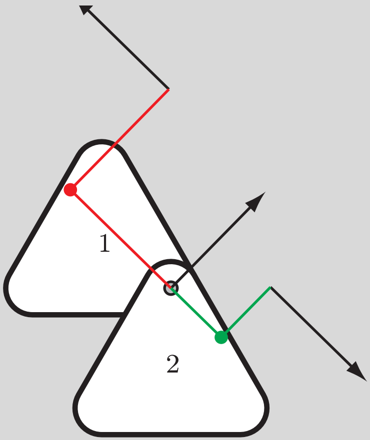


$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_2 \\ \omega_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



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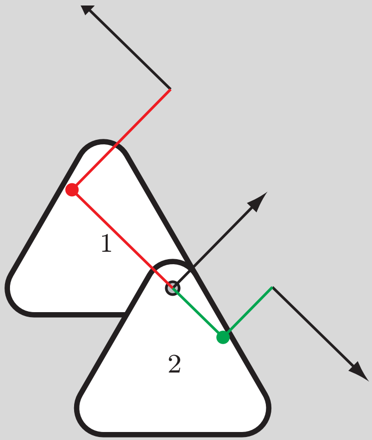
$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_2 \\ \omega_2 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -q & -1 & 0 & q \\ 0 & 1 & p & 0 & -1 & -p \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ \omega_1 \\ a_2 \\ b_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



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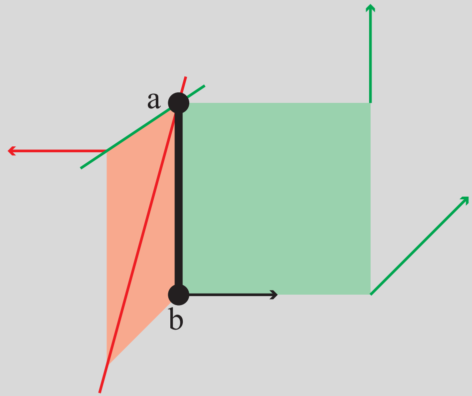
In 3d

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \tau_1' \\ \tau_2' \\ \tau_3' \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3' & \omega_2' \\ \omega_3' & 0 & -\omega_1' \\ -\omega_2' & \omega_1' & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\tau + \omega \times \mathbf{a} = \tau' + \omega' \times \mathbf{a} \qquad \tau + \omega \times \mathbf{b} = \tau' + \omega' \times \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 & \mathbf{a}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{a}\times\mathbf{i} \\ 0 & 1 & 0 & \mathbf{a}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{a}\times\mathbf{j} \\ 0 & 0 & 1 & \mathbf{a}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{a}\times\mathbf{k} \\ 1 & 0 & 0 & \mathbf{b}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}\times\mathbf{i} \\ 0 & 1 & 0 & \mathbf{b}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}\times\mathbf{j} \\ 0 & 0 & 1 & \mathbf{b}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{b}\times\mathbf{k} \end{bmatrix} \begin{bmatrix} \tau \\ \omega \\ \tau' \\ \omega' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



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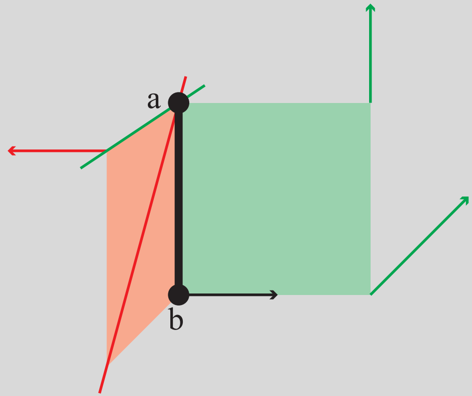
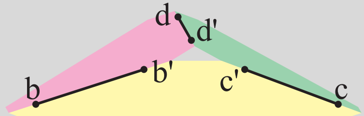
Row dependence

$$[(\mathbf{b}-\mathbf{a}) \quad (\mathbf{a}-\mathbf{b})] \begin{bmatrix} 1 & 0 & 0 & \mathbf{a}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{a}\times\mathbf{i} \\ 0 & 1 & 0 & \mathbf{a}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{a}\times\mathbf{j} \\ 0 & 0 & 1 & \mathbf{a}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{a}\times\mathbf{k} \\ 1 & 0 & 0 & \mathbf{b}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}\times\mathbf{i} \\ 0 & 1 & 0 & \mathbf{b}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}\times\mathbf{j} \\ 0 & 0 & 1 & \mathbf{b}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{b}\times\mathbf{k} \end{bmatrix}$$

$$= \left[\, \mathbf{0} \;\; (\mathbf{b}-\mathbf{a})\times(\mathbf{b}-\mathbf{a}) \;\; \mathbf{0} \;\; (\mathbf{a}-\mathbf{b})\times(\mathbf{a}-\mathbf{b}) \,\right]$$



$$\begin{bmatrix}
1 & 0 & 0 & \mathbf{b}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} \\
0 & 1 & 0 & \mathbf{b}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} \\
0 & 0 & 1 & \mathbf{b}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{b}\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} \\
1 & 0 & 0 & \mathbf{b}'\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}'\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} \\
0 & 1 & 0 & \mathbf{b}'\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}'\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} \\
0 & 0 & 1 & \mathbf{b}'\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{b}'\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} \\
1 & 0 & 0 & \mathbf{c}\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} & -1 & 0 & 0 & -\mathbf{c}\times\mathbf{i} \\
0 & 1 & 0 & \mathbf{c}\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} & 0 & -1 & 0 & -\mathbf{c}\times\mathbf{j} \\
0 & 0 & 1 & \mathbf{c}\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & -1 & -\mathbf{c}\times\mathbf{k} \\
1 & 0 & 0 & \mathbf{c}'\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} & -1 & 0 & 0 & -\mathbf{c}'\times\mathbf{i} \\
0 & 1 & 0 & \mathbf{c}'\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} & 0 & -1 & 0 & -\mathbf{c}'\times\mathbf{j} \\
0 & 0 & 1 & \mathbf{c}'\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & -1 & -\mathbf{c}'\times\mathbf{k} \\
0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & \mathbf{d}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{d}\times\mathbf{i} \\
0 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 & \mathbf{d}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{d}\times\mathbf{j} \\
0 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & \mathbf{d}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{d}\times\mathbf{k} \\
0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & \mathbf{d}'\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{d}'\times\mathbf{i} \\
0 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 & \mathbf{d}'\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{d}'\times\mathbf{j} \\
0 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & \mathbf{d}'\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{d}'\times\mathbf{k}
\end{bmatrix}$$



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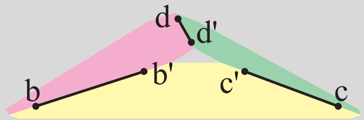
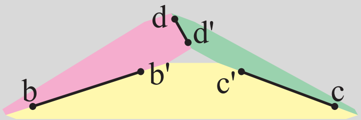
$$
\begin{bmatrix}
1 & 0 & 0 & \mathbf{b}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} \\
0 & 1 & 0 & \mathbf{b}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} \\
0 & 0 & 1 & \mathbf{b}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{b}\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} \\
1 & 0 & 0 & \mathbf{b}'\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}'\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} \\
0 & 1 & 0 & \mathbf{b}'\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}'\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} \\
0 & 0 & 1 & \mathbf{b}'\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{b}'\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} \\
1 & 0 & 0 & \mathbf{c}\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} & -1 & 0 & 0 & -\mathbf{c}\times\mathbf{i} \\
0 & 1 & 0 & \mathbf{c}\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} & 0 & -1 & 0 & -\mathbf{c}\times\mathbf{j} \\
0 & 0 & 1 & \mathbf{c}\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & -1 & -\mathbf{c}\times\mathbf{k} \\
1 & 0 & 0 & \mathbf{c}'\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} & -1 & 0 & 0 & -\mathbf{c}'\times\mathbf{i} \\
0 & 1 & 0 & \mathbf{c}'\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} & 0 & -1 & 0 & -\mathbf{c}'\times\mathbf{j} \\
0 & 0 & 1 & \mathbf{c}'\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & -1 & -\mathbf{c}'\times\mathbf{k} \\
0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & \mathbf{d}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{d}\times\mathbf{i} \\
0 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 & \mathbf{d}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{d}\times\mathbf{j} \\
0 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & \mathbf{d}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{d}\times\mathbf{k} \\
0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & \mathbf{d}'\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{d}'\times\mathbf{i} \\
0 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 & \mathbf{d}'\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{d}'\times\mathbf{j} \\
0 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & \mathbf{d}'\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{d}'\times\mathbf{k}
\end{bmatrix}
$$



$$
\left[\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & \mathbf{b}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0}\\
0 & 1 & 0 & \mathbf{b}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0}\\
0 & 0 & 1 & \mathbf{b}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{b}\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0}\\
1 & 0 & 0 & \mathbf{b}'\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}'\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0}\\
0 & 1 & 0 & \mathbf{b}'\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}'\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0}\\
1 & 0 & 0 & \mathbf{c}\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} & -1 & 0 & 0 & -\mathbf{c}\times\mathbf{i}\\
0 & 1 & 0 & \mathbf{c}\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} & 0 & -1 & 0 & -\mathbf{c}\times\mathbf{j}\\
0 & 0 & 1 & \mathbf{c}\times\mathbf{k} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & -1 & -\mathbf{c}\times\mathbf{k}\\
1 & 0 & 0 & \mathbf{c}'\times\mathbf{i} & 0 & 0 & 0 & \mathbf{0} & -1 & 0 & 0 & -\mathbf{c}'\times\mathbf{i}\\
0 & 1 & 0 & \mathbf{c}'\times\mathbf{j} & 0 & 0 & 0 & \mathbf{0} & 0 & -1 & 0 & -\mathbf{c}'\times\mathbf{j}\\
0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & \mathbf{d}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{d}\times\mathbf{i}\\
0 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 & \mathbf{d}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{d}\times\mathbf{j}\\
0 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & \mathbf{d}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{d}\times\mathbf{k}\\
0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & \mathbf{d}'\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{d}'\times\mathbf{i}\\
0 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 & \mathbf{d}'\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{d}'\times\mathbf{j}
\end{array}\right]
$$



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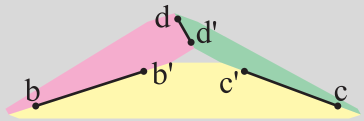
$$\left[\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & \mathbf{b}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}\times\mathbf{i} & & & & \\
0 & 1 & 0 & \mathbf{b}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}\times\mathbf{j} & & & & \\
0 & 0 & 1 & \mathbf{b}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{b}\times\mathbf{k} & & & & \\
1 & 0 & 0 & \mathbf{b}'\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{b}'\times\mathbf{i} & & & & \\
0 & 1 & 0 & \mathbf{b}'\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{b}'\times\mathbf{j} & & & & \\
1 & 0 & 0 & \mathbf{c}\times\mathbf{i} & & & & & -1 & 0 & 0 & -\mathbf{c}\times\mathbf{i} \\
0 & 1 & 0 & \mathbf{c}\times\mathbf{j} & & & & & 0 & -1 & 0 & -\mathbf{c}\times\mathbf{j} \\
0 & 0 & 1 & \mathbf{c}\times\mathbf{k} & & & & & 0 & 0 & -1 & -\mathbf{c}\times\mathbf{k} \\
1 & 0 & 0 & \mathbf{c}'\times\mathbf{i} & & & & & -1 & 0 & 0 & -\mathbf{c}'\times\mathbf{i} \\
0 & 1 & 0 & \mathbf{c}'\times\mathbf{j} & & & & & 0 & -1 & 0 & -\mathbf{c}'\times\mathbf{j} \\
 & & & & 1 & 0 & 0 & \mathbf{d}\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{d}\times\mathbf{i} \\
 & & & & 0 & 1 & 0 & \mathbf{d}\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{d}\times\mathbf{j} \\
 & & & & 0 & 0 & 1 & \mathbf{d}\times\mathbf{k} & 0 & 0 & -1 & -\mathbf{d}\times\mathbf{k} \\
 & & & & 1 & 0 & 0 & \mathbf{d}'\times\mathbf{i} & -1 & 0 & 0 & -\mathbf{d}'\times\mathbf{i} \\
 & & & & 0 & 1 & 0 & \mathbf{d}'\times\mathbf{j} & 0 & -1 & 0 & -\mathbf{d}'\times\mathbf{j}
\end{array}\right]$$



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Example: Let G be a triangle, whose edges, a, b, c are representing pins and are located at $(x_a, y_a), (x_b, y_b), (x_c, y_c)$ respectively. The vertex set represents the bodies, each body has two pins on it.

$$\begin{bmatrix} 1 & 0 & -x_a & -1 & 0 & x_a & 0 & 0 & 0 \\ 0 & 1 & y_a & 0 & -1 & -y_a & 0 & 0 & 0 \\ 1 & 0 & -x_b & 0 & 0 & 0 & -1 & 0 & x_b \\ 0 & 1 & y_b & 0 & 0 & 0 & 0 & -1 & -y_b \\ 0 & 0 & 0 & 1 & 0 & -x_c & -1 & 0 & x_c \\ 0 & 0 & 0 & 0 & 1 & y_a & 0 & -1 & -y_a \end{bmatrix}$$

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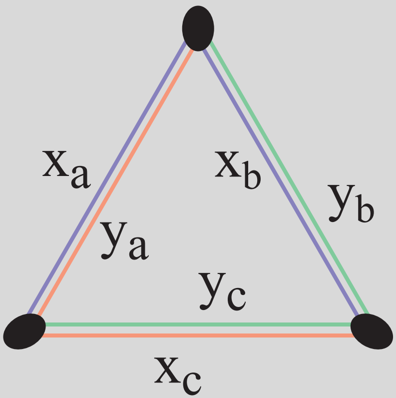
re-arrange the rows/columns:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -x_a & x_a & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -x_b & 0 & x_b \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & -x_c & x_c \\ 0 & 0 & 0 & 1 & -1 & 0 & y_a & -y_a & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & y_b & 0 & -y_b \\ 0 & 0 & 0 & 0 & 1 & -1 & -y_c & y_c & 0 \end{bmatrix}$$



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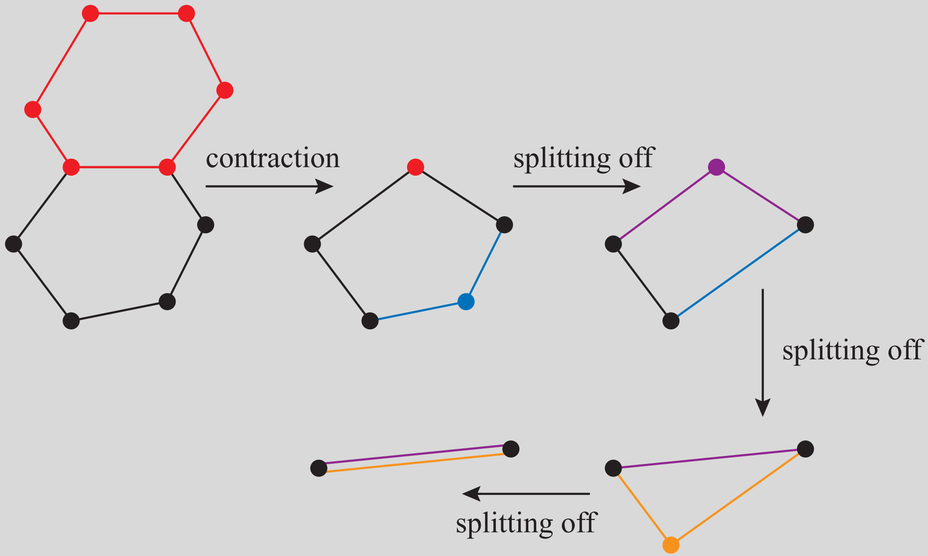
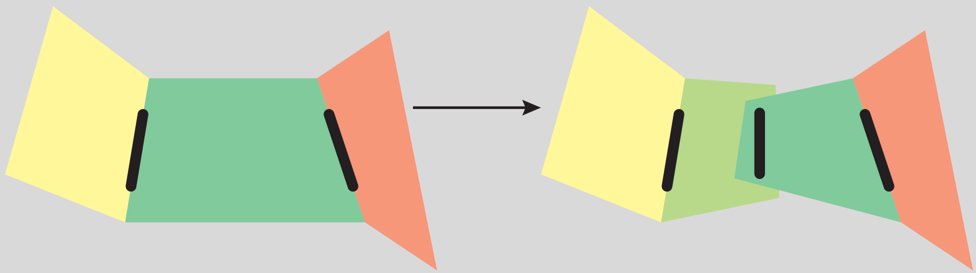
Now use the inductive constructions to construct embeddings of the correct rank.



The crucial move is:



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4. Back to 2-d

Conjecture: A generically rigid body pin framework, remains first-order rigid for realizations generic under the condition that all pins of each body are collinear, without restriction of how many bodies a pin is incident to.

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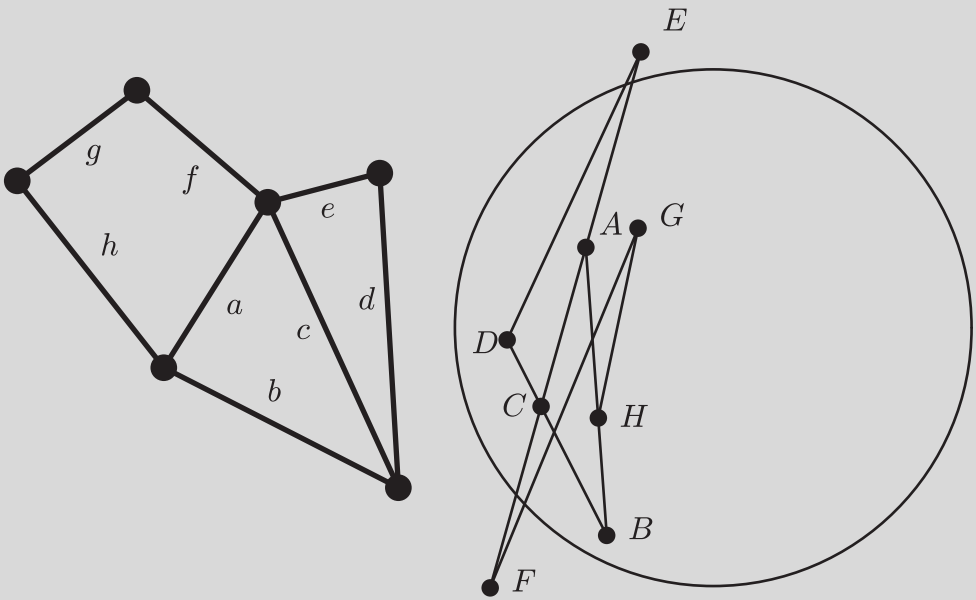
5. Realization in the Plane.

Theorem 3 *If $G = (V, E)$ is simple, then a pin collinear structure exists.*

Take any generic embedding of the structure graph $G = (V, E)$ in $\mathbb{R}^2$. Form the polar of that embedding.



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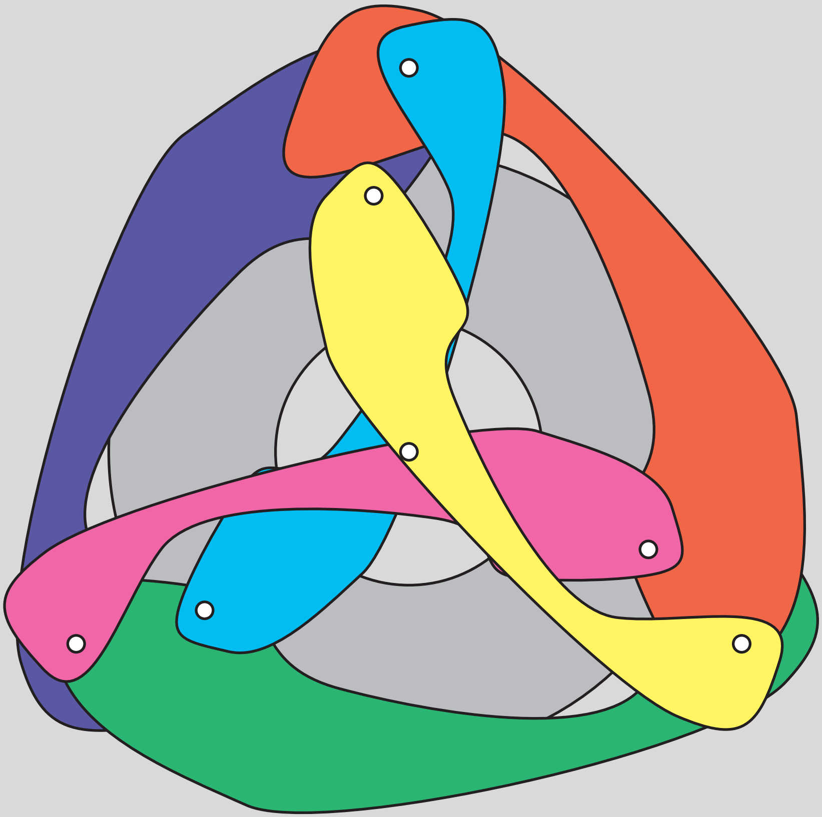
A general body-pin structure:



The incidence structure is a hyper-graph. Does it have a pin collinear realization?

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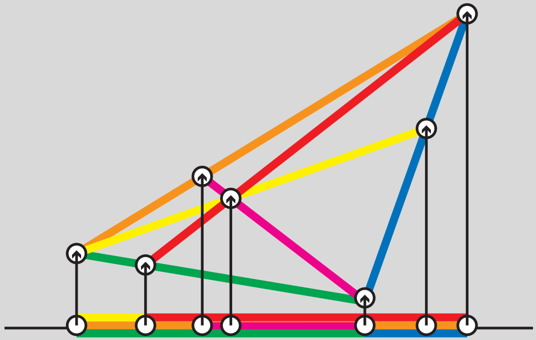

Theorem 4 *If the 2-plane matroid of an incidence structure has rank $a + 2b - 2$, then placing a points on any line in the plane with generic x-coordinates and joining them appropriately with b rigid bars gives a structure which is infinitesimally rigid in the plane.*

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6. The k-plane matroids

Given: A Hypergraph: $(A, B; I)$
The *k-plane matroid* on I has has independent sets $I' \subseteq I$ defined via:

For all $I'' \subseteq I'$, we have

$$|I''| \leq |A(I'')| + k|B(I'')| - k$$

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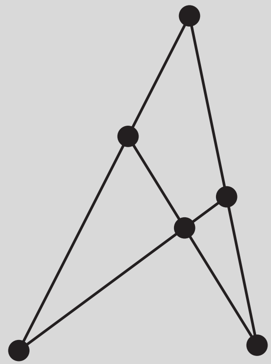
The 2-plane matroids

Given: A Hypergraph: $(A, B; I)$
The 2-*plane matroid* on I has has independent sets $I' \subseteq I$ defined via:

For all $I'' \subseteq I'$, we have

$$|I''| \leq |A(I'')| + 2|B(I'')| - 2$$



A: The lines
B: the points
I the incidence relation

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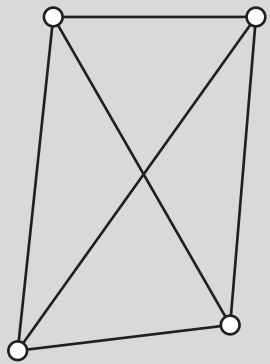
The 2-plane matroids

Given: A Hypergraph: $(A, B; I)$
The 2-*plane matroid* on I has has independent sets $I' \subseteq I$ defined via:

For all $I'' \subseteq I'$, we have

$$|I''| \leq |A(I'')| + 2|B(I'')| - 2$$



A: The points
B: the lines
I the incidence relation

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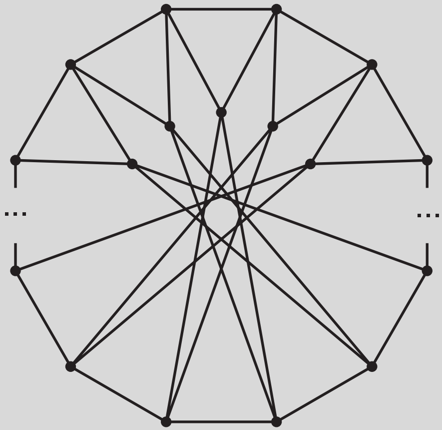
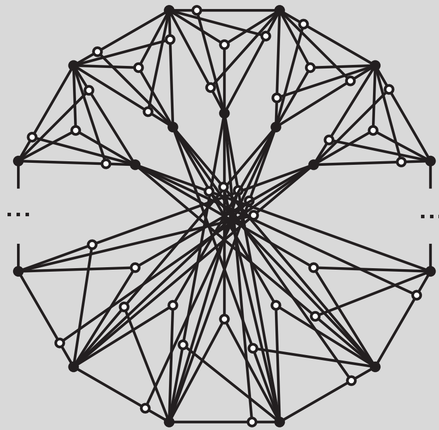
7. 2-plane matroid and connectivity of the incidence graph

Theorem 5 *Let $G = (A, B, I)$ be an incidence graph. If G is vertex 4-connected then I is 2-tight.*

We can construct 3-connected incidence structures which are not tight.





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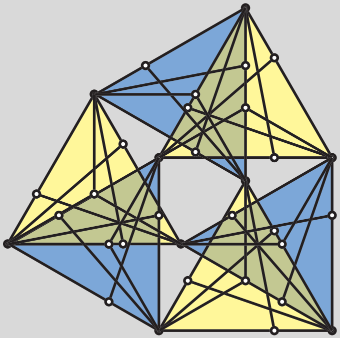
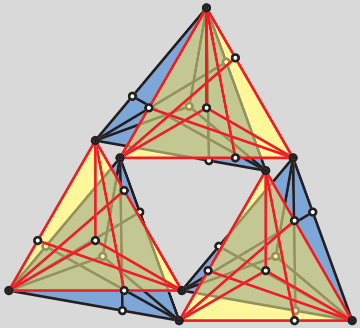
Theorem 6 *Let $G = (A, B, I)$ be an incidence graph. If G is vertex 4-connected then I is 2-tight.*

We can construct 3-connected incidence structures which are not tight.





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8. Whiteley's Theorem

Given an incidence graph $G = (B, J; I)$ the following are equivalent:

(i) G has a realization as an independent (isostatic) identified body and joint framework in the plane.

(ii) G satisfies

$$2i \leq 3b + 2j - 3(=)$$

and, for every subset of bodies and induced subgraph of attached joints,

$$2i' \leq 3b' + 2j' - 3.$$

(iii) G has an independent (isostatic) realization as an identified body and joint framework in the plane such that each body has all its joints collinear.

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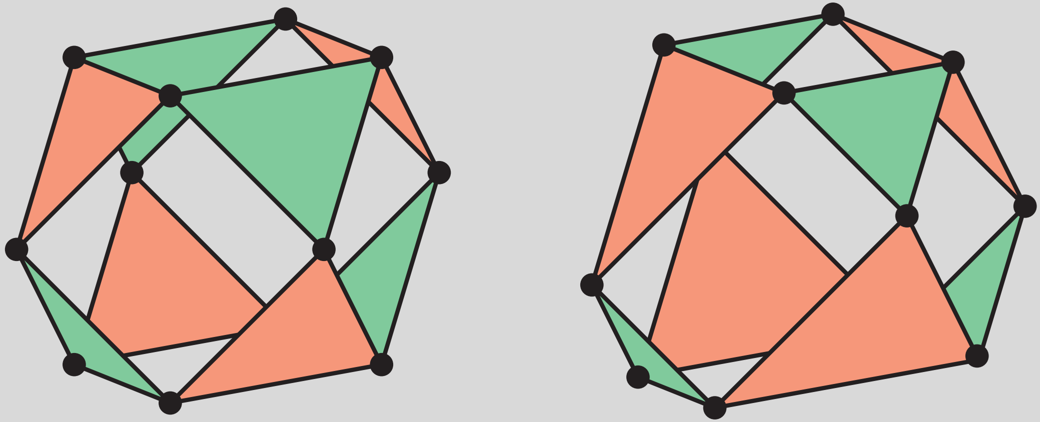
In the plane the parallel drawing and rigidity are equivalent. A parallel-body pin framework in d-space is a collection of points (pins) and bodies (collections of the pins) and a realization of the pins as points in d-space. The body is assumed to group its pins into a parallel-tight unit, with only translations.



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Theorem 7 *Given an incidence graph $G = (B, J; I)$ the following are equivalent:*
(i) G has a realization as an independent (tight) identified parallel-body pin framework in d-space.
(ii) G satisfies $di \leq (d+1)b + dj - d + 1 (=)$ and, for every subset of bodies and induced subgraph of attached joints, $di' \leq (d+1)b' + dj' - d + 1$.
(iii) G has an independent (tight) realization as a parallel-body pin framework in d-space such that each body has all its joints collinear.

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Conjecture 2 (Parallel Drawing Flattening Conjecture) *If a parallel body-and-pin incidence structure is generically tight for generic configurations in d-space, then it will remain tight for realizations generic under the constraint that all pins for a body are coplanar.*