Graph groups 2010

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1. What is a graph group

$$\Gamma = (V, E)$$

$$V = \{a, b, c, \ldots\} \qquad E = \{(x, y), (z, w), \ldots\}$$

$$F(\Gamma) = \langle a, b, c, \ldots \mid xy = yx, zw = wz \ldots \rangle$$

$\Gamma = K_n \implies F(\Gamma)$ free abelian of rank k.

$\Gamma = K_n^c \implies F(\Gamma)$ free of rank k.

Theorem: (Droms 1983) [3] $F(\Gamma) \cong F(\Gamma') \iff \Gamma \cong \Gamma'$

Combinatorics of Graph Groups

Word problem, conjugacy problem, centralizer problem are all solved by algorithms linear in the number of edges.

Algorithmic questions are quite tractable.

Theorem: (Laurence 1995) [6] *The automorphisms of a graph group are generated by*

Partially Inner Automorphisms
Graph Automorphisms
Generator Inversions
Pendant Translations

What are Artin groups ?

Commutation Relation: $ab = ba$

Braid Relation: $aba = bab$

Artin Relation: $aba \cdots = bab \cdots$

Coxeter adds $a^2 = 1$.

Coherence

A group is *coherent* if every finitely generated subgroup is finitely presented.
Free groups are coherent.
Free abelian groups a coherent.

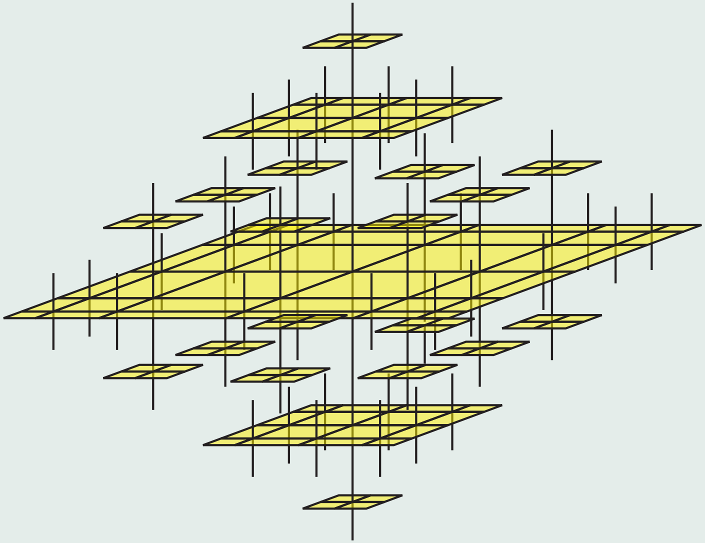
Theorem 1 *(Droms-1983) [2] A graph group $G(\Gamma)$ is coherent if and only each cycle of length greater than 3 has a chord.*

2. Graph Groups in $1D$

The Cayley graphs can be easily constructed.

The Cayley complex can be easily constructed and have a nice geometry.

The Eilenberg-MacLane spaces are also easy to construct.



3. Graph Groups in $2D$

The graph group $F(\{a, b\}, \{(a, b)\})$ is the fundamental group of the torus.
Can we find other surface groups?

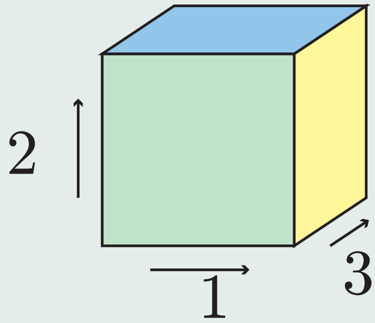
The following pullback diagram realizes the commutator subgroup of $F(\Gamma)$.

$$\begin{tikzcd} U_\Gamma \arrow[r] \arrow[d] & U_K \arrow[d] \\ X_\Gamma \arrow[r] & X_K \end{tikzcd}$$

$$X_K = [(S_1)^n]_2 \qquad U_K = [\mathbb{R}^n]_2$$

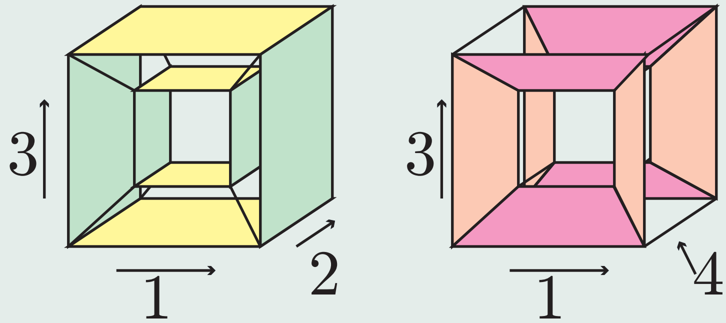
Theorem 2 *Let Z be a cover of the Cayley complex of F_Γ, and let Y be a subcomplex of Z with the property that any face of Z which contains at least two incident edges of Y is also a face of Y. Then the inclusion map $i : Y \to Z$ induces a monomorphism $i_* : \pi_1(Y) \to \pi_1(Z)$.*

Γ a triangle. 3D Cube

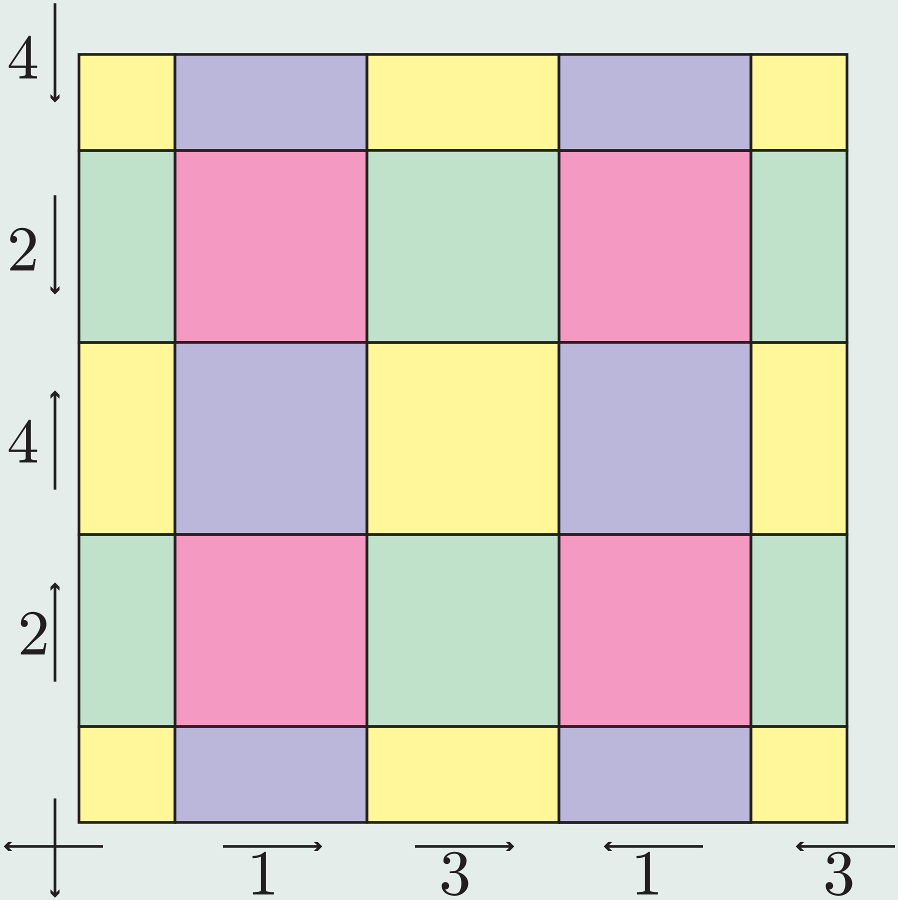


Conclusion: The free abelian group of rank three contains a trivial subgroup.

Γ 4-gon. 4D Cube

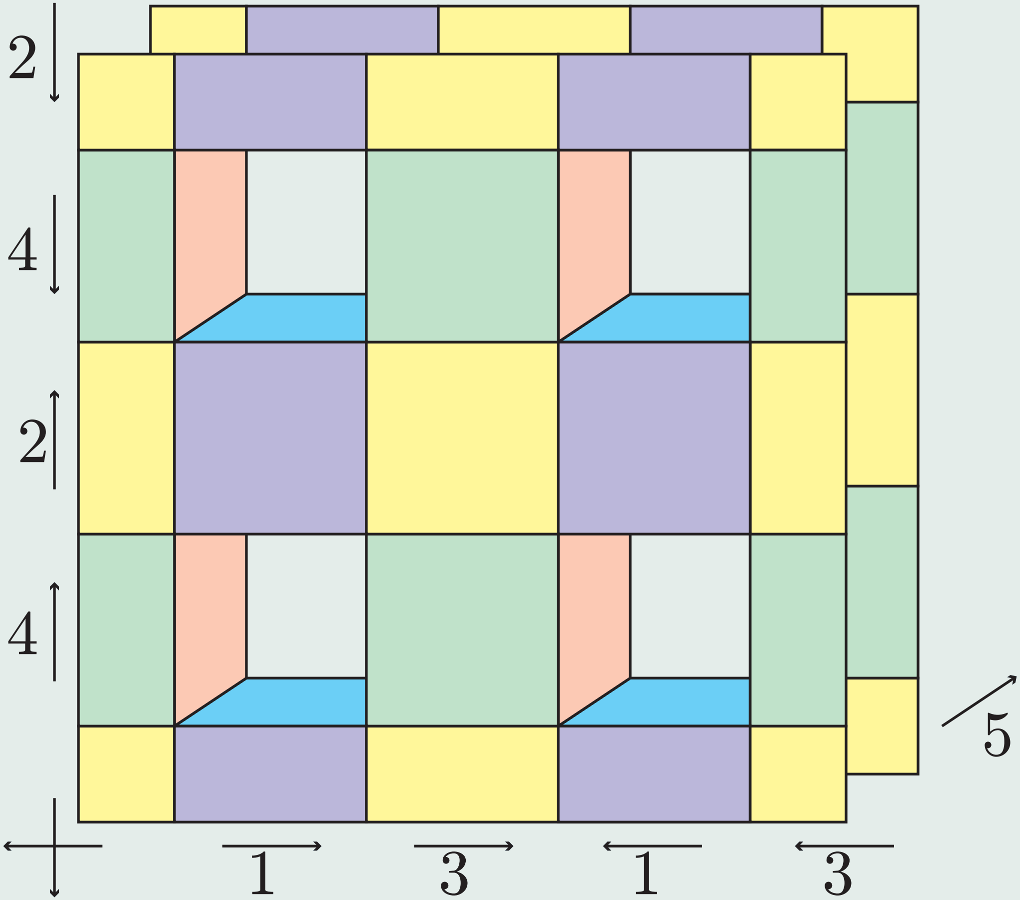


$$F(\Gamma) = \langle a, c\rangle \oplus \langle a, c\rangle$$



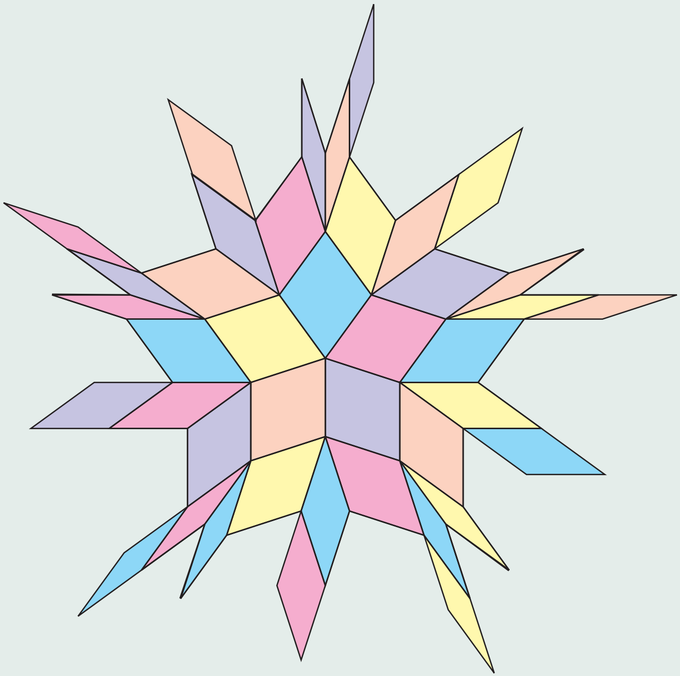
Conclusion: The direct sum of two free groups of rank 2 contains a free abelian group of rank 2.

Γ 5-gon. 5D Cube



Theorem (DSS 1990) [7] *Let F_{Γ_n} be the graph group of the n-gon graph. F'_{Γ_n} contains a subgroup isomorphic to the fundamental group of the orientable surface of genus $1 + (n-4)2^{n-3}$.*

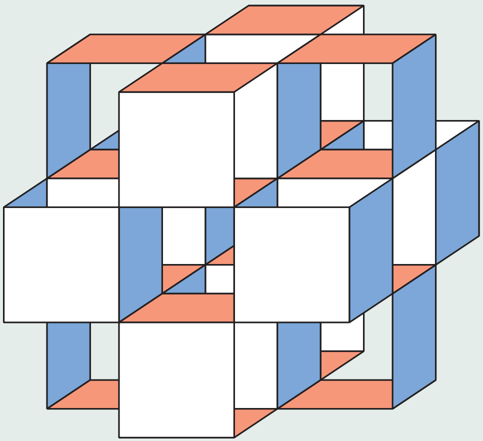
Theorem (Crisp Wiest 2004) [1] *All but finitely many surface groups embed in graph groups.*



A *Commutator Map* is a map on a surface such that: Every face is a quadrilateral The oriented edges are colored such that the boundary of each quadrilateral is a commutator in the color labels If two colors form a commutator, then any two edges of those colors which are incident at a vertex are incident at a face.

Theorem

Every Eulerian graph embeds in a surface as a commutator map.



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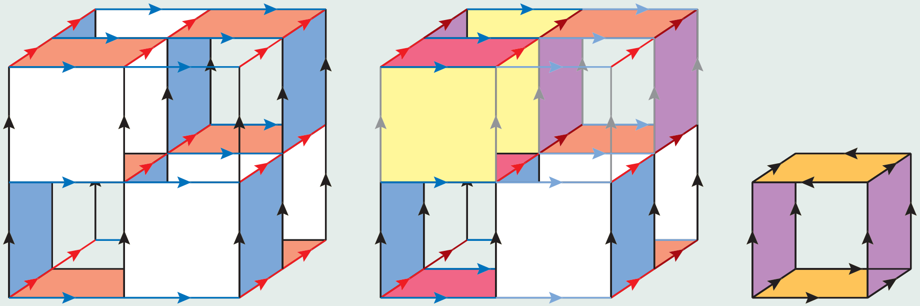
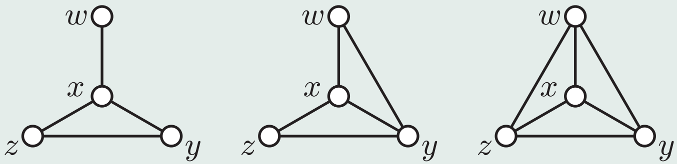


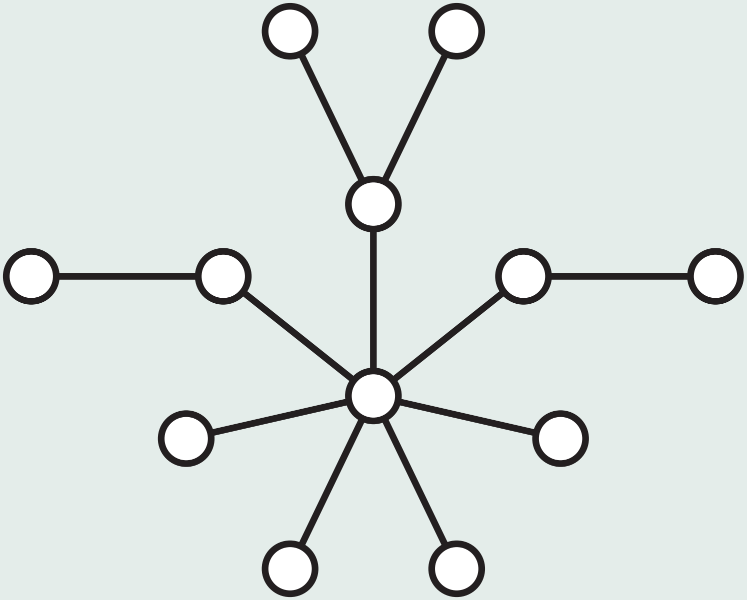
4. Graph Groups in $3D$

Theorem 3 *(Droms-1983) [2] A graph group $G(\Gamma)$ is a 3-manifold group if and only if every connected component of Γ is either a tree or a triangle.*

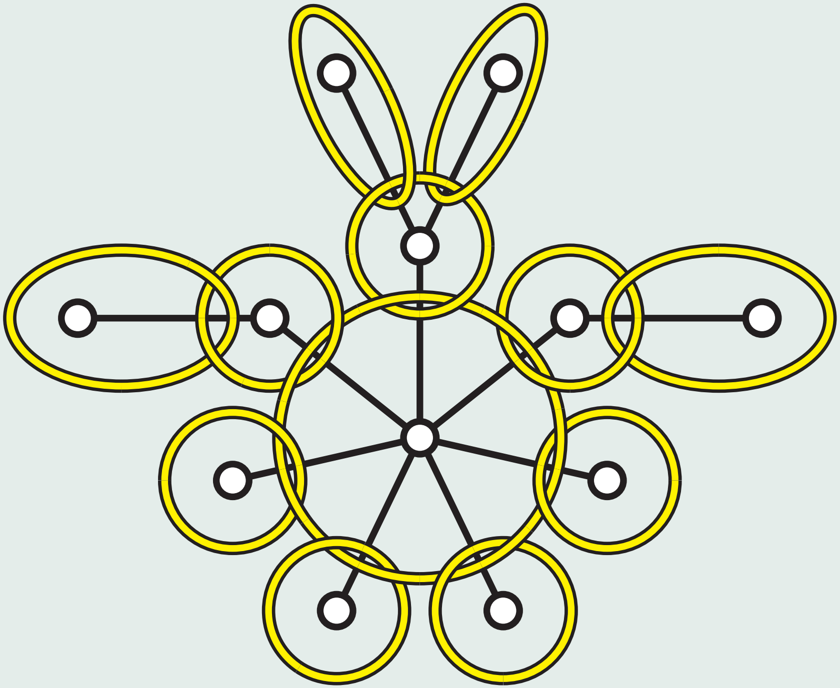
Necessity:
(Jaco and Shalen) [5]A three-manifold group is coherent.



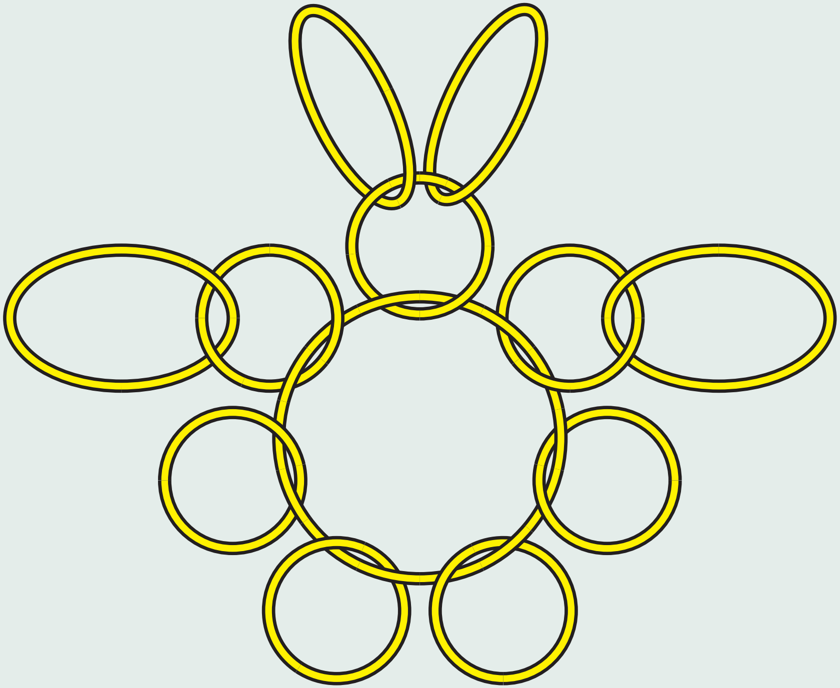
(Shalen) [8] If $\langle x \rangle \oplus X$ is a 3-manifold group, then X is a surface group.
(Hoare Karrass Solitar) [4] Every subgroup of infinite index in a surface group is free.



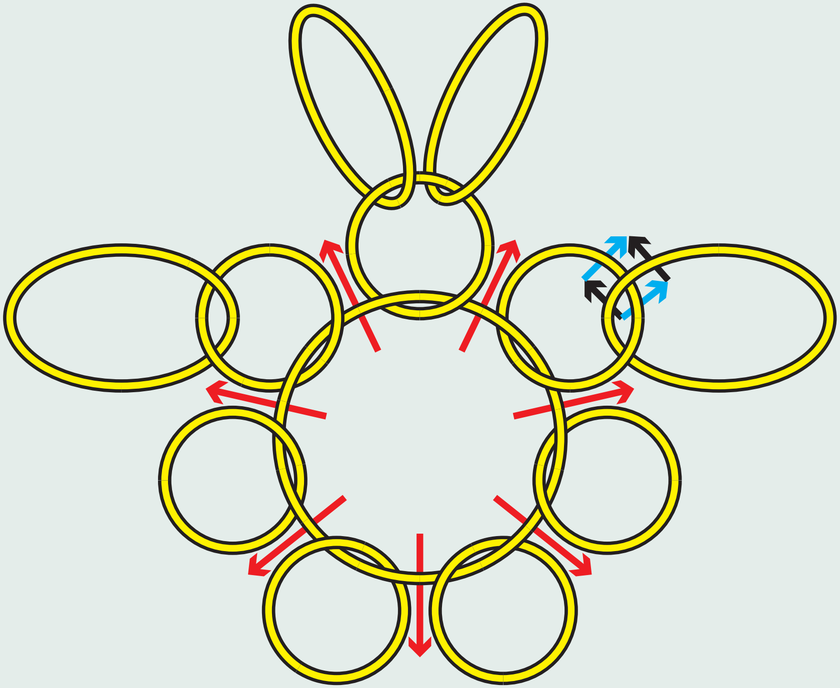
For those components which are trees



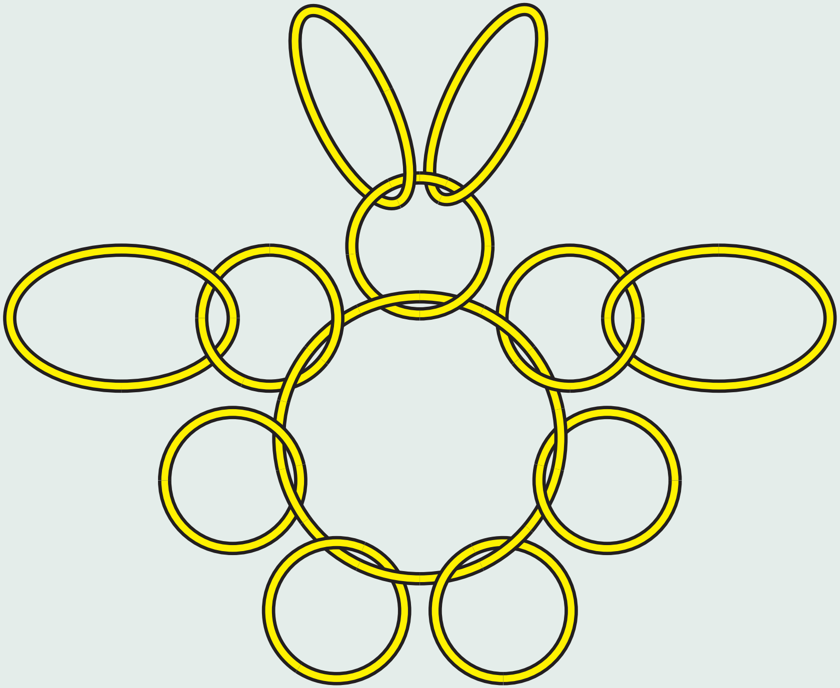
For those components which are trees



For those components which are trees



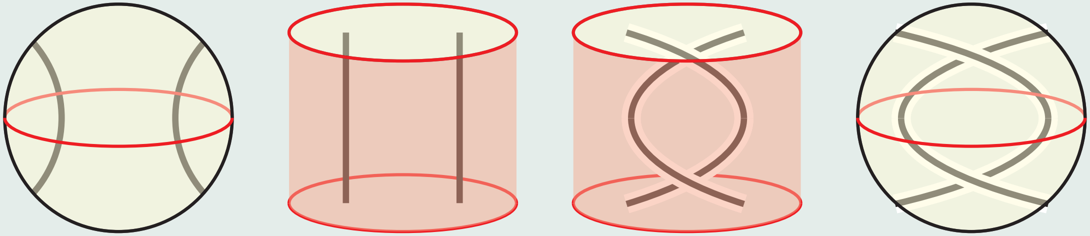
For those components which are trees

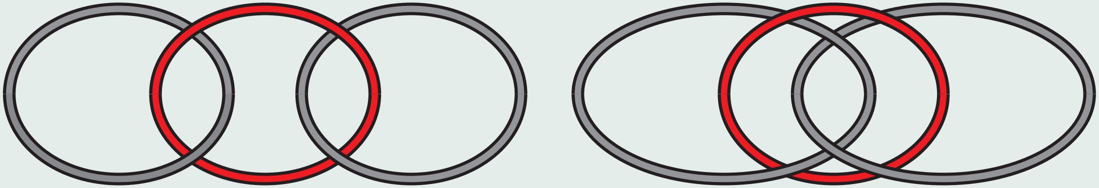


There are uncountably many 3-manifolds whose fundamental groups are graph groups.

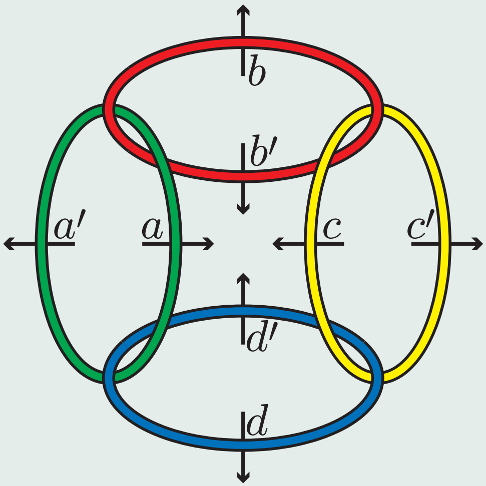
When is uncountable not enough?

Are there other links in S_3 whose groups are graph groups?





Dehn twist on circle d



$$
\begin{array}{ll}
ab = ba' & ba = ab' \\
bc = cb' & cb = bc' \\
cd = dc' & dc = cb' \\
da = ad' & ad = da'
\end{array}
$$

A graph group link group.

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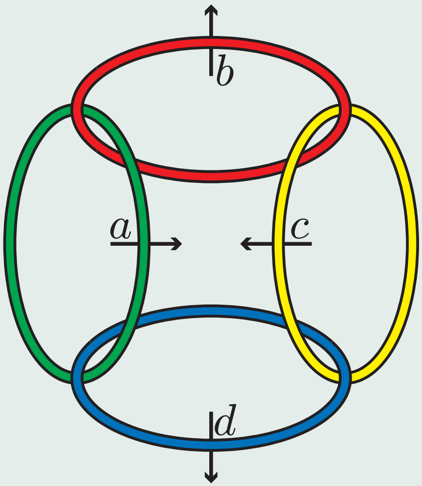
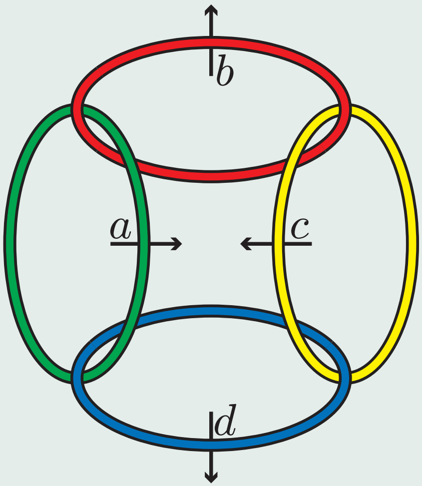
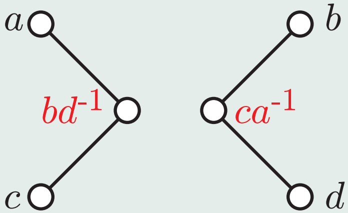


$$
\begin{aligned}
b^{-1}ab &= d^{-1}ad \\
c^{-1}bc &= a^{-1}ba \\
d^{-1}cd &= b^{-1}cb \\
a^{-1}da &= c^{-1}dc
\end{aligned}
$$

So both a and c commute with bd^{-1}. And both b and d commute with ca^{-1}.



$$\left\langle a, b, c, d, \mid \begin{array}{c} [a, bd^{-1}] = 1, [b, ca^{-1}] = 1, \\ [c, db^{-1}] = 1, [d, ac^{-1}] = 1 \\ [ac^{-1}, bd^{-1}] = 1 \end{array} \right\rangle$$



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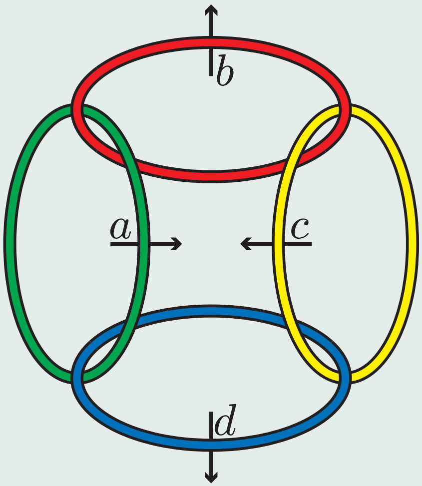

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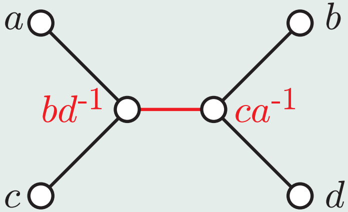

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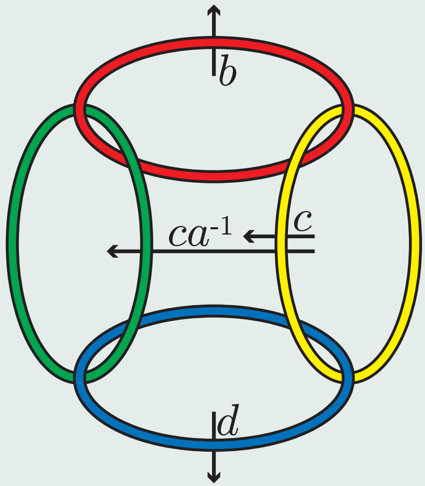
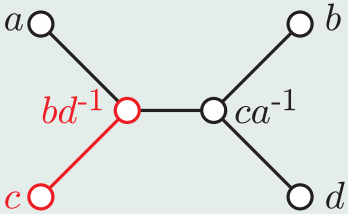
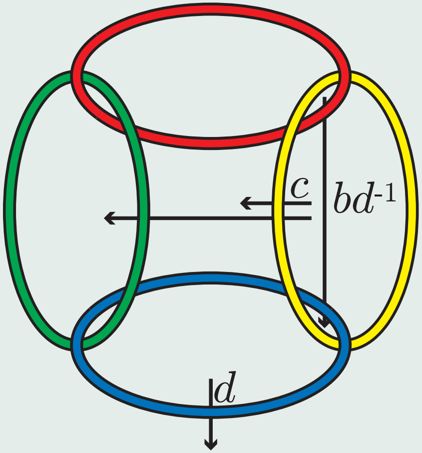
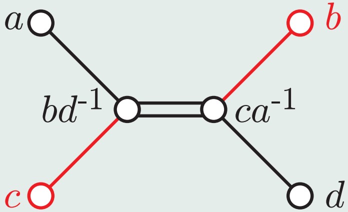


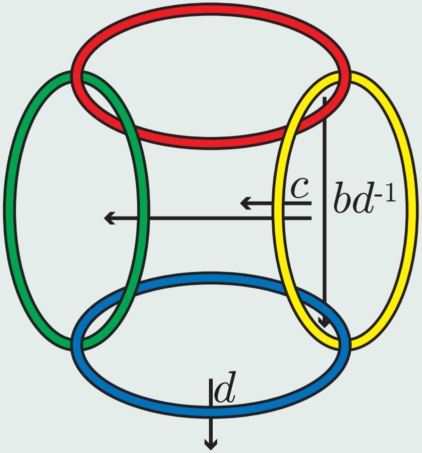
$$\left\langle a, b, ca^{-1}, d, \mid \begin{array}{ll} [a, bd^{-1}] = 1, & [b, ca^{-1}] = 1, \\ [ca^{-1}, db^{-1}] = 1, & [d, ac^{-1}] = 1 \end{array} \right\rangle$$





$$\left\langle a, bd^{-1}, ca^{-1}, d, \mid \begin{array}{ll} [a, bd^{-1}] = 1, & [bd^{-1}, ca^{-1}] = 1, \\ [ca^{-1}, db^{-1}] = 1, & [d, ac^{-1}] = 1 \end{array} \right\rangle$$





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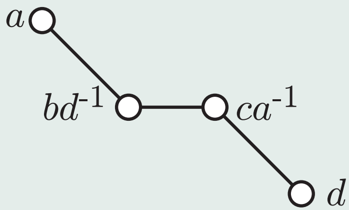
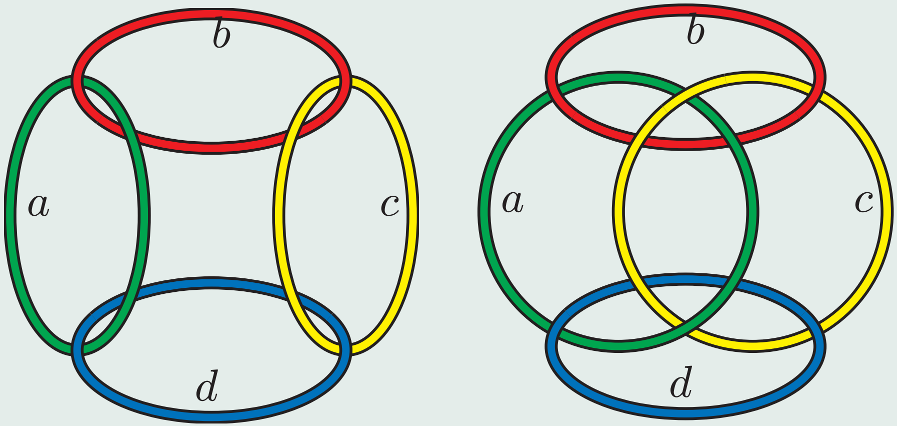
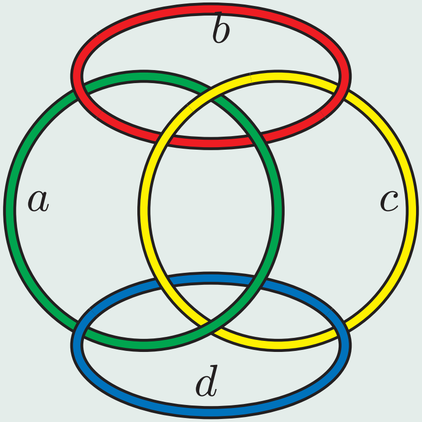
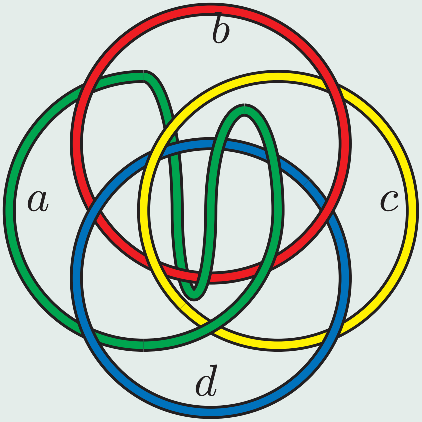


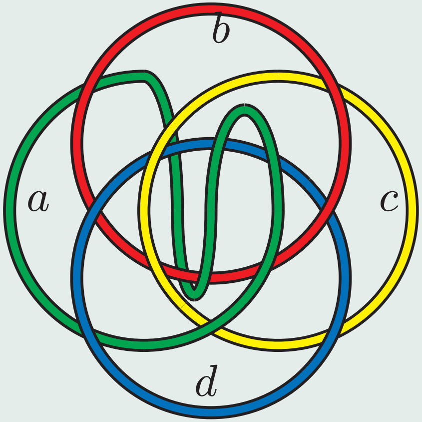


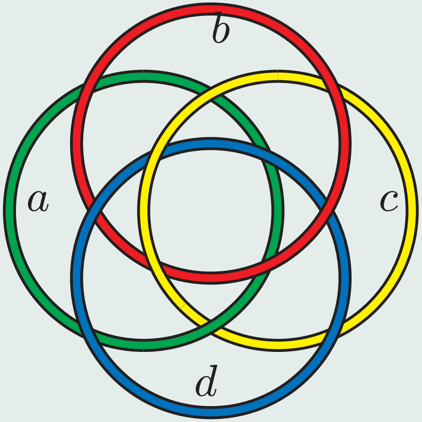
Dehn twist on circle d



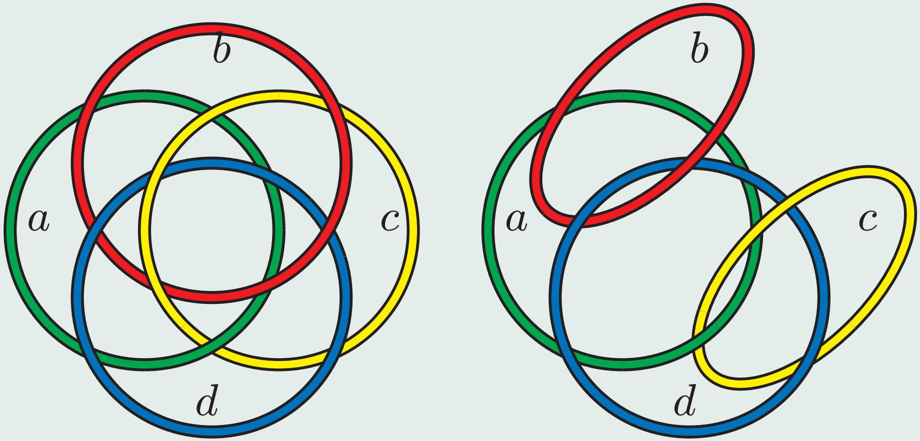


Dehn twist on circle c.

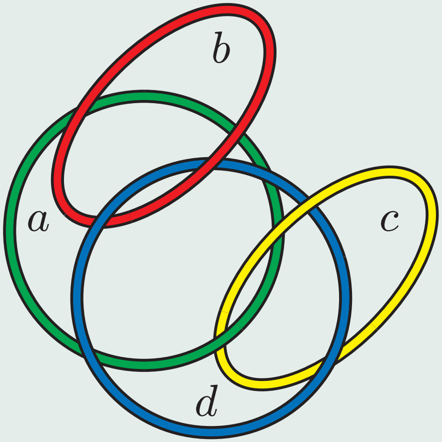


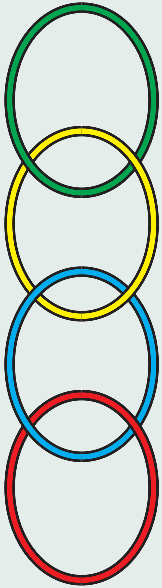


Isotope circle a.

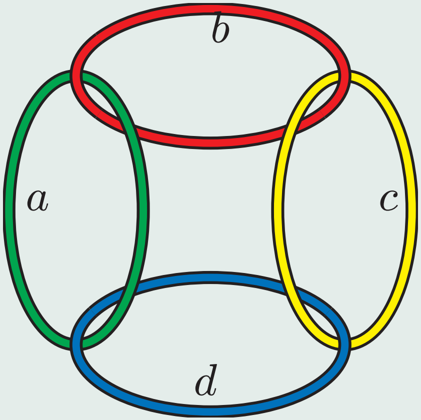


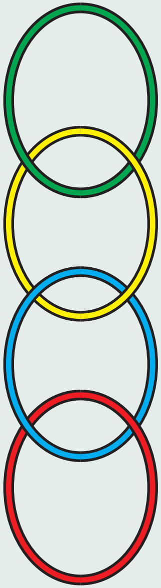
Dehn twist on circle d in opposite direction.





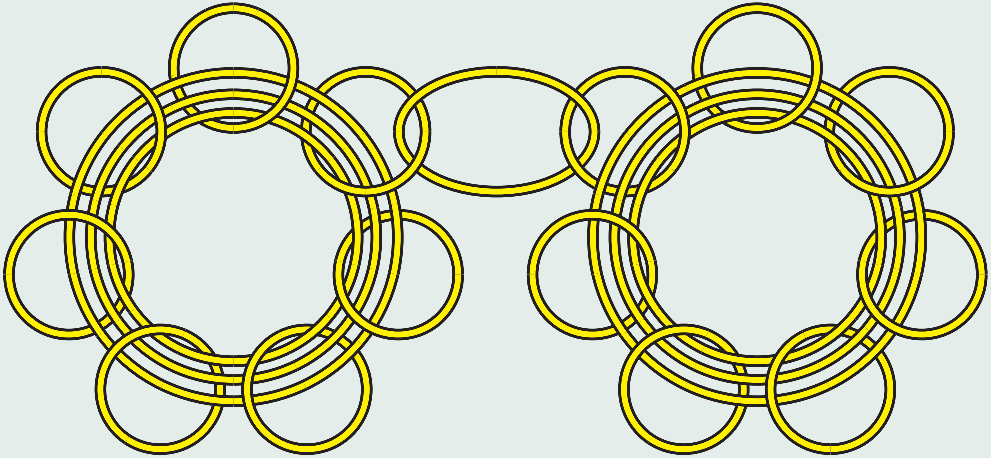
Pull taut for the result. a——c——d——b.

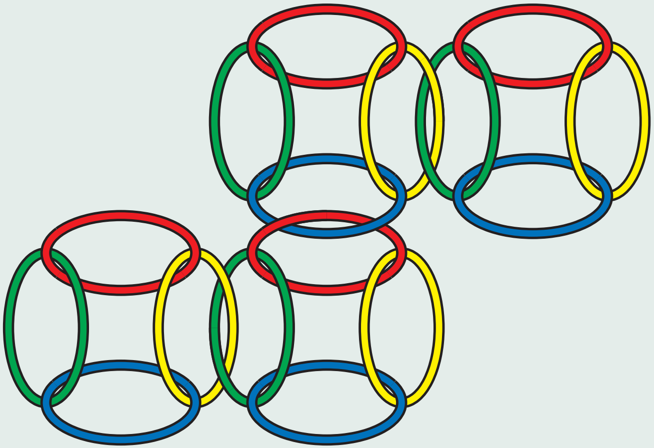




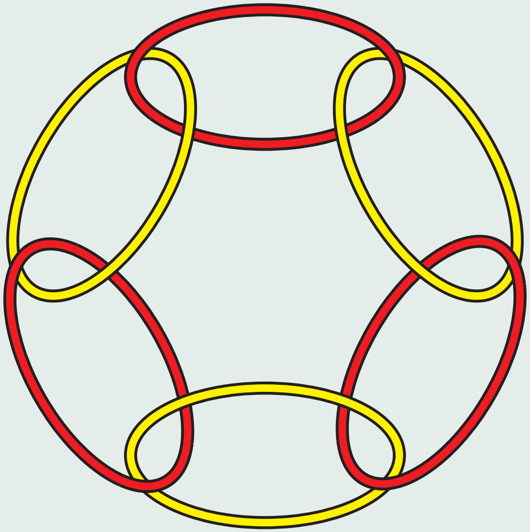
So these links have homeomorphic complements

Which links of unknotted circles yield graph groups



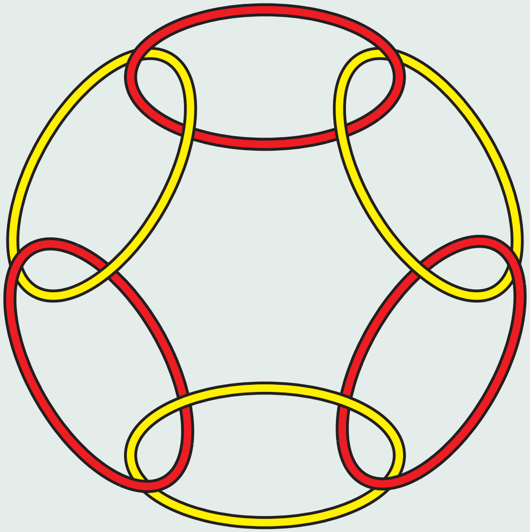


Fundamental group a graph group?



If so, what is the graph? It would have to have six vertices.

Fundamental group a graph group?



Theorem (Thurston 198?) 1982 [9] *The complement of the six link anklet in S_3 has a hyperbolic structure.*

Hyperbolic 3-manifold groups

Theorem 4 *A graph group $G(\Gamma)$ which is a hyperbolic 3-manifold group has no component which is not complete.*

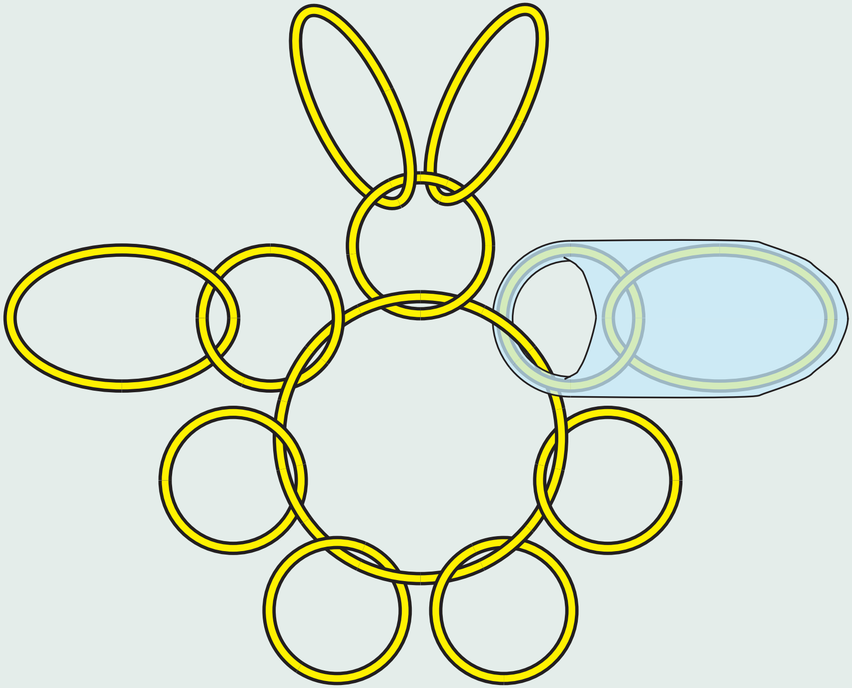
Theorem 5 *$PSL(2, \mathbb{C})$ has no discrete subgroup isomorphic to $G(\circ \text{———} \circ \text{———} \circ)$.*

$$\begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{bmatrix}$$

Theorem 6 *The group of the six link anklet is not a graph group.*

Geometric Structure of Graph Links

The an epsilon neighborhood of any link is *incompressible.*



The graph link manifolds which are *atoroidal* are
$L(a)$ - solid torus with geometry $\mathbb{R}^3$
$L(a$——$b)$ - thickened torus with geometry $\mathbb{R}^3$
$L(a$——b——$c)$ - $S^1 \times (\mathbb{R}^2 - \{p, q, r\})$ with geometry $\mathbb{R} \times \mathbb{H}$.

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References

[1] John Crisp and Bert Wiest. Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups. *Algebr. Geom. Topol.*, 4:439–472, 2004.

[2] Carl Droms. Graph groups, coherence, and three-manifolds. *J. Algebra*, 106(2):484–489, 1987.

[3] Carl Droms. Isomorphisms of graph groups. *Proc. Amer. Math. Soc.*, 100(3):407–408, 1987.

[4] A. Howard M. Hoare, Abraham Karrass, and Donald Solitar. Subgroups of infinite index in Fuchsian groups. *Math. Z.*, 125:59–69, 1972.

[5] William Jaco and Peter B. Shalen. Seifert fibered spaces in 3-manifolds. In *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977)*, pages 91–99. Academic Press, New York, 1979.

[6] Michael R. Laurence. A generating set for the automorphism group of a graph group. *J. London Math. Soc. (2)*, 52(2):318–334, 1995.

[7] Herman Servatius, Carl Droms, and Brigitte Servatius. Surface subgroups of graph groups. *Proc. Amer. Math. Soc.*, 106(3):573–578, 1989.

[8] Peter B. Shalen. Representations of 3-manifold groups. In *Handbook of geometric topology*, pages 955–1044. North-Holland, Amsterdam, 2002.

[9] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.