Delta-matroids and rigidity matroids

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1. Rigidity of Bar and Joint Frameworks

We want to consider the rigidity of a framework $F = ((V, E), \mathbf{p})$ in d-space with

- nodes corresponding to the vertices V,
- length constraints corresponding to the edges E,
- a placement of the vertices into Euclidean space by $\mathbf{p} : V \to \mathbb{R}^{|V|d}$.

Forms of rigidity:

- local
- global
- infinitesimal
- generic....

A framework is *rigid* if every placement $\mathbf{q} : V \to \mathbb{R}^d$ sufficiently close to $\mathbf{p}$ and preserving the distances between the placements of adjacent vertices, is related to $\mathbf{p}$ by a congruence of $\mathbb{R}^d$.

2. Generic Rigidity Matroids

Ground set: The edge set K of a (large enough) complete graph. The rigidity matroid for a graph $G = (V, E)$ is the restriction to E.

- d=1: rigidity=connectivity.
- d=2: If $E \subseteq K$ is independent, then $|F| \leq 2|V(F)| - 3$, for each nonempty subset F of E.
- d=3: ??

"Counting" matroids

Graphic matroids

A matroid is *graphic* if it is isomorphic to the cycle matroid on a graph G. Non-isomorphic graphs may have the same cycle matroid, but 3-connected graphs are uniquely determined by their cycle matroids.

M is co-graphic if M^* is graphic.

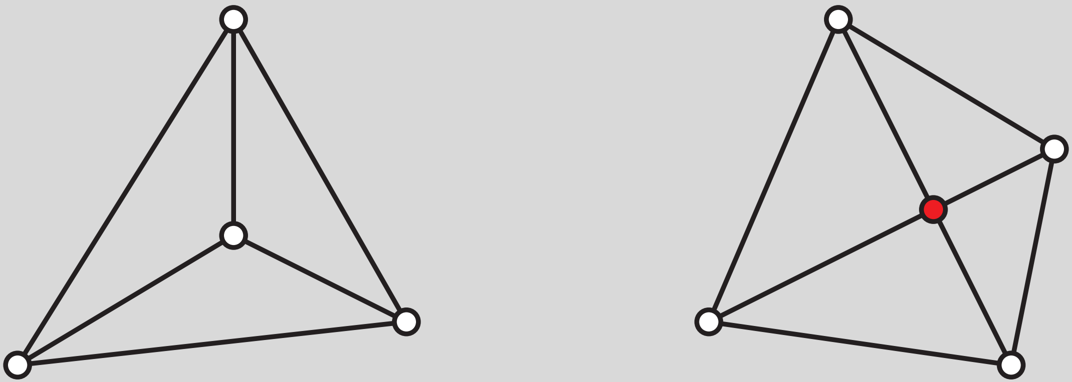
M is graphic as well as co-graphic iff G is planar. Map duality (geometric duality) agrees with matroid duality.

If $G(V, E)$ is planar and connected, its cycle matroid has rank $|V| - 1$, its co-cycle matroid has rank $|F| - 1$, so $|V| - 1 + |F| - 1 = |E|$.

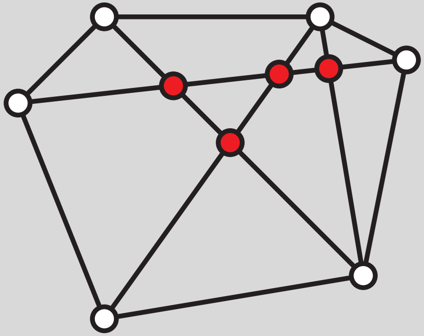
The facial cycles generate the cycle space.

Miracle in the Plane

Planar rigidity cycles dualize into rigidity cycles.



Theorem 1 *Let C be a cycle in the generic 2-dimensional rigidity matroid such that $(V(C), C)$ is planar. Then the edge set of the geometric dual of $(V(C), C)$ is also a generic cycle.*



Maxwell's reciprocal figures, parallel re-drawings, Assur graphs.

New rigidity results in dimension 3

Several recent papers characterize rigidity of graphs whose vertices are constrained to move on a sphere, concentric spheres, or concentric cylinders [5, 7, 6, 4, **?**]
Geometric/algebraic methods are used.
Can we use just combinatorics?

3. Combinatorial Maps

Tutte [8] defined maps axiomatically. If we have three fixed point free permutations τ_0, τ_2, and $\mathcal{V}$ on a set Φ of *flags* such that

A1 $\tau_0^2 = \tau_2^2 = \mathrm{Id}$

A2 $\tau_0\tau_2 = \tau_2\tau_0$

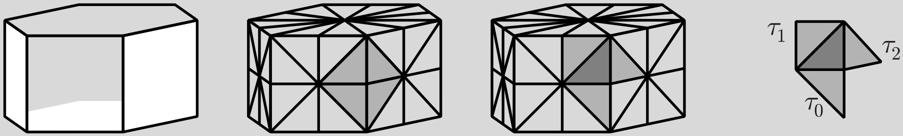
A3 $\mathcal{V}\tau_2 = \tau_2\mathcal{V}^{-1}$

A4 $\{\mathcal{V}^i\phi\} \cap \{\mathcal{V}^i\tau_2\phi\} = \emptyset$

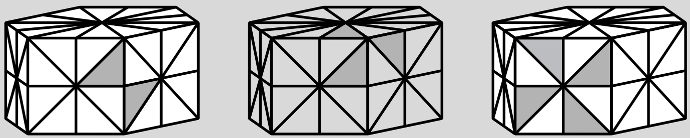
A5 τ_0, τ_2 and $\tau_0\tau_2$ are fixed point free

A6 $\langle\tau_0, \tau_2, \mathcal{V}\rangle$ acts transitively on Φ

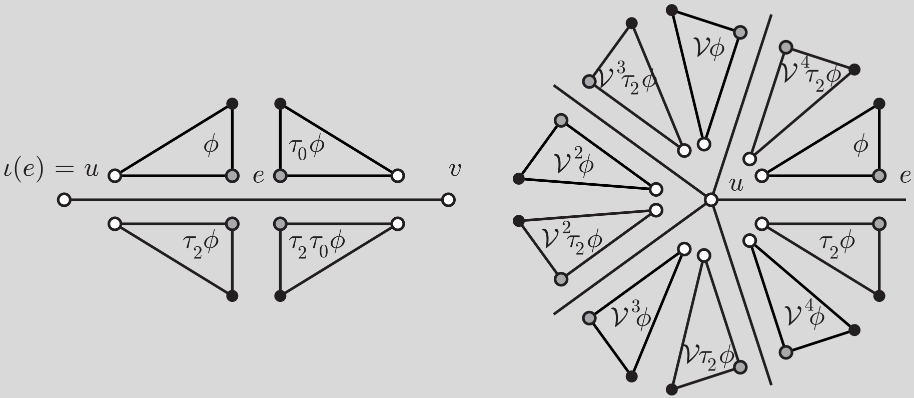
then we can define a graph G whose vertices are the orbits of Φ under $\langle\tau_2, \mathcal{V}\rangle$ and whose edges are the orbits of Φ under $\langle\tau_0, \tau_2\rangle$. The orbits of $\langle\tau_0, \tau_2\rangle$ each have four elements and intersect either one or two orbits of $\langle\tau_2, \mathcal{V}\rangle$, defining the endpoints of the edge. $M(G, \tau_0, \tau_2, \mathcal{V})$ is a combinatorial map.



Decomposition of the hexagonal prism into flags.



Orbits under $\tau_2\tau_0$, $\tau_2\tau_1$ and $\tau_0\tau_1$.



Flags at an edge, and at a vertex.

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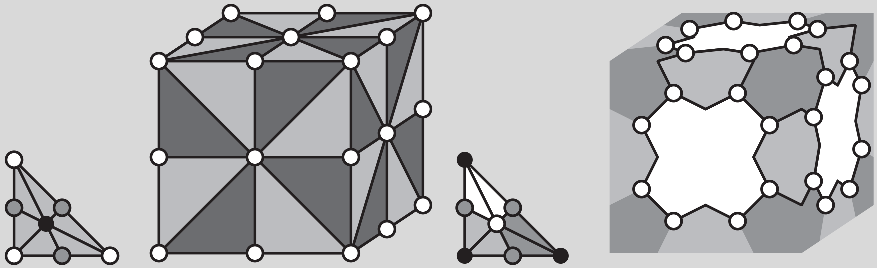
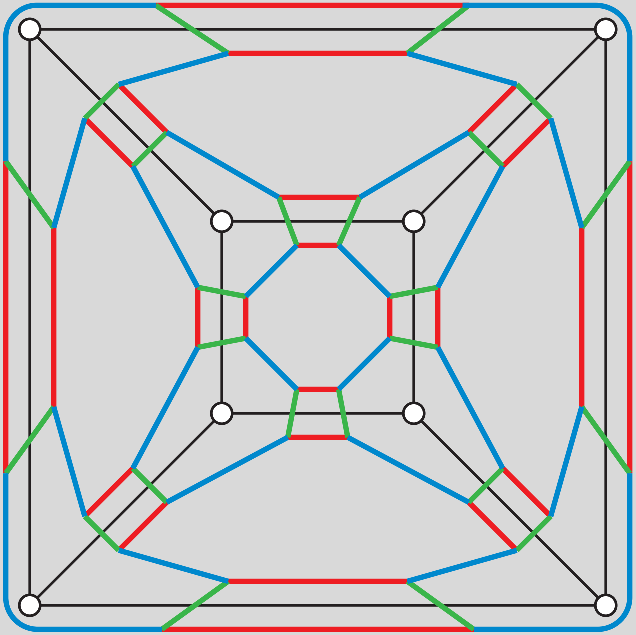
4. The flag graph



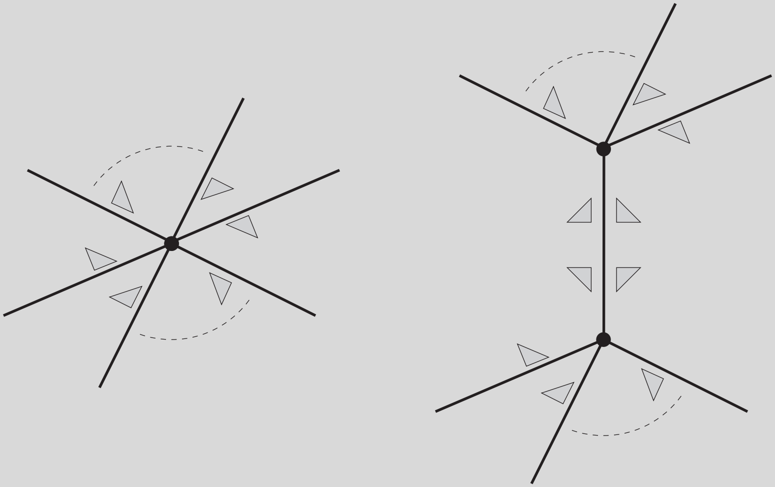
$BS(\mathcal{M})$ and $Co(\mathcal{M})$.

A map is orientable if and only if its flag graph is bipartite.



τ_0 red
τ_2 green
τ_1 blue

5. Vertex splitting and edge contraction



Vertex split/edge contraction.

Number of faces stays the same. Contraction of a loop is undefined.

6. Reduction to unitary maps

A map is called *unitary* if it has exactly one vertex and one face. By a sequence of edge contractions one can reduce the number of vertices of a map M, or the number of faces of M^*.

The flags $\{x, \tau_0 x, \tau_2 x, \mathcal{E}x\}$ of a unitary map are distributed among the two disjoint cycles of $\mathcal{V}$ by

$$(x, A, \mathcal{E}x, B)(\tau_2 B, \tau_0 x, \tau_2 A, \tau_2 x)$$

or

$$(x, A, \tau_0 x, B)(\tau_2 B, \mathcal{E}x, \tau_2 A, \tau_2 x)$$

Crosscap

A crosscap is assembled if A or B is empty. Crosscaps may be assembled by a sequence of vertex splits and edge contractions.

Rigidity of Bar . . .
Generic Rigidity . . .
Combinatorial Maps
The flag graph
Vertex splitting . . .
Reduction to . . .
Classification of . . .
Δ-matroids

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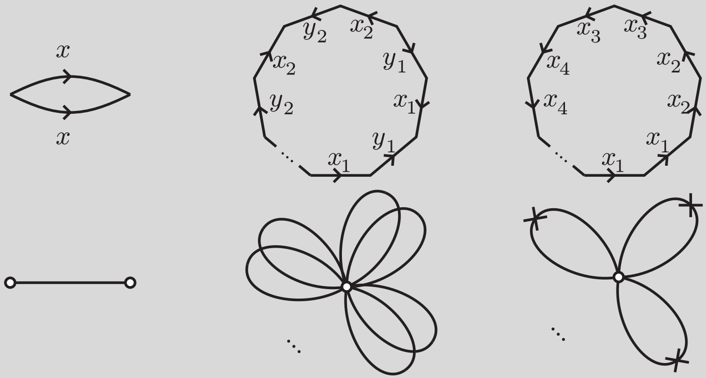
7. Classification of Surfaces

Normal forms



Map projection and polygon models of canonical normal forms.

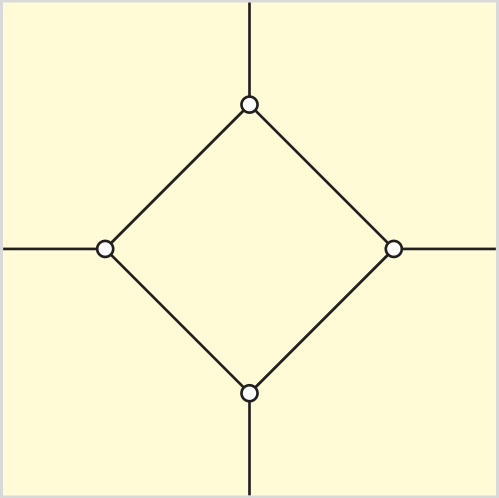
Theorem 2 *Every closed surface has the topological type of either*

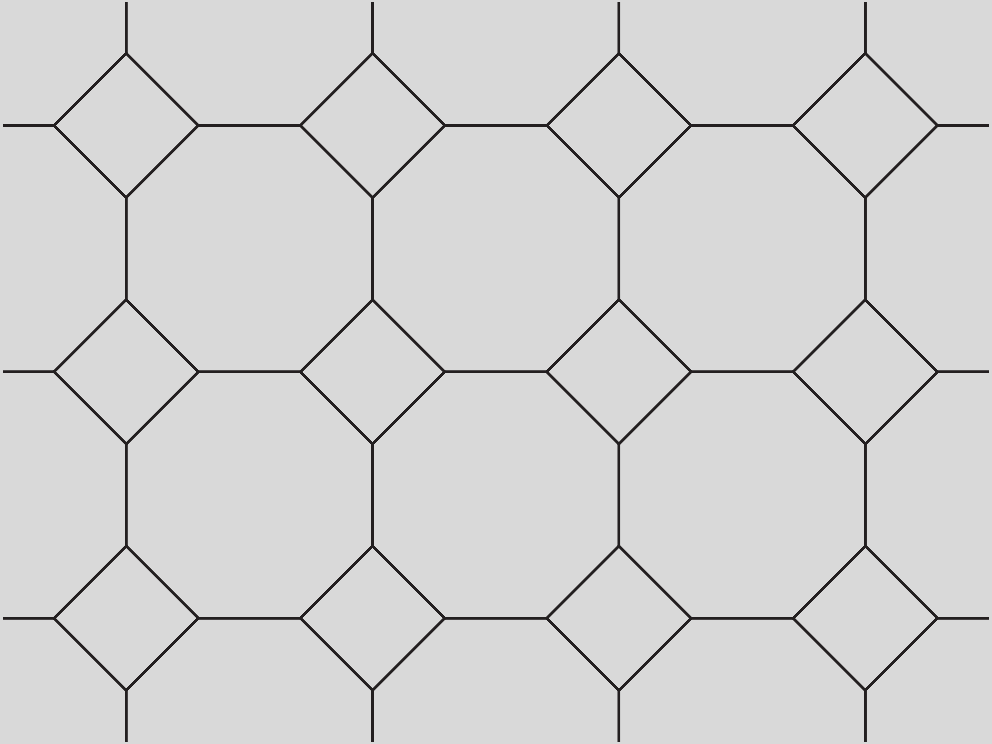
1. *The sphere.* $(\chi(S) = 2)$.
2. *A connected sum of n tori.* $(\chi(T^n) = 2(n-1))$.
3. *A connected sum of n projective planes.* $(\chi(P^n) = 2-n)$.

8. Δ-matroids

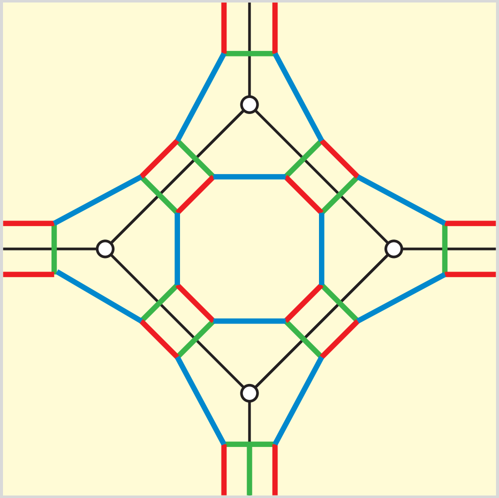
Introduced by Bouchet as set systems satisfying the symmetric exchange axiom
For $F', F'' \in \mathcal{F}$, $x \in F' \Delta F''$, there exists $y \in F'' \Delta F'$ such that $F' \Delta \{x, y\} \in \mathcal{F}$.
Feasible sets need not be equicardinal.
In [2] Bouchet associates a Δ-matroid to a map on a topological surface S by defining edge sets F feasible if $S - cl(F \cup \overline{F}^*)$ is connected. This easily translates to connectivity properties of the flag graph.

For example consider K_4 embedded on a torus.





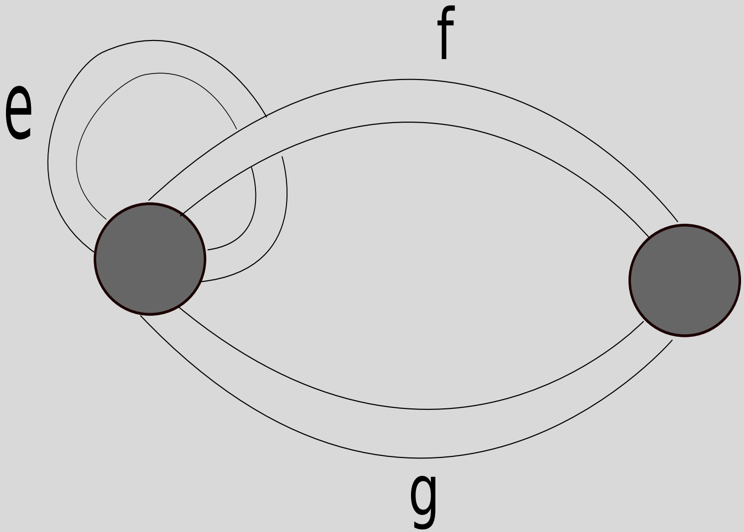
From each red-green square both edges of one color must be deleted without destroying connectivity of the flag graph.



Theorem 3 *Let $D(M)$ be the Δ-matroid of a map $M(G, S)$ and let $M^*(G^*, S)$ be the dual map. Then*

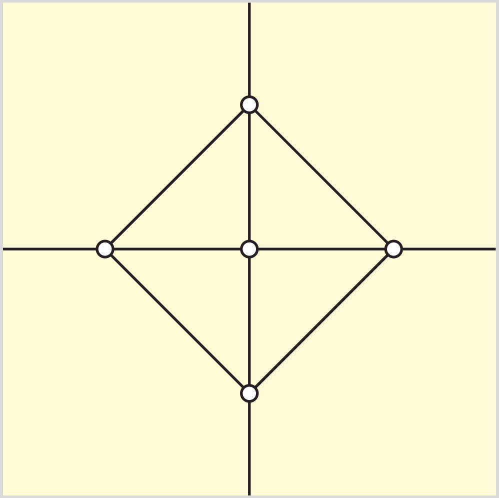
- $D(M^*) = D(M)^*$;
- *the lower matroid of D is the cycle matroid of G;*
- *the upper matroid of D is the co-cycle matroid of G^*;*
- $w(D) = 2 - \chi(S)$.

Note that there are connections to 2-matroids and ribbon graphs. A good reference for ribbon graphs is [3], see also [1].



A ribbon graph.

K_5 on the torus



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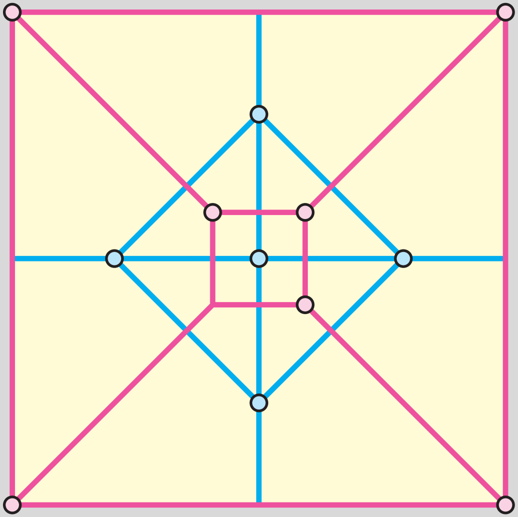

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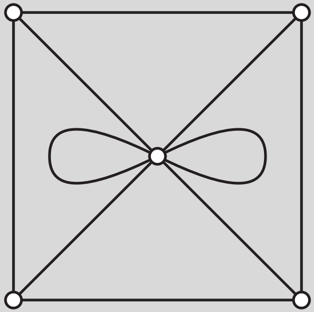
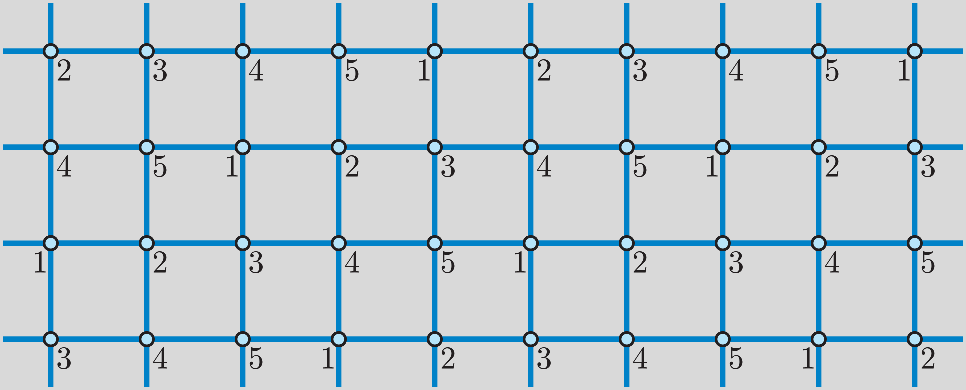


Another way to embed K_5 on the torus



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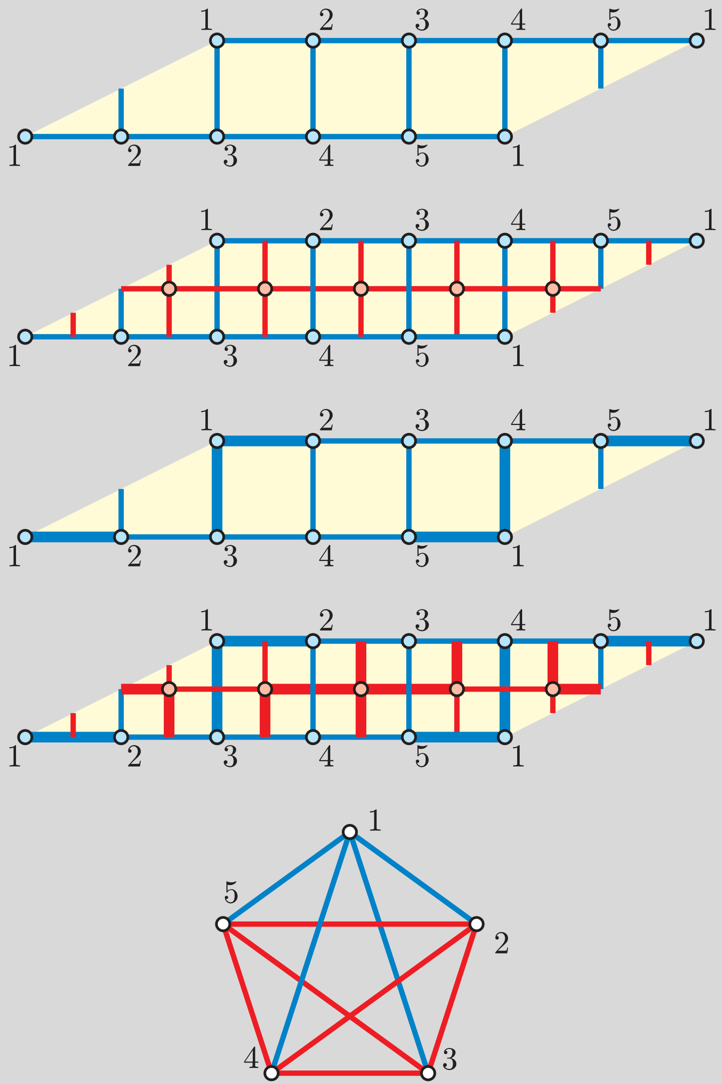

Observations

- The upper and lower matroid of the Δ-matroid of a map on the sphere are identical.
- The 2-dimensional generic rigidity matroid may be considered a Dilworth truncation of two connectivity matroids.
- For graphs embedded on other surfaces we may define a rigidity matroid as (truncation) of two upper matroids of the corresponding Δ-matroid.

To appear: Handbook of geometric constraint systems

References

[1] Remi Cocou Avohou. Polynomial invariants for arbitrary rank D weakly-colored stranded graphs. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 12:Paper No. 030, 23, 2016.

[2] André Bouchet. Maps and $\triangle$-matroids. *Discrete Math.*, 78(1-2):59–71, 1989.

[3] Joanna A. Ellis-Monaghan and Iain Moffatt. *Graphs on surfaces*. SpringerBriefs in Mathematics. Springer, New York, 2013. Dualities, polynomials, and knots.

[4] B. Jackson, T. A. McCourt, and A. Nixon. Necessary conditions for the generic global rigidity of frameworks on surfaces. *Discrete Comput. Geom.*, 52(2):344–360, 2014.

[5] Bill Jackson and Anthony Nixon. Stress matrices and global rigidity of frameworks on surfaces. *Discrete Comput. Geom.*, 54(3):586–609, 2015.

[6] A. Nixon, J. C. Owen, and S. C. Power. A characterization of generically rigid frameworks on surfaces of revolution. *SIAM J. Discrete Math.*, 28(4):2008–2028, 2014.

[7] Anthony Nixon and Elissa Ross. Periodic rigidity on a variable torus using inductive constructions. *Electron. J. Combin.*, 22(1):Paper 1.1, 30, 2015.

[8] William T. Tutte. What is a map? In *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971)*, pages 309–325. Academic Press, New York, 1973.