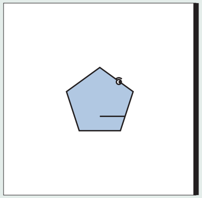
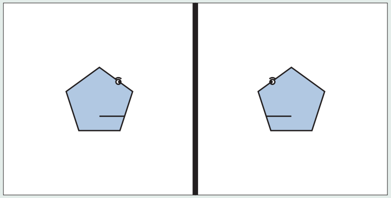
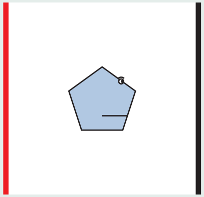
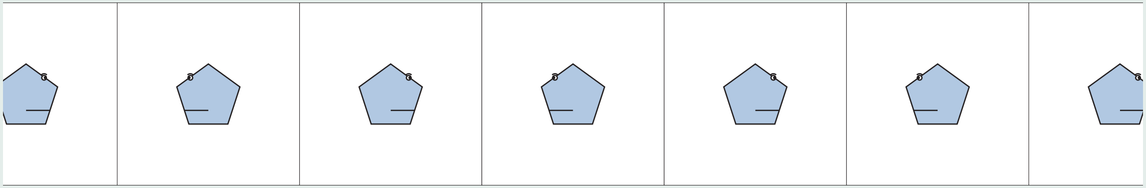
1. Mirrors and Symmetry



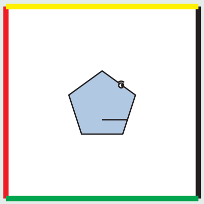




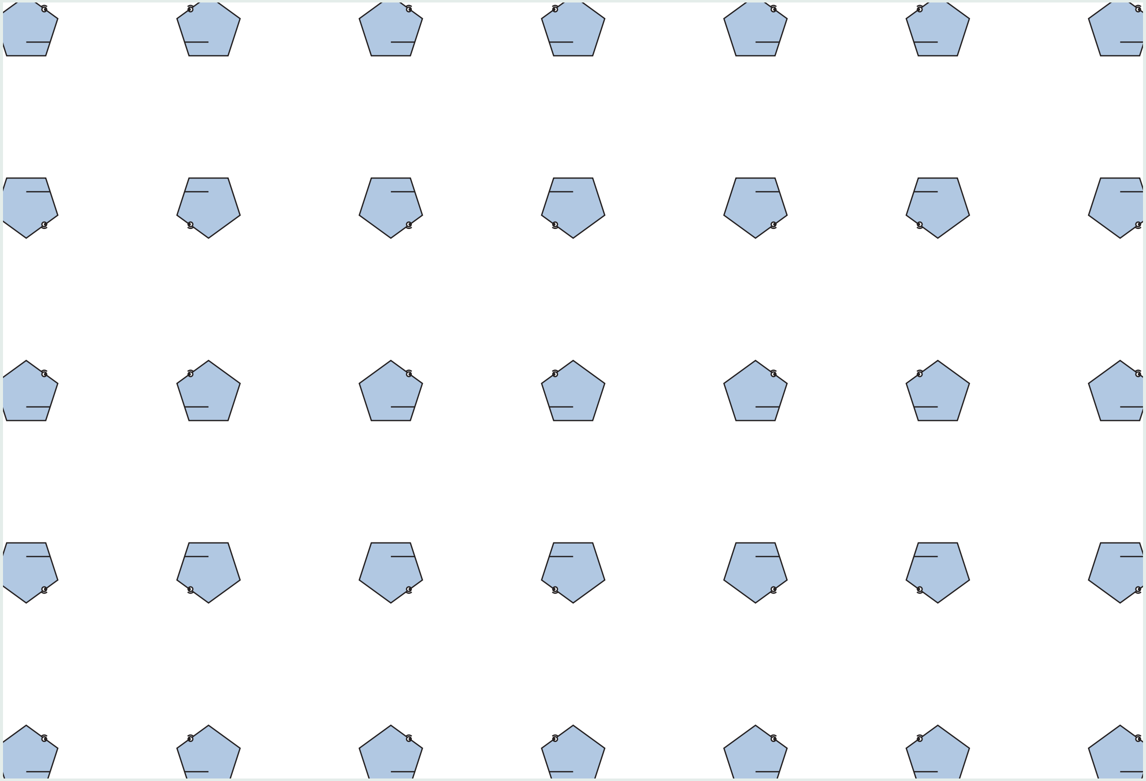
Two mirrored walls



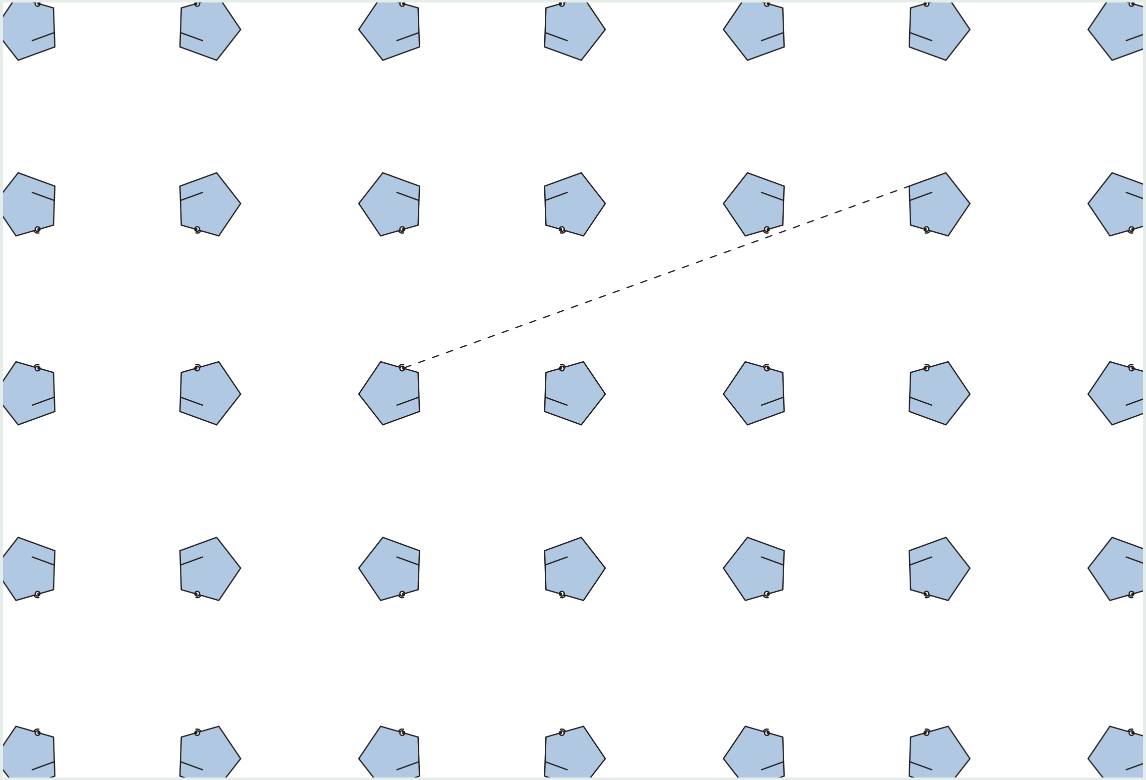
A glimpse of infinity - a repeating linear pattern



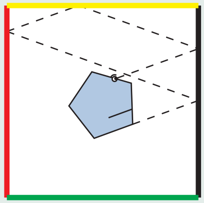
Four mirrored walls



A doubly periodic pattern with with four orientations.



Mr. Pentagon looks at one of his chins.



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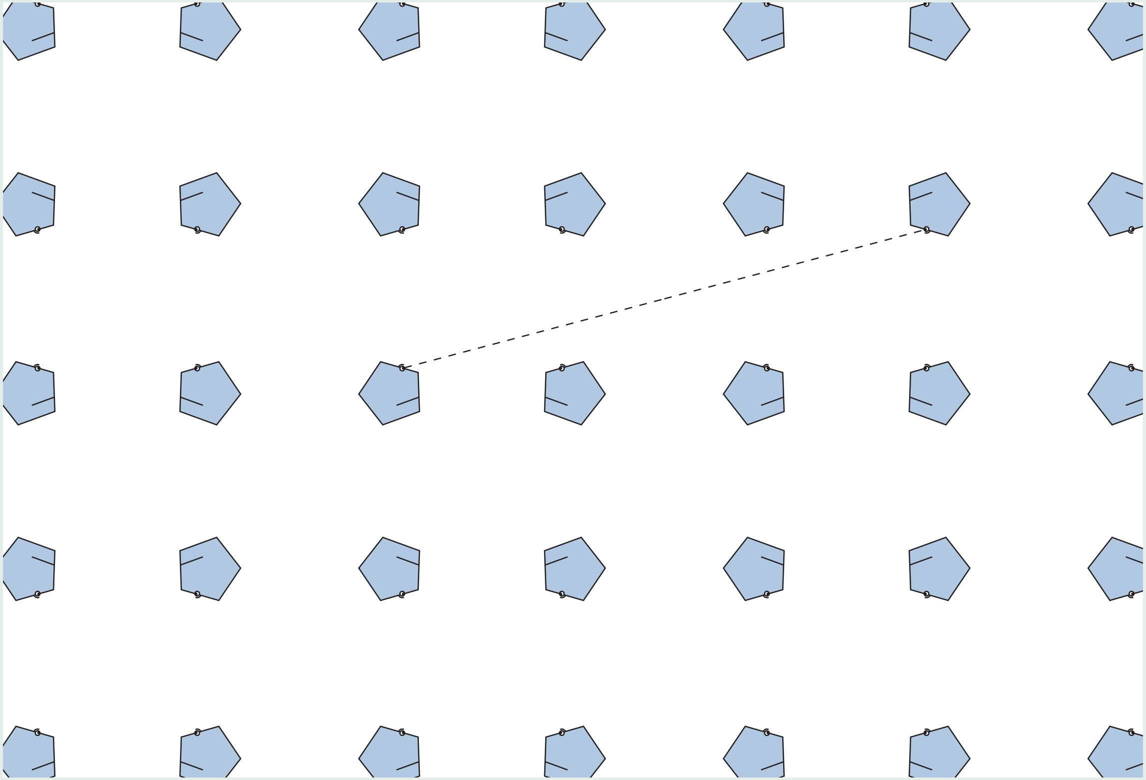
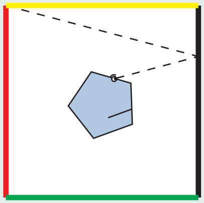
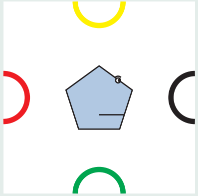


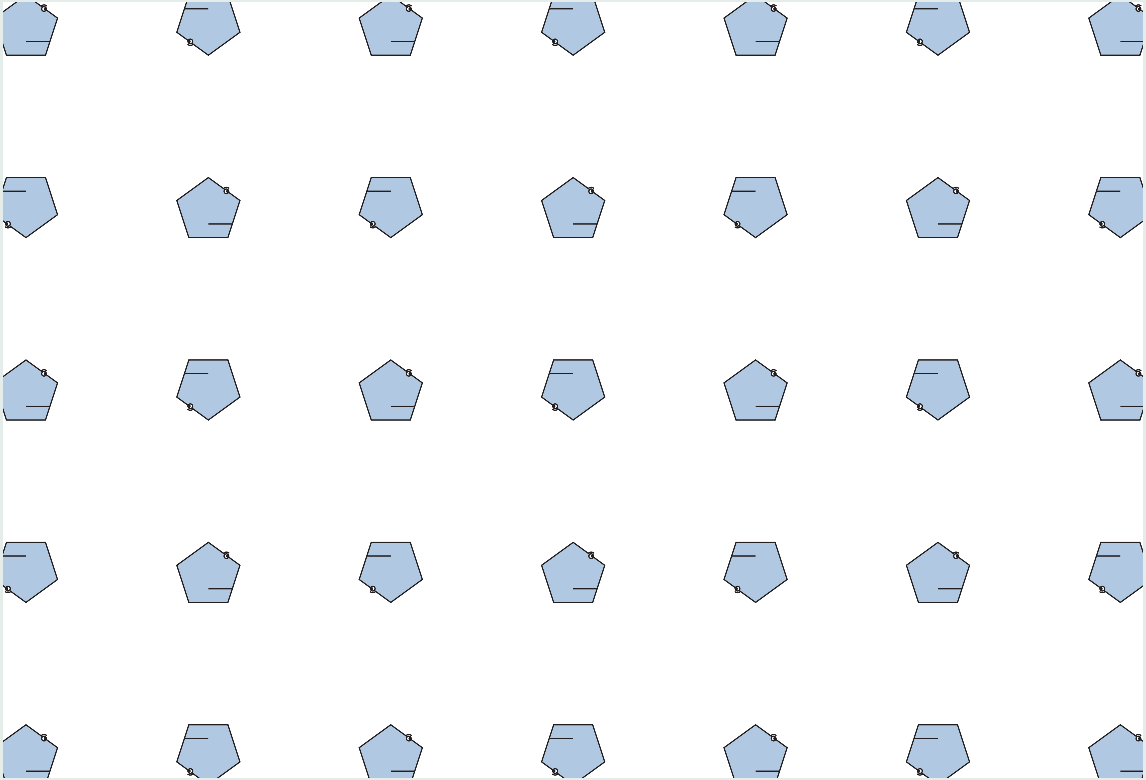
Mr. Pentagon looks himself in the eye. . .



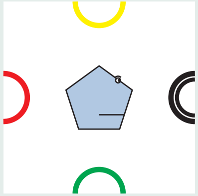
Does it always go through a corner?



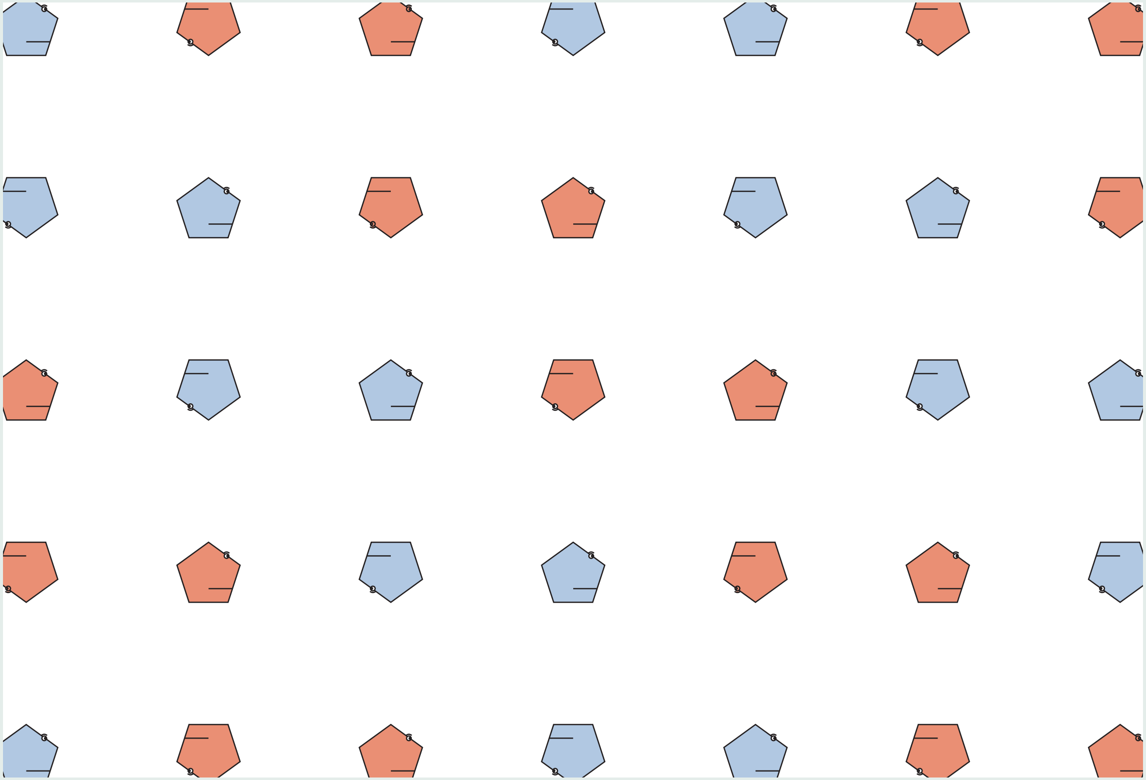
A room with 180° rotational views



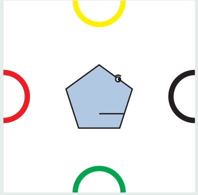
A room with 180° rotational views (how many orientations?)

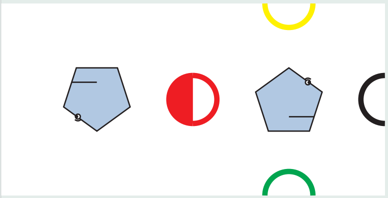


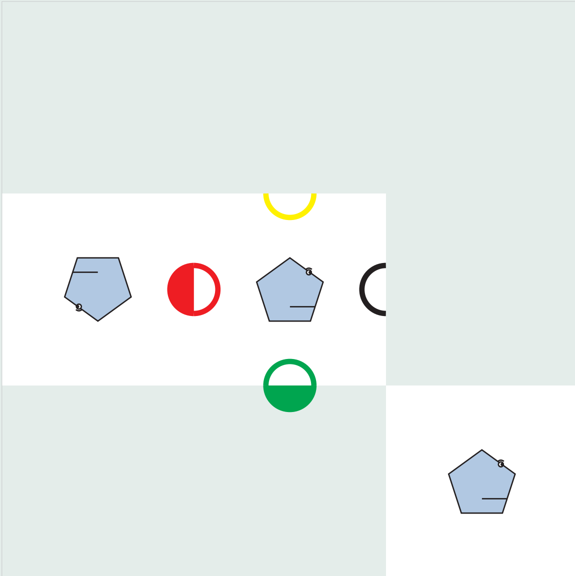
One of the walls inverts colors . . .

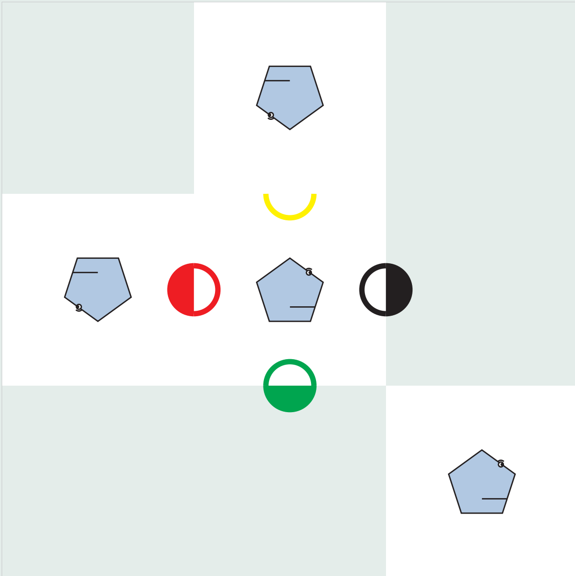


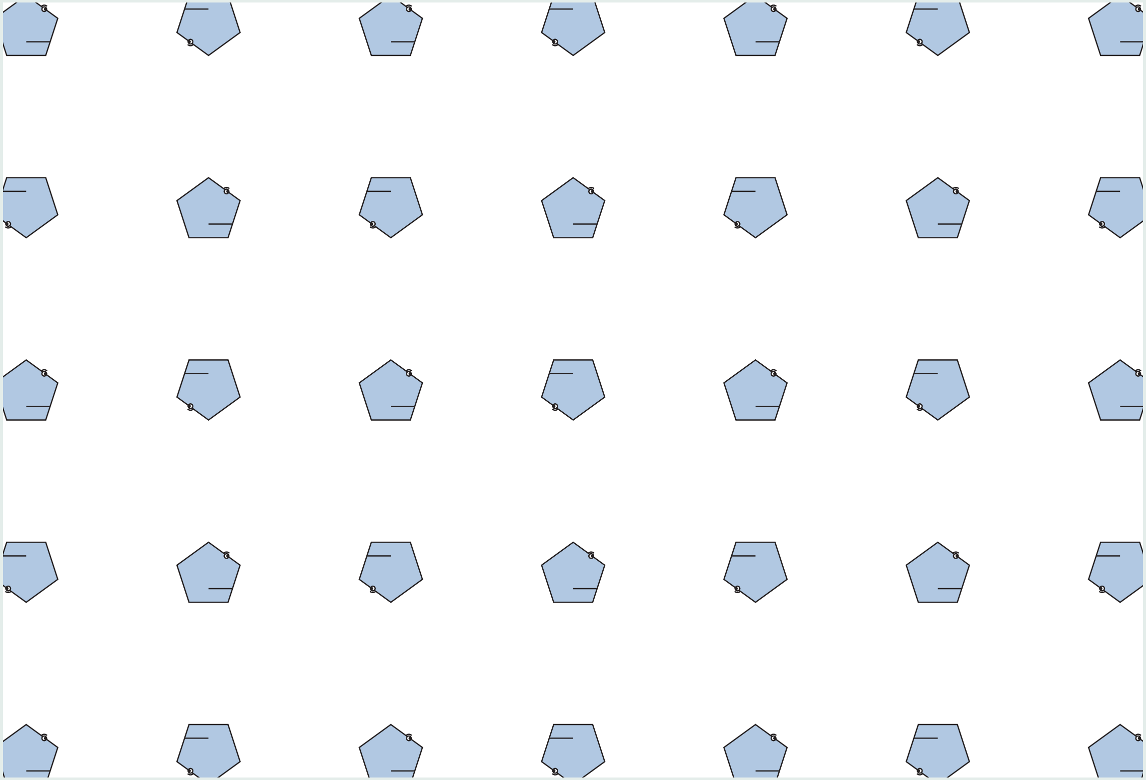
One of the walls inverts colors . . . or is it two . . .











So how many such colored patterns are there?

Mirrors and Symmetry
Geometry to Graphs
Connectivity
The Program
Bibliography

Home Page

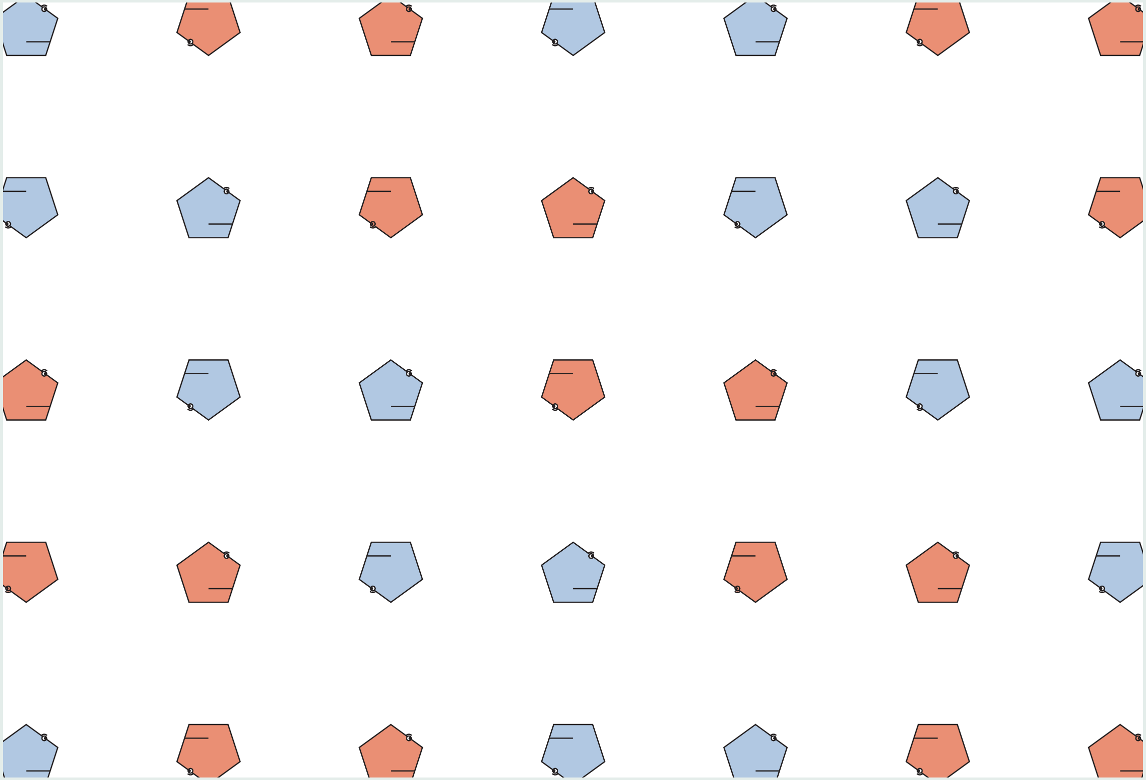
Title Page

Go Back

Full Screen

Close

Quit



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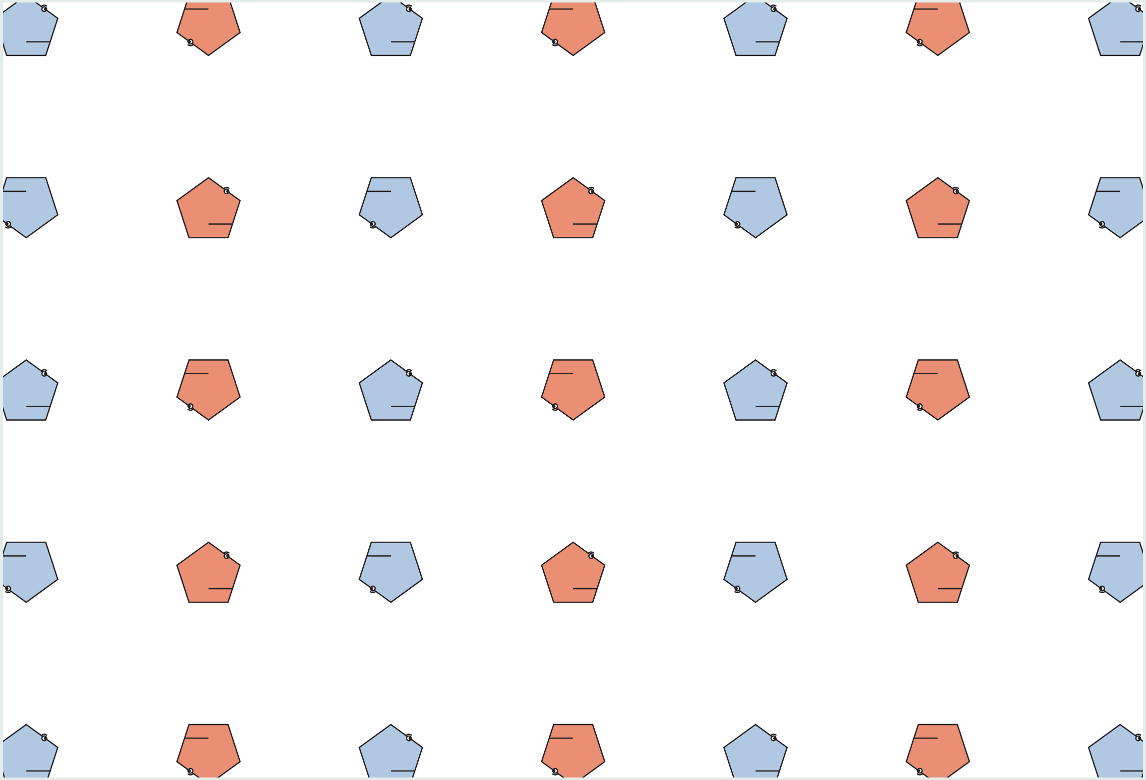
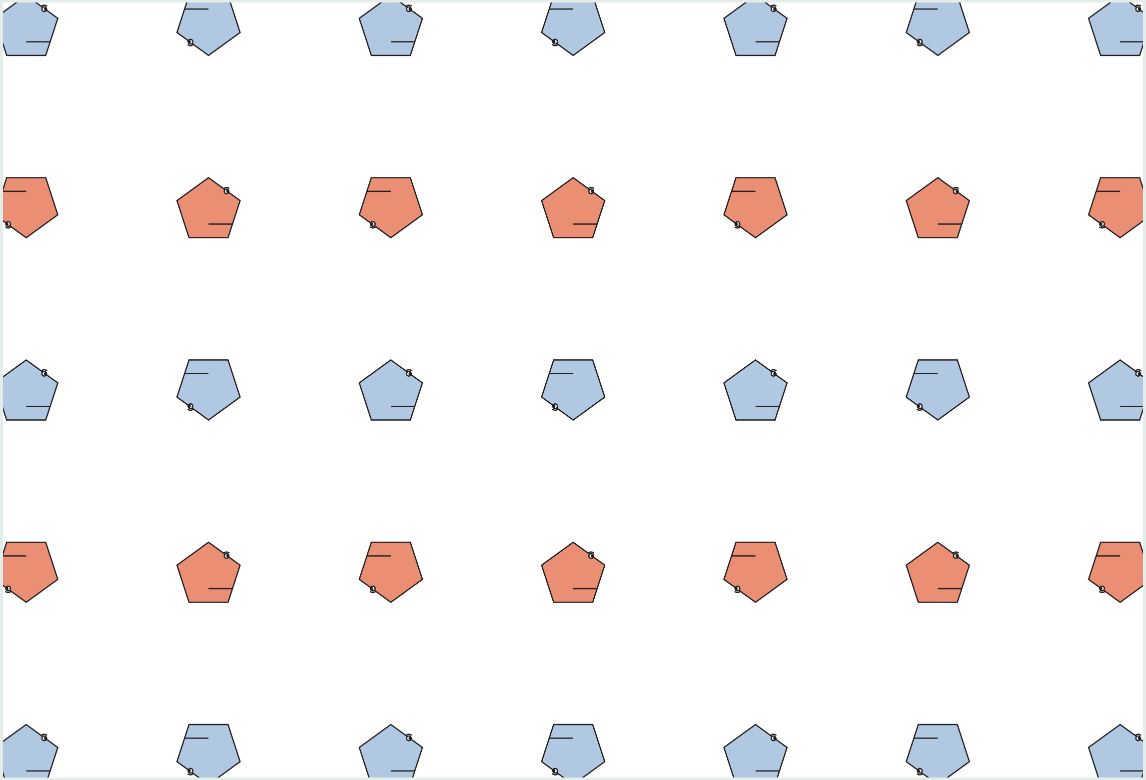


So how many such colored patterns are there?



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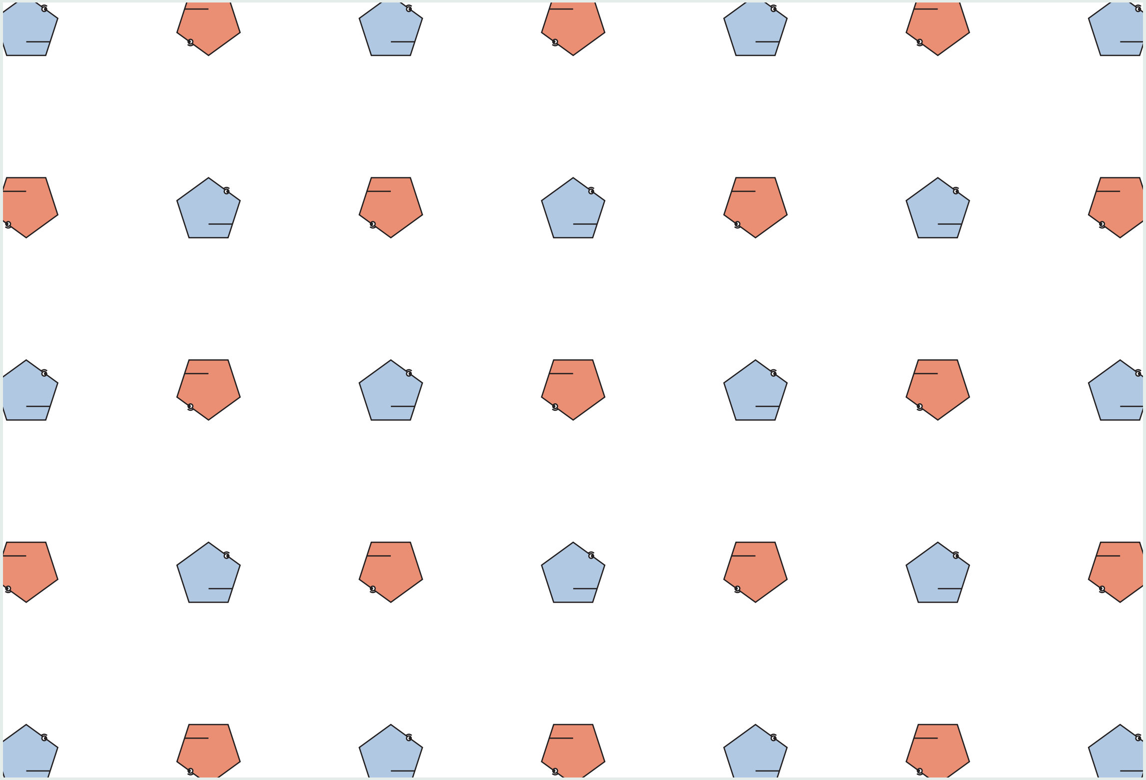

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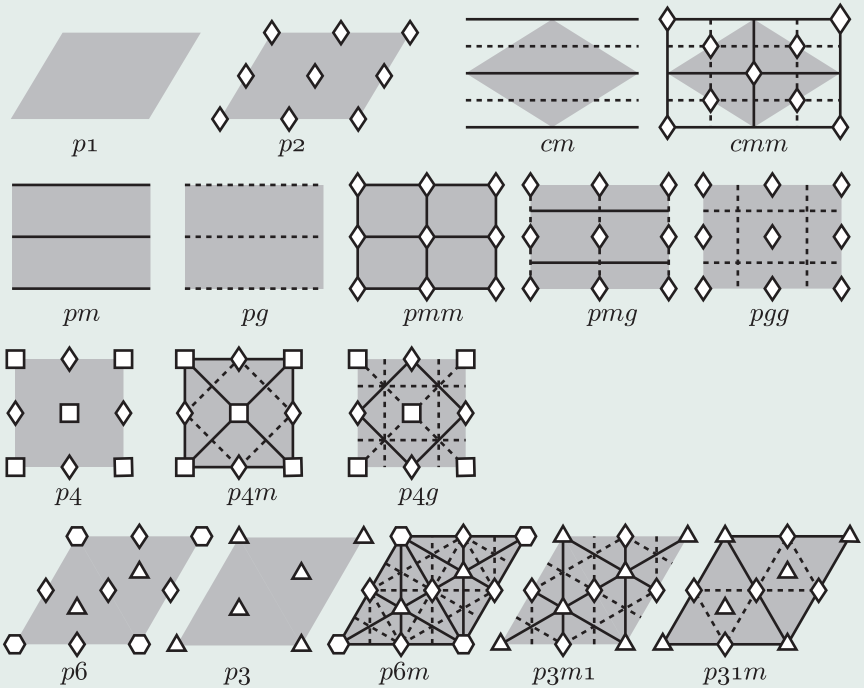
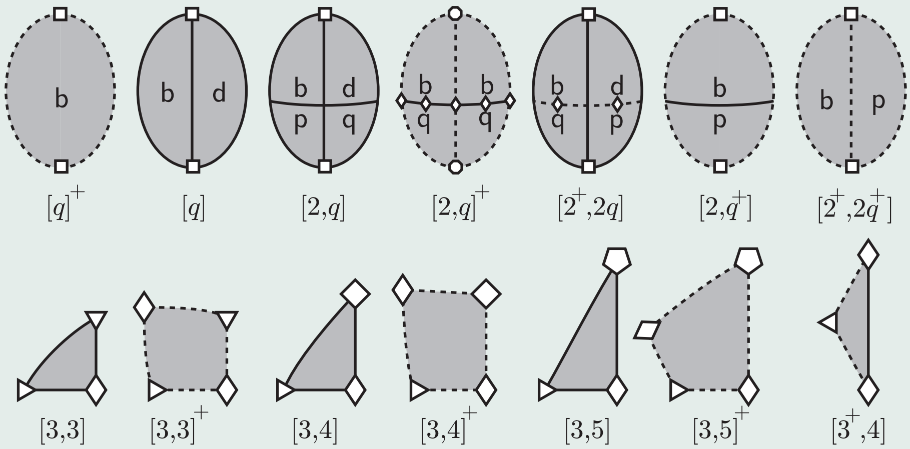


17 plane crystallographic groups [Fedorov [3]]

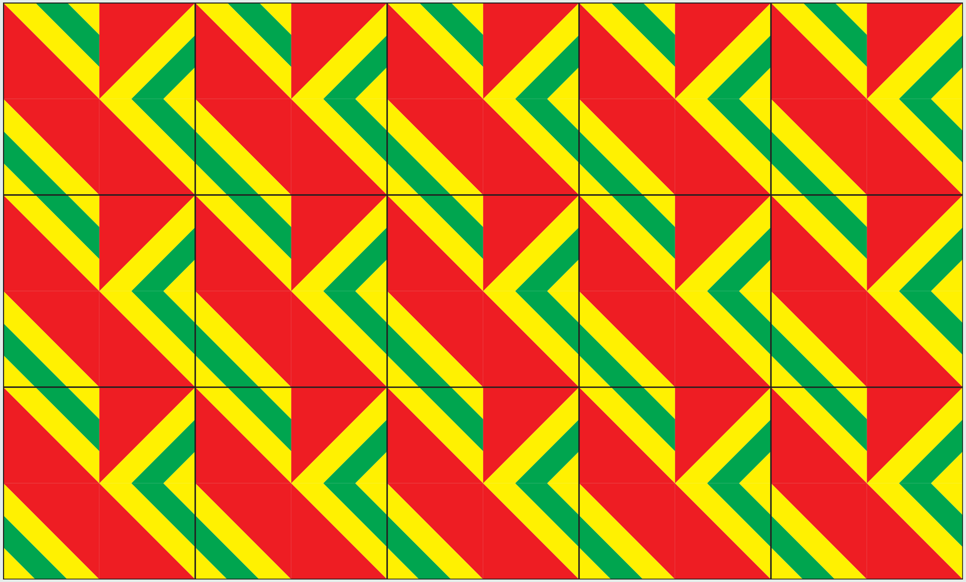
46 two-colored crystallographic groups [Coxeter [1]]



Spherical groups

Application 1: Pattern counting

M. C. Escher [2, 8, 6] laboriously examined multitudes of sketches to determine how many different patterns would result by repeatedly translating a 2×2 square having its four unit squares filled with copies of an asymmetric motif in any of four aspects.

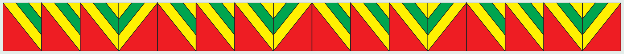


The 1×4 case: Escher revisited

1 *bbbb*

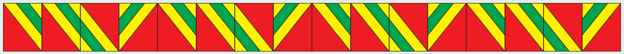

2 *bbbq*


3 *bbbp*


4 *bbbd*


5 *bbqq*


6 *bbqp*


7 *bbqd*


8 *bbpq*


9 *bbpp*


10 *bbpd*


11 $bbdq$ 

12 $bbdp$ 

13 $bbdd$ 

14 $bqbq$ 

15 $bqbp$ 

16 $bqbd$ 

17 $bqpd$ 

18 $bqdp$ 

19 $bpbp$ 

20 $bdbp$ 

21 $bdbd$ 

22 $bdqp$ 

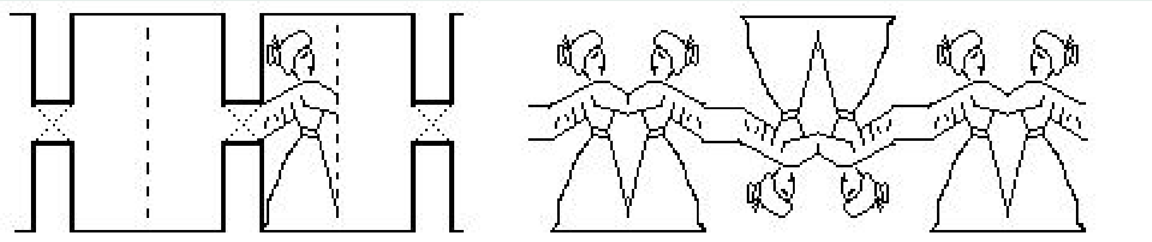
23 $bdpq$ 

Summary [7, 8]

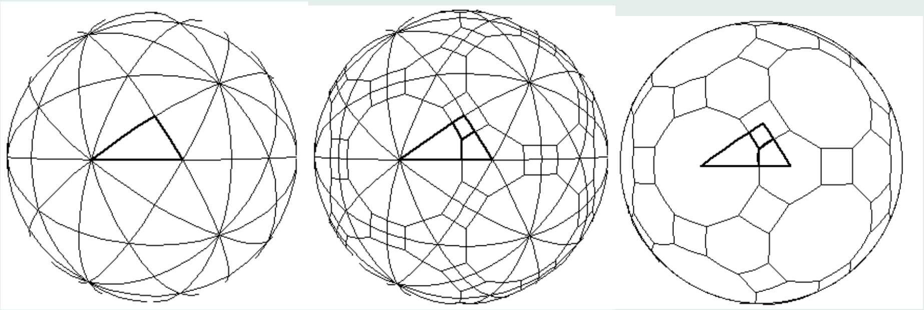
1. $f(n)$ – A053656 in Sloan's On-Line Encyclopedia of integer sequences [15]
2. $G(n) \approx 2F(n)$
3. $G(n) \approx 4^n/(4n)$ $(G(p) = \lceil 4^n/(4n) \rceil$ for $n = p$ prime)
4. Symmetric motifs
5. Over/Under weave motifs
6. Multiple motifs

Strip patterns

All frieze patterns may be realized in folding paper dolls [9] and [4].

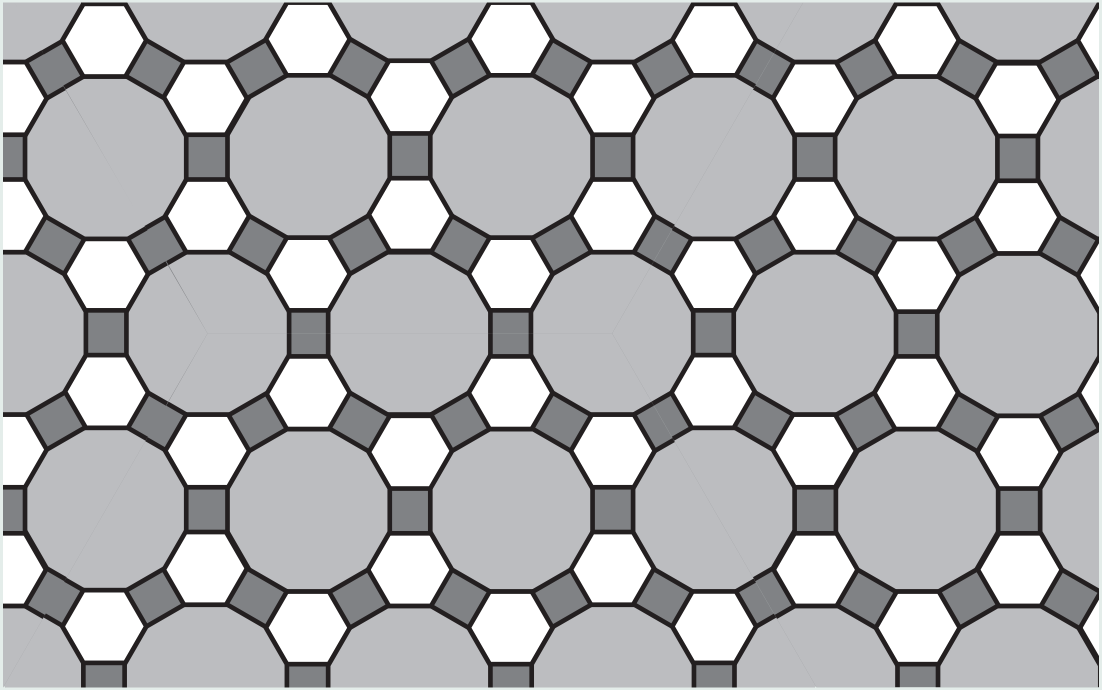


Application 2: Cayley Graphs

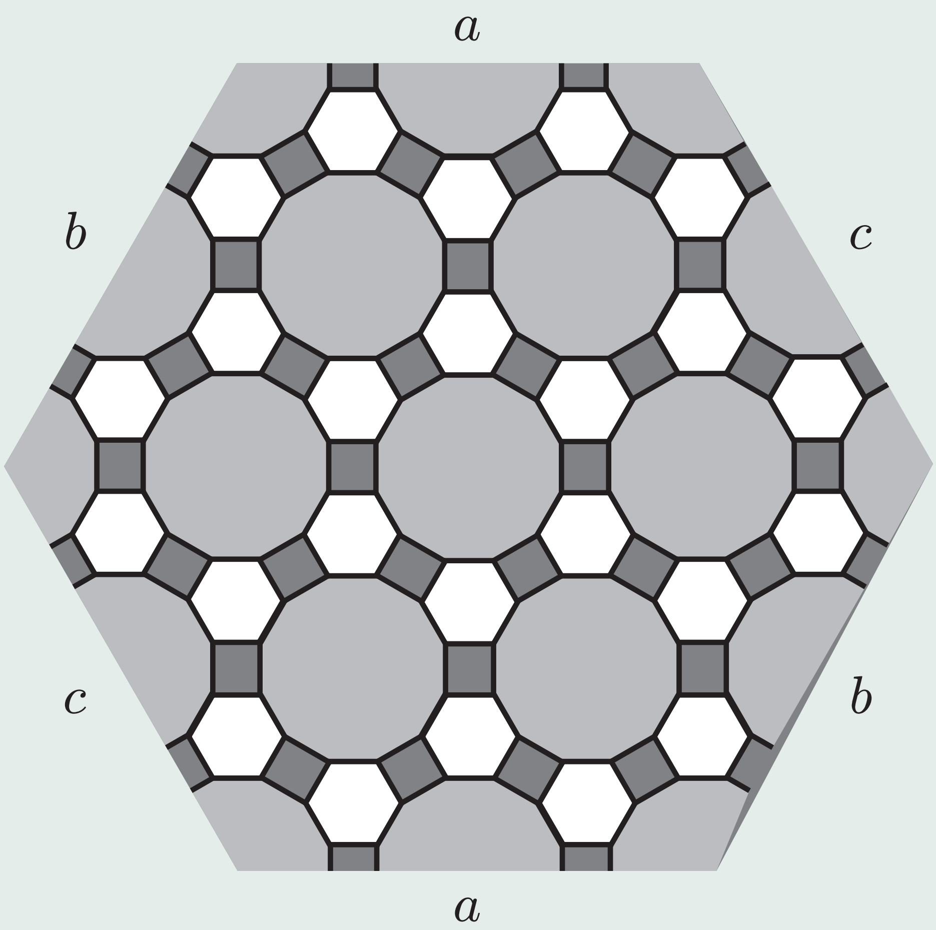


Vertices: The orbit of any point in a fundamental region
Edges: Generators mapping to incident fundamental regions

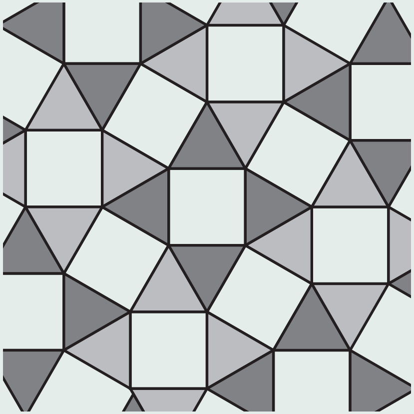
Result: A symmetrically embedded Cayley graph.

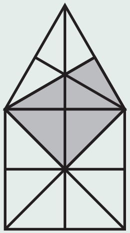


Analogous construction in the Euclidean plane
The Cayley graph of the $30 - 60 - 90$ triangle group.

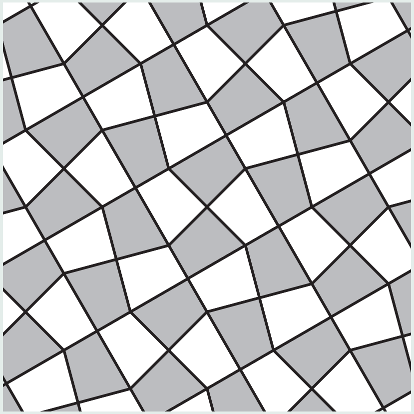


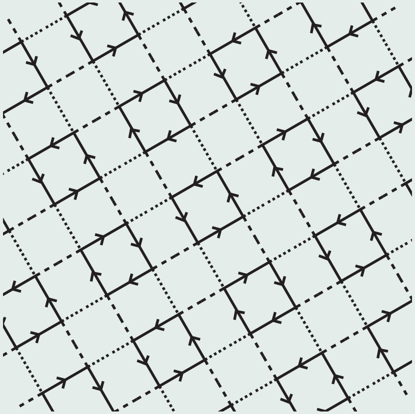
Analogous construction on the torus





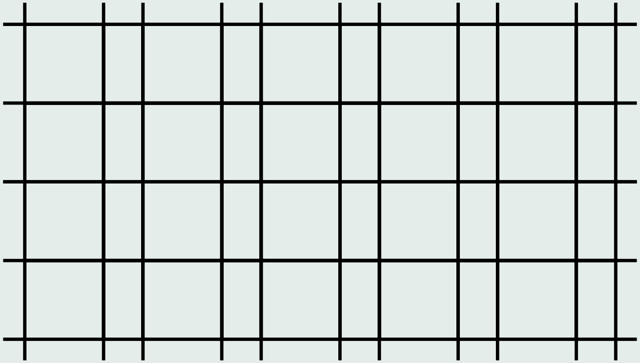
A symmetric tiling - a choice of fundamental region.

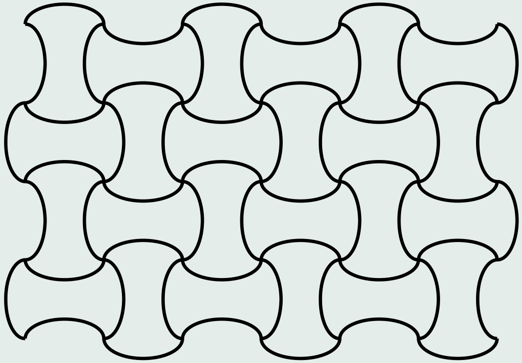




Fundamental region tiling – Cayley graph of symmetry group

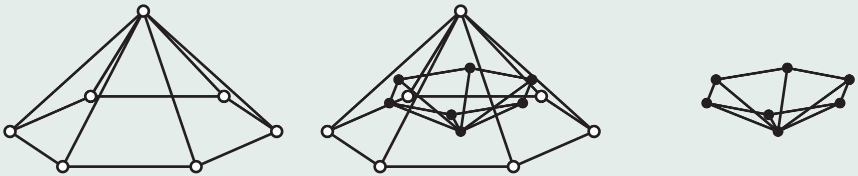
Homework





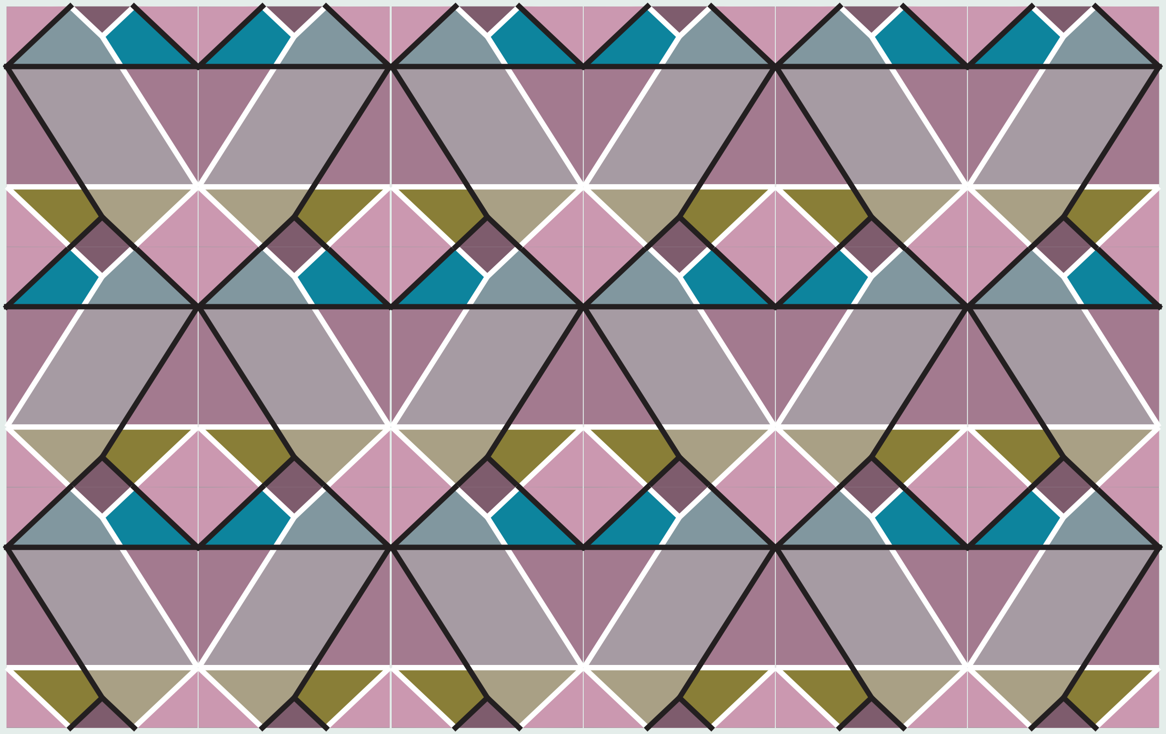
Application 3: Self–Duality

[13, 11, 10, 12, 14]



A self-dual polyhedron.

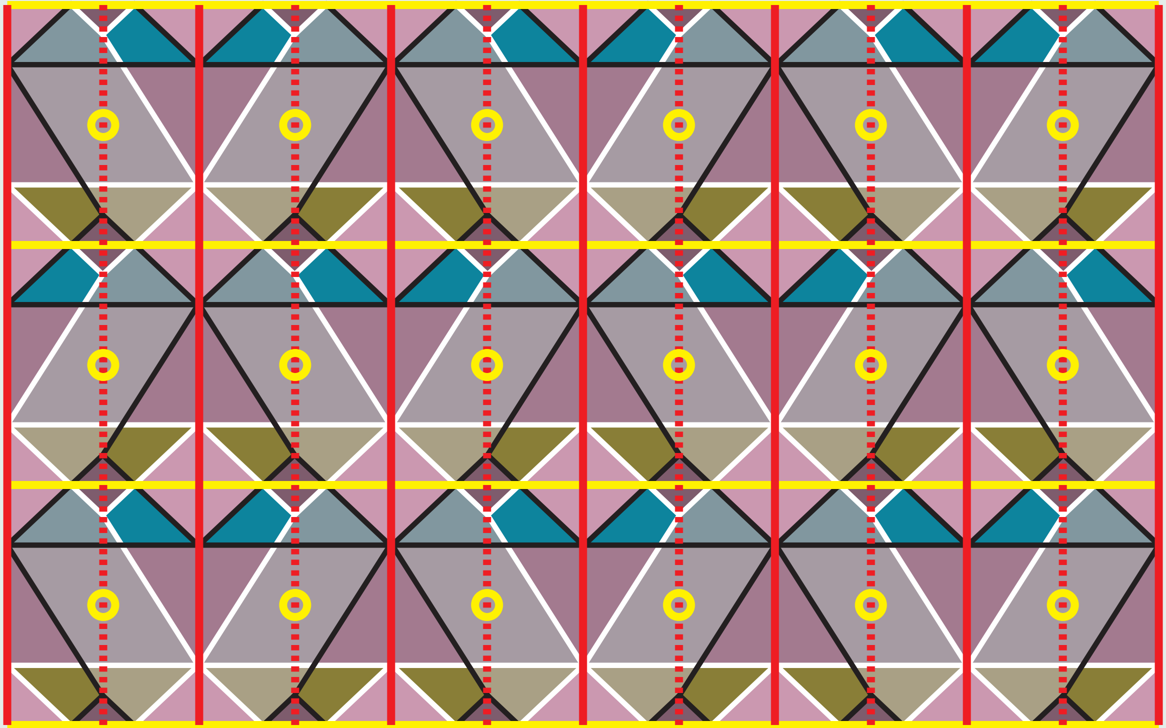
Add polarity to the Euclidian group.



Self–dual tilings –

$$cmm \triangleright cm$$

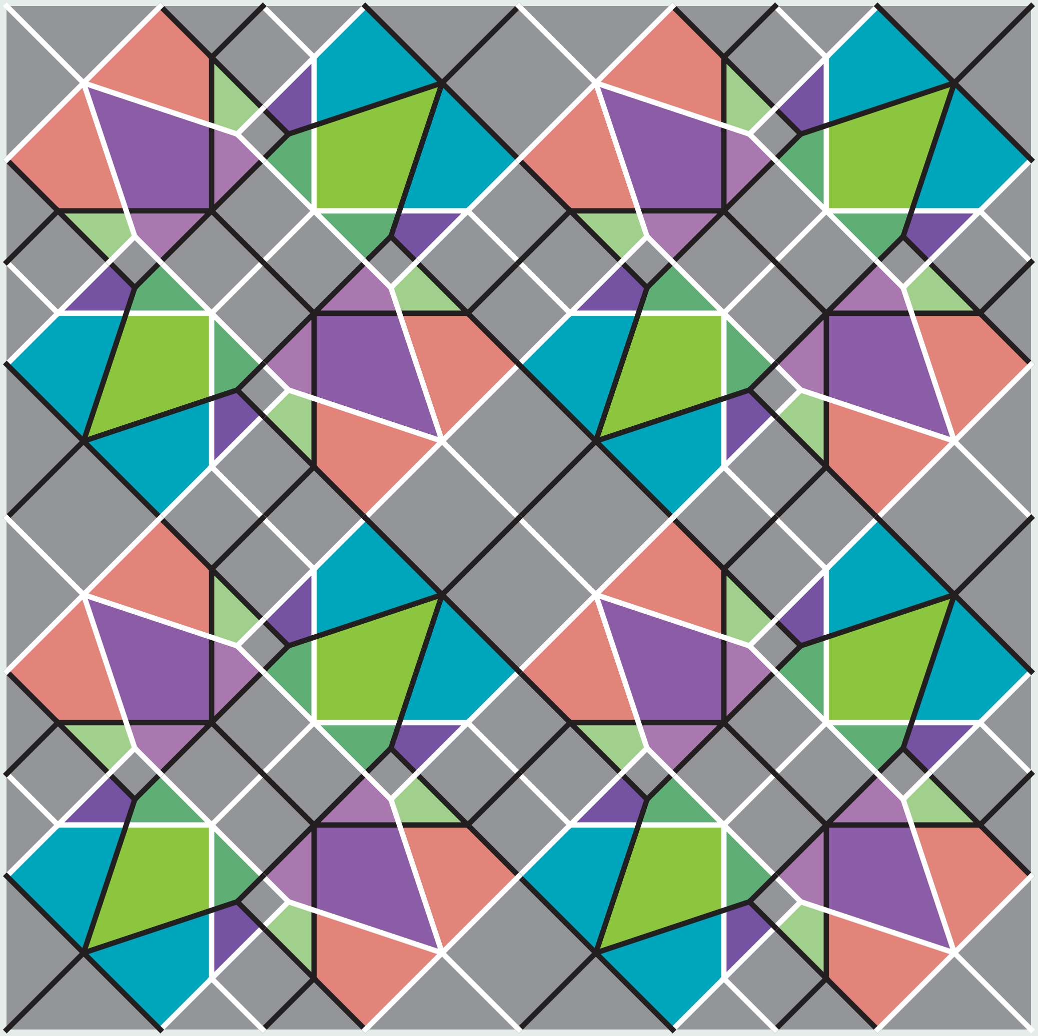
.

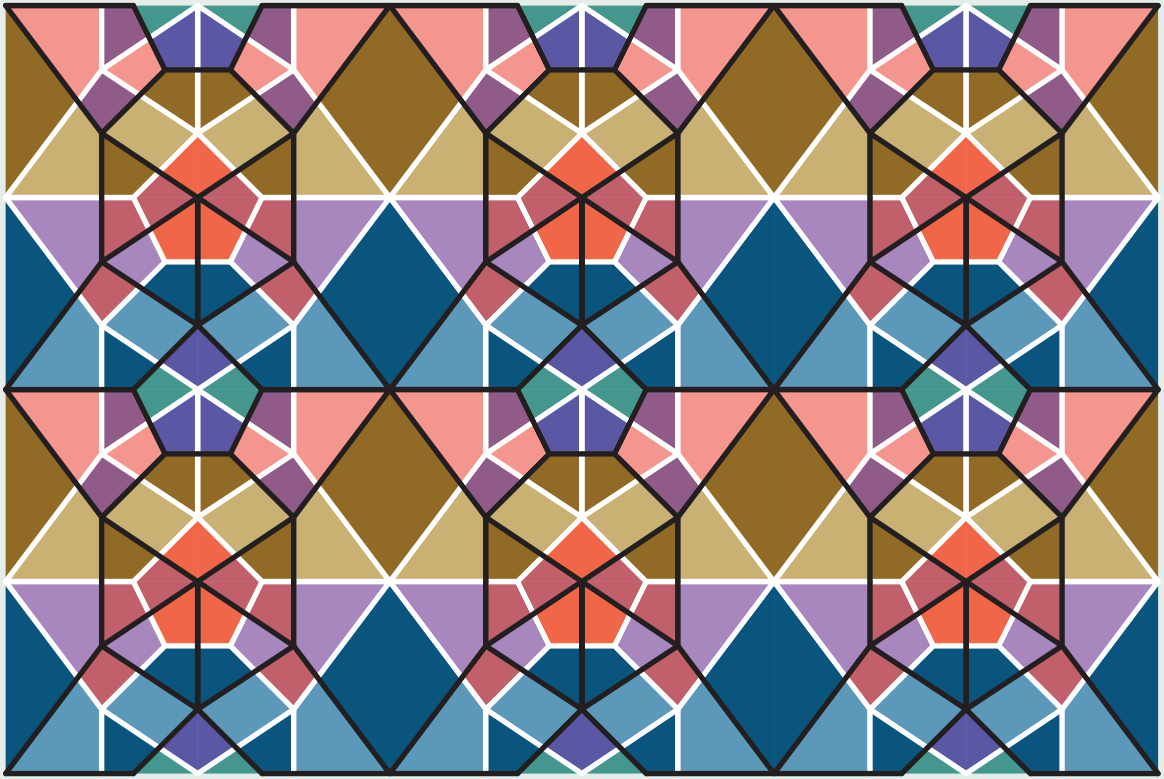


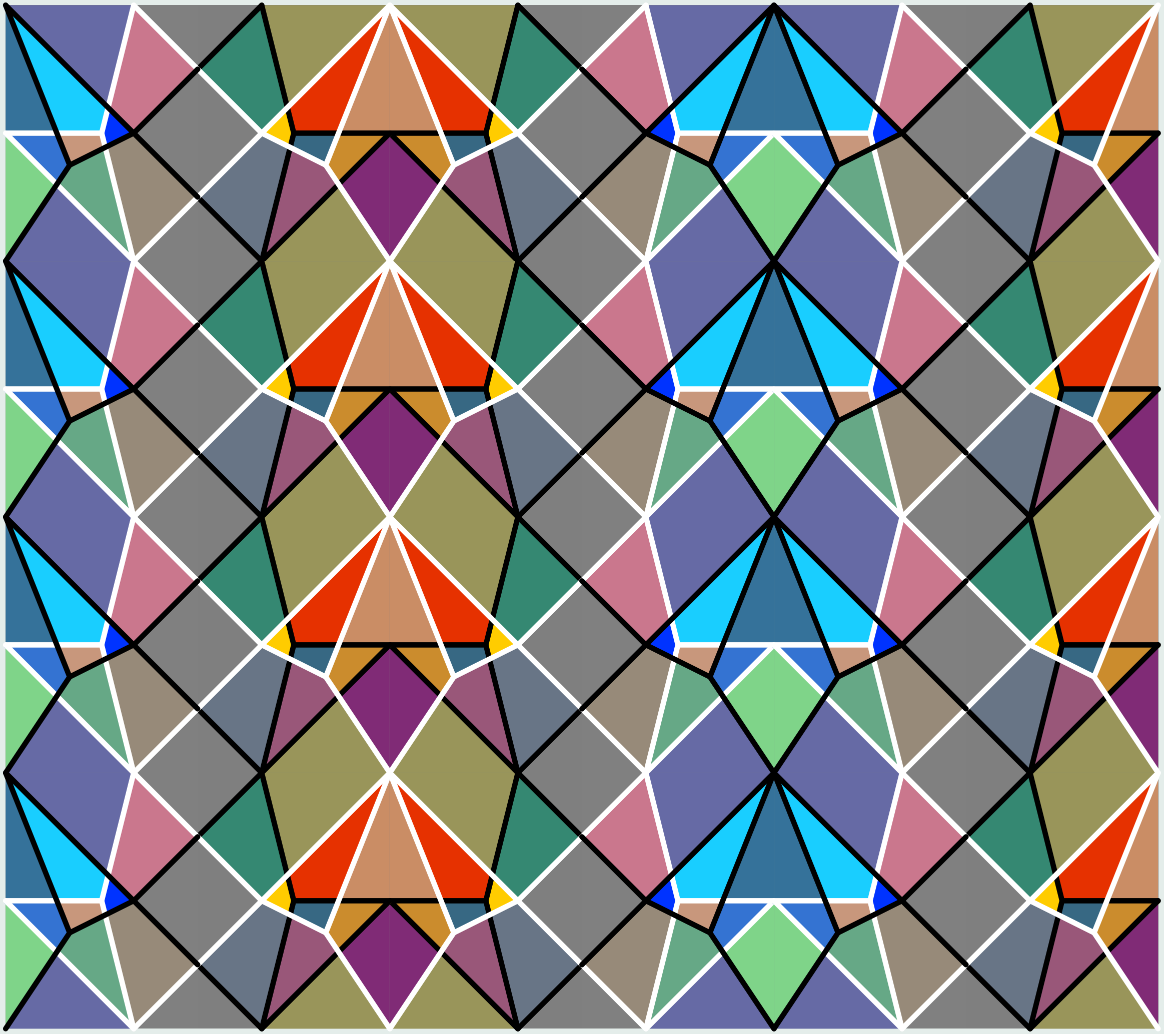
Self–dual tilings –

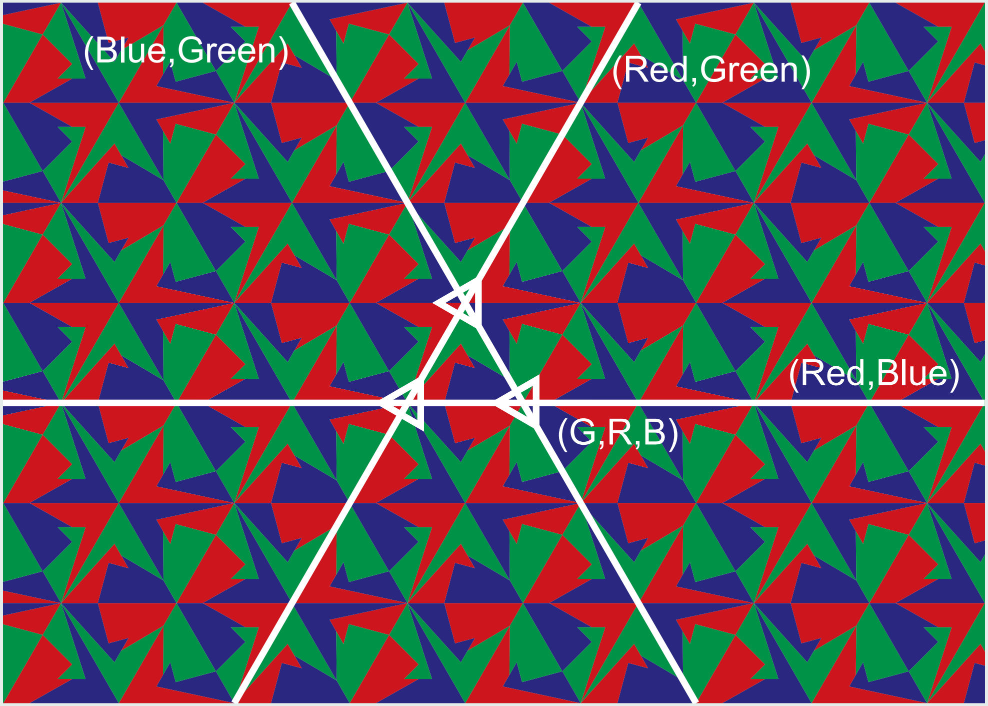
$$cmm \triangleright cm$$

.



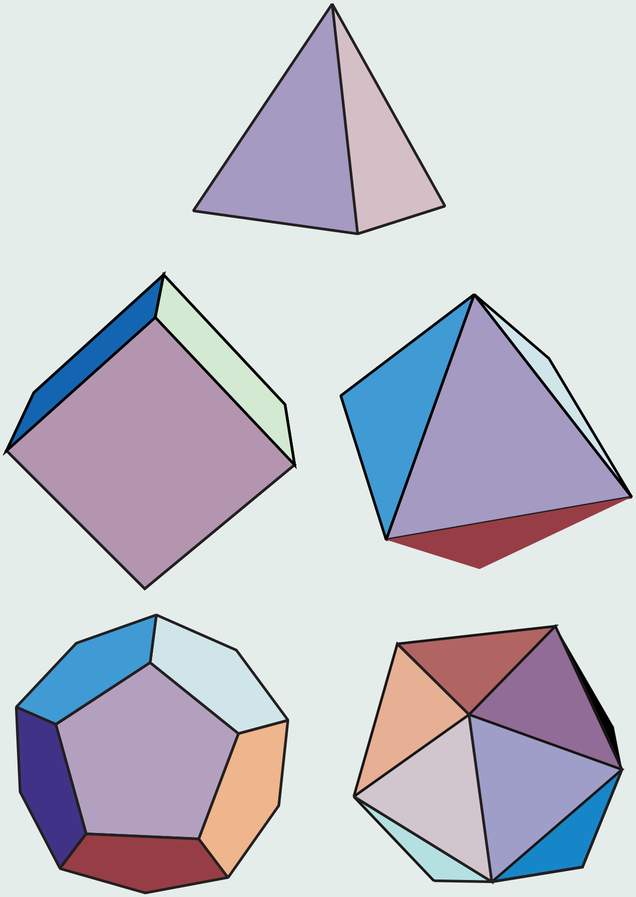




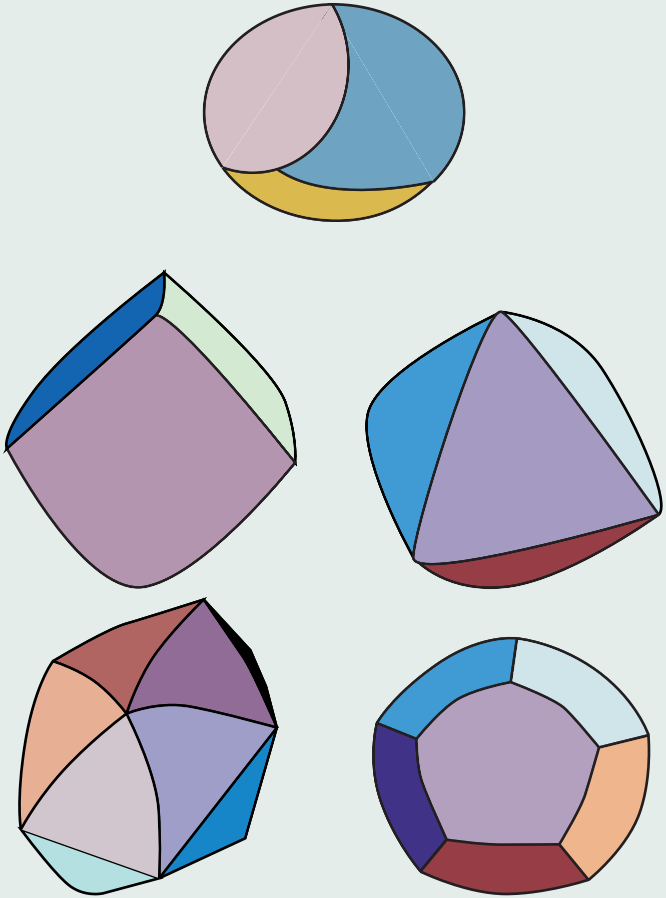


There are also three-color groups

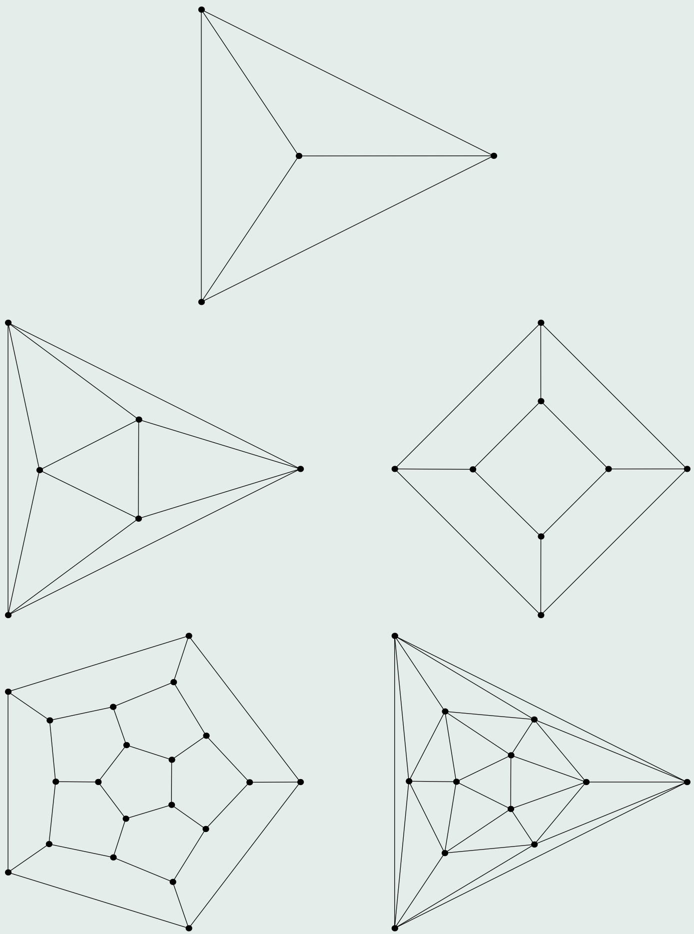
2. Geometry to Graphs



–To a geometer - Platonic Solids

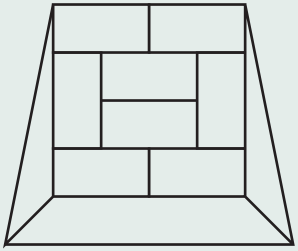


–To a topologist - Platonic Maps

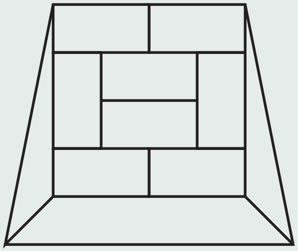


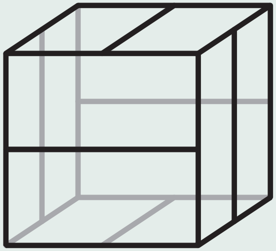
–To a graph theorist - Platonic Graphs

How many automorphisms?

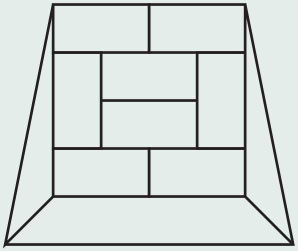


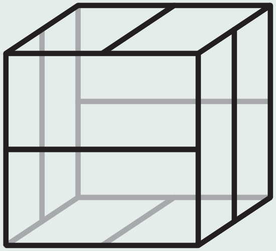
How many automorphisms?

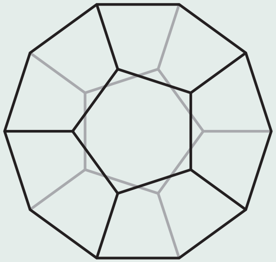


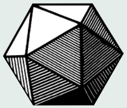


How many automorphisms?

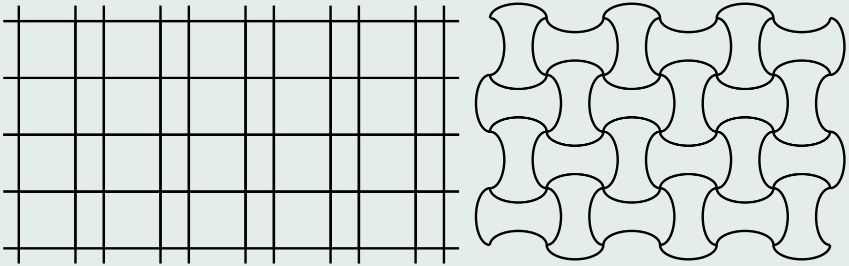








Straightening Lemma



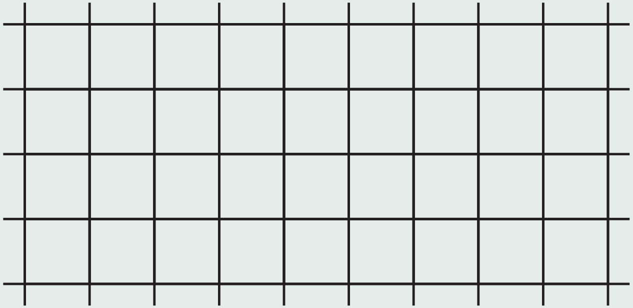
Theorem (H. Maschke)[5]

The automorphism group of a finite planar graph is isomorphic to one of the discrete groups of the sphere.

Straightening Lemma

Given a locally finite planar 3-connected graph, then it has a drawing on either the sphere, the Euclidean plane or the hyperbolic plane such that all its automorphism group is realized by isometries. [14].

Straightening Lemma



Theorem (Mashke)

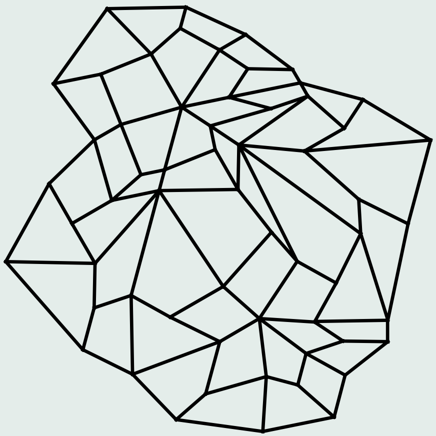
The automorphism group of a finite planar graph is isomorphic to one of the discrete groups of the sphere [5].

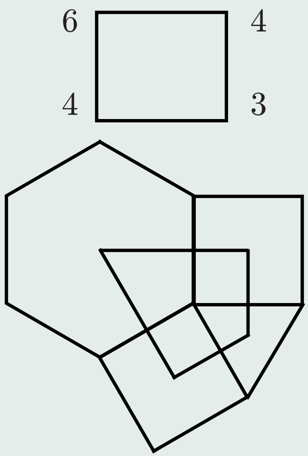
Straightening Lemma

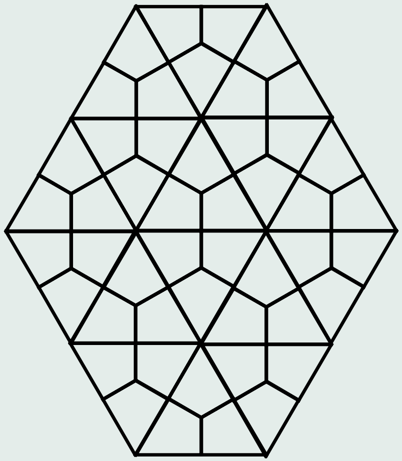
Given a locally finite planar 3-connected graph, then it has a drawing on either the sphere, the Euclidean plane or the hyperbolic plane such that all its automorphism group is realized by isometries.

Algorithm

1. Choose a connected fundamental region
   you may need a barycentric subdivision first.
2. Record the intersection numbers at each vertex.
3. Assemble at a vertex regular n-gons.
4. Take the dual at the vertex







Mirrors and Symmetry
Geometry to Graphs
Connectivity
The Program
Bibliography

Home Page

Title Page

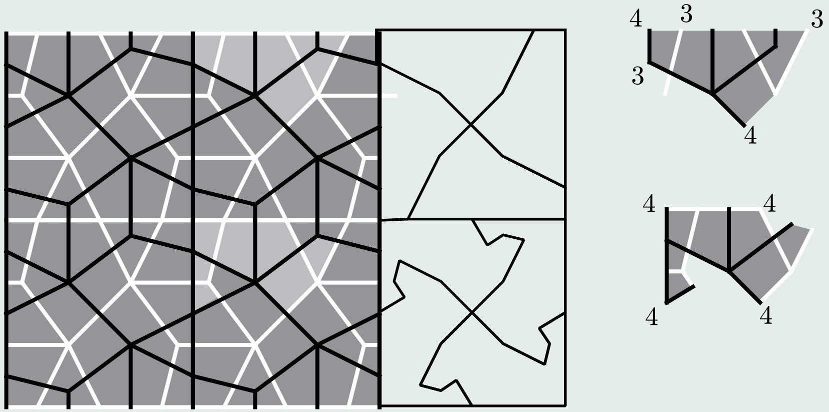
Page 52 of 71

Go Back

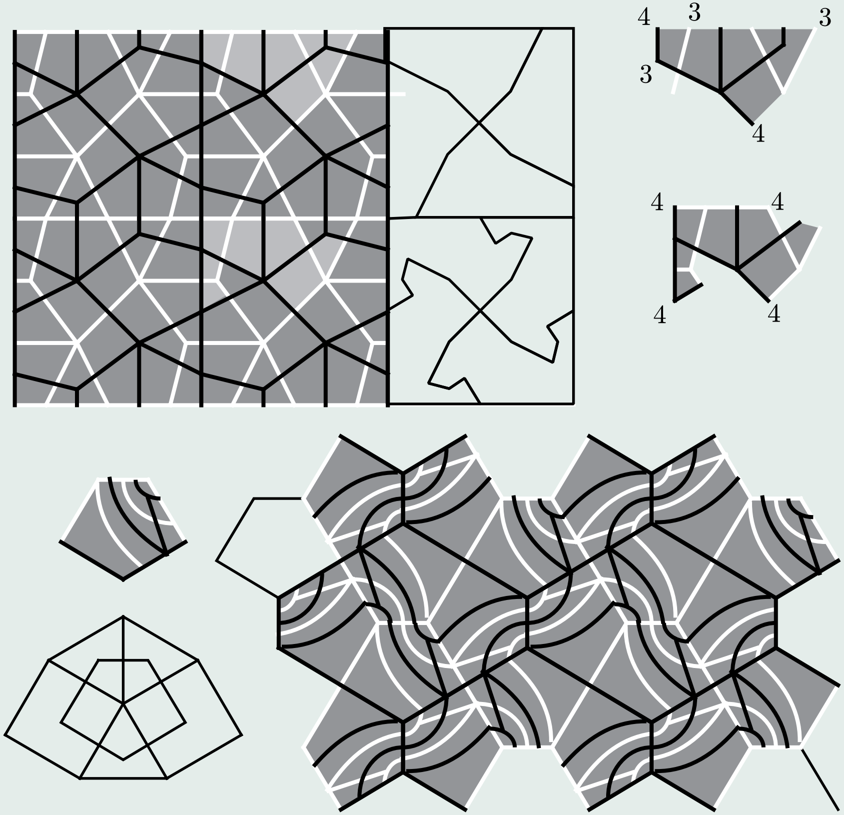
Full Screen

Close

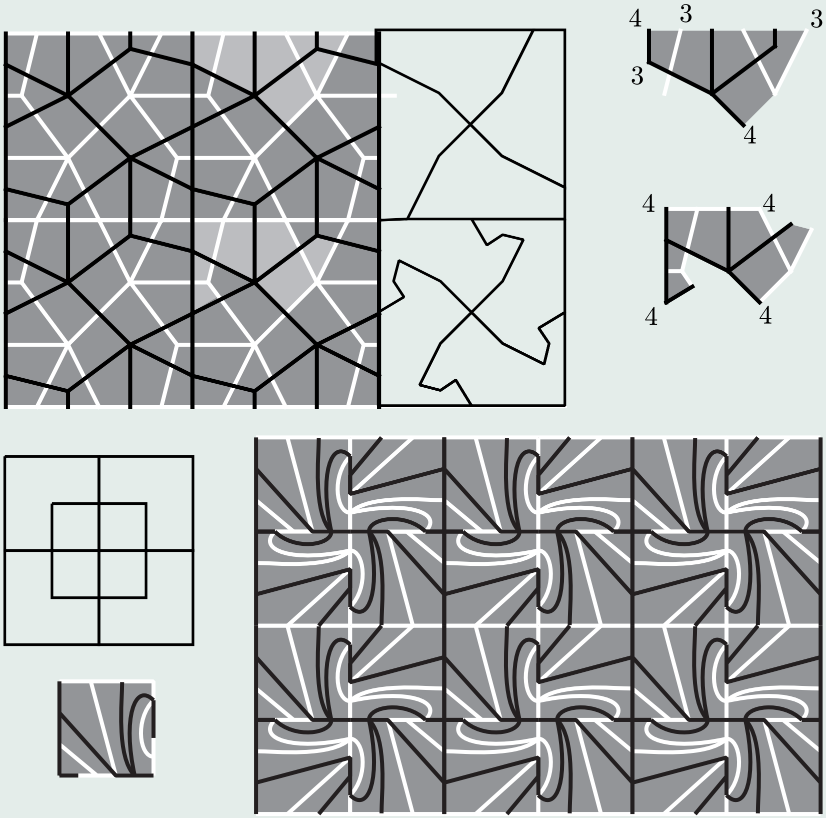
Quit



Different choices of fundamental region give different straightenings.



Using the 4–3–3–4–3

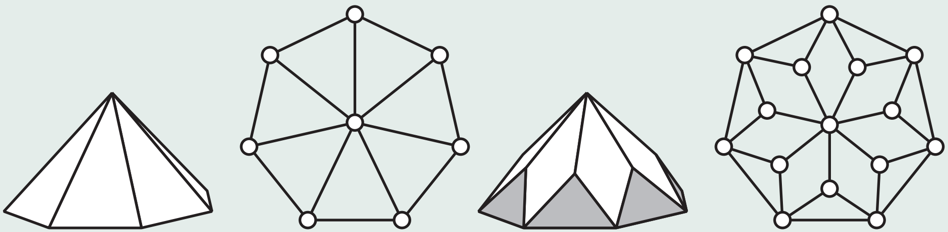


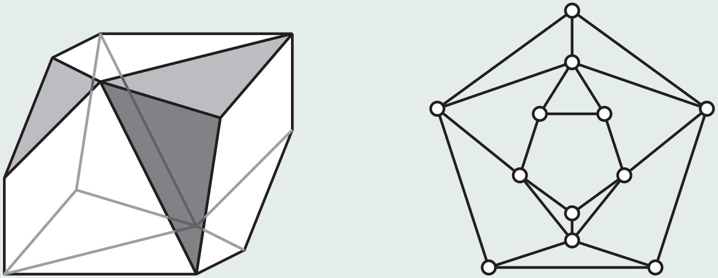
Using the 4–4–4–4

Moral

We can study automorphisms of 3-connected planar graphs via geometry:

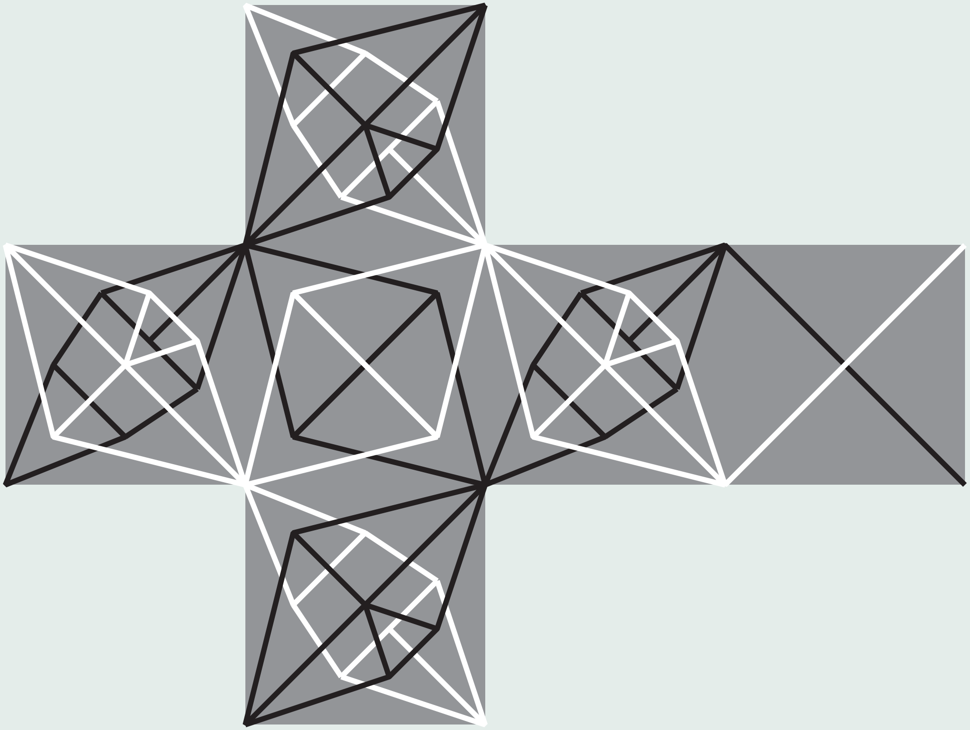
Example: classifying self-dual graphs



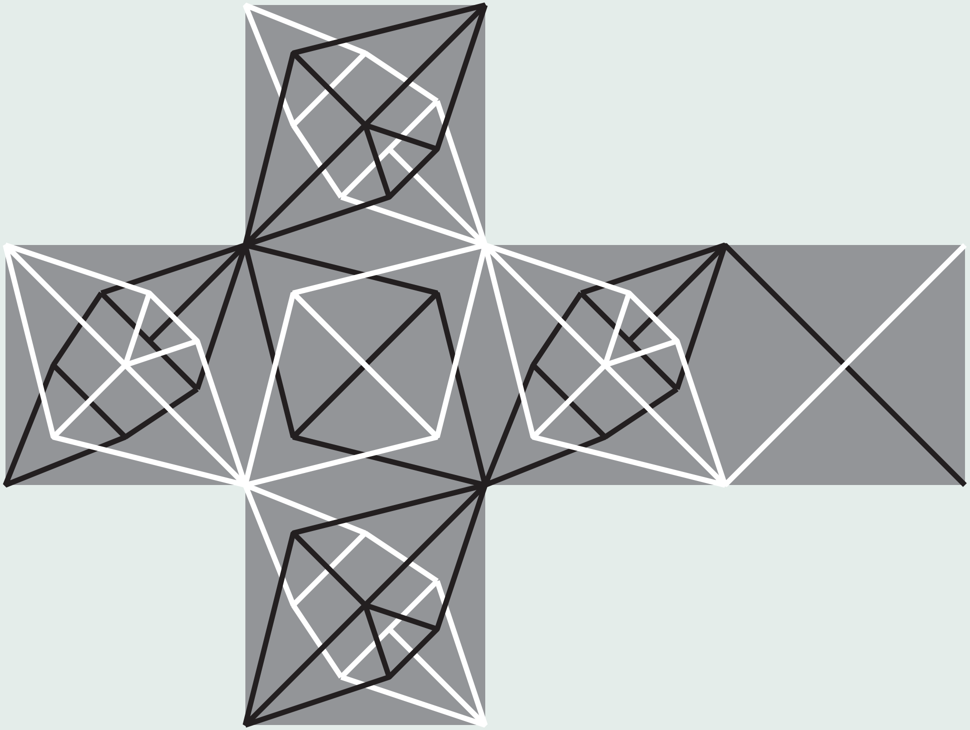


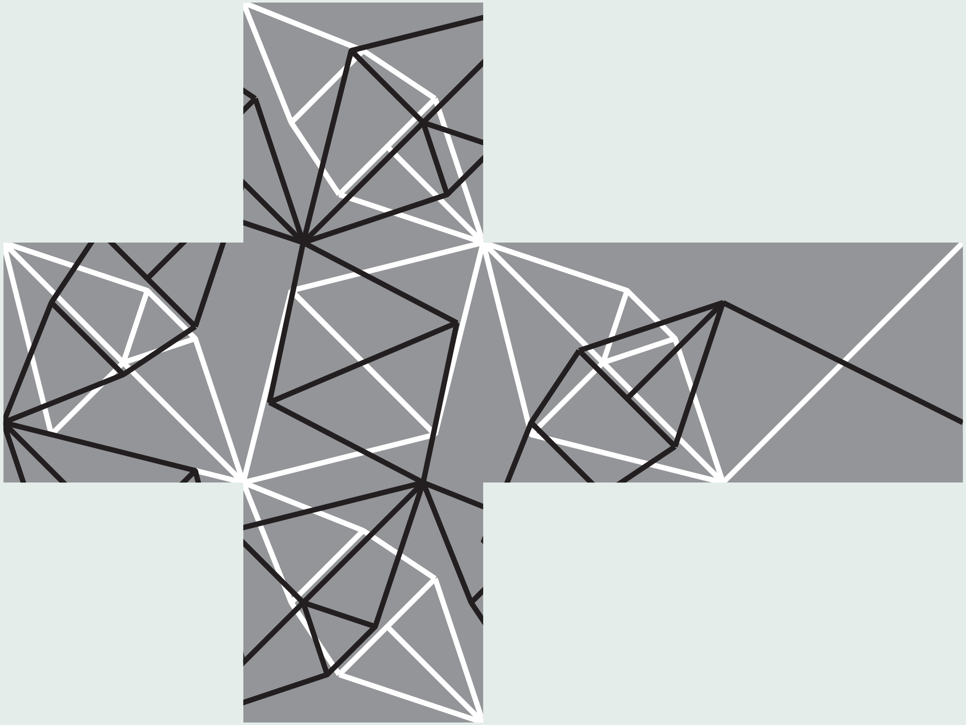
3. Connectivity

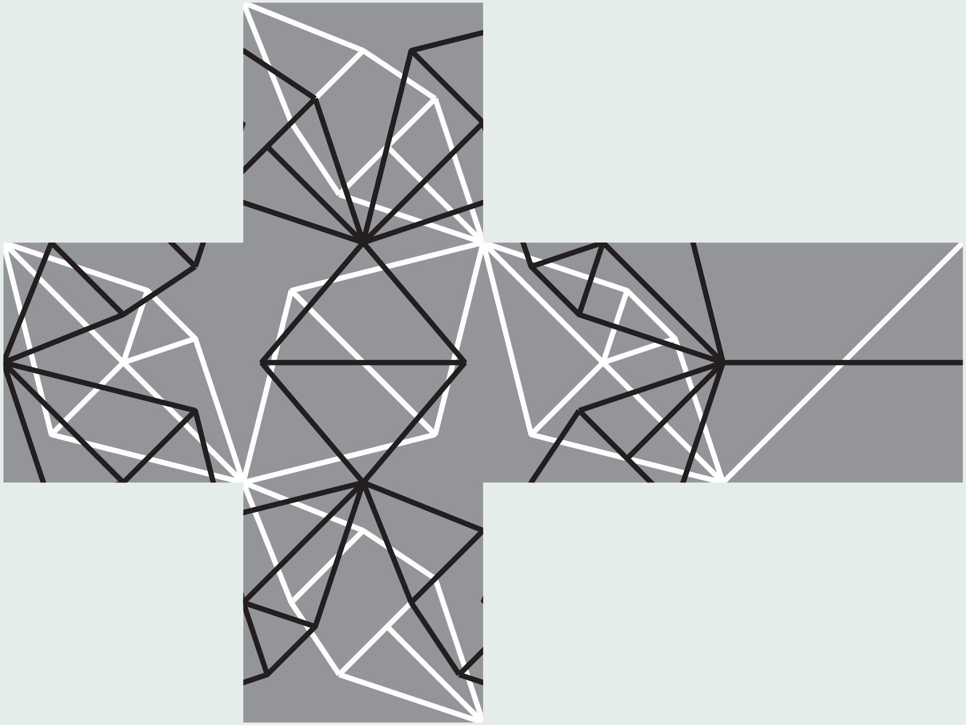
Maps are not enough

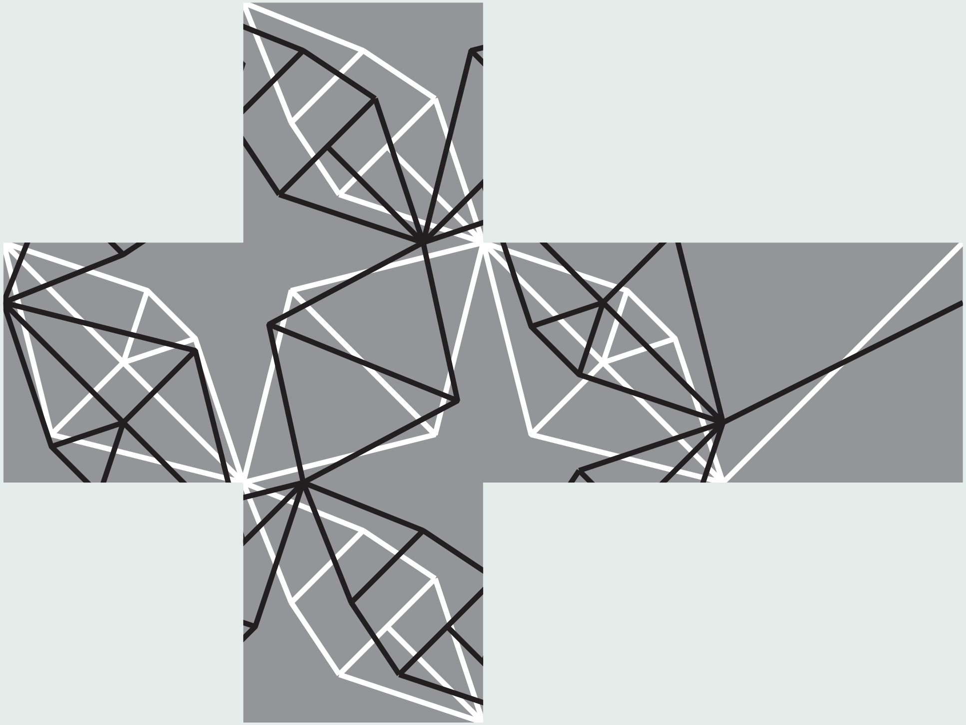


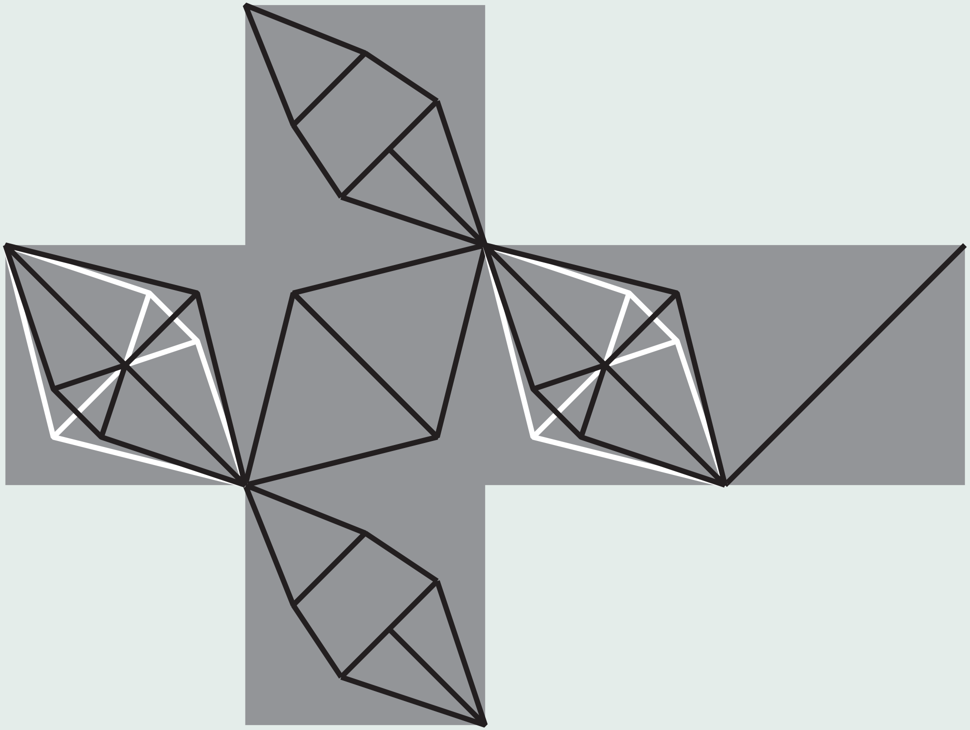
A self-dual graph with no corresponding self-dual map.

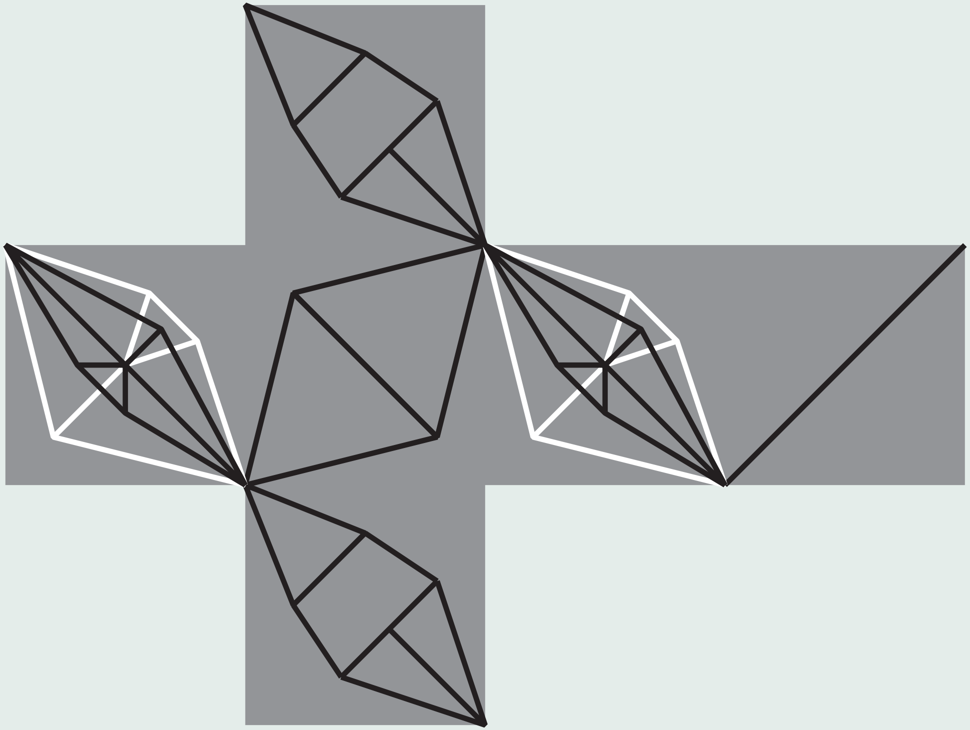


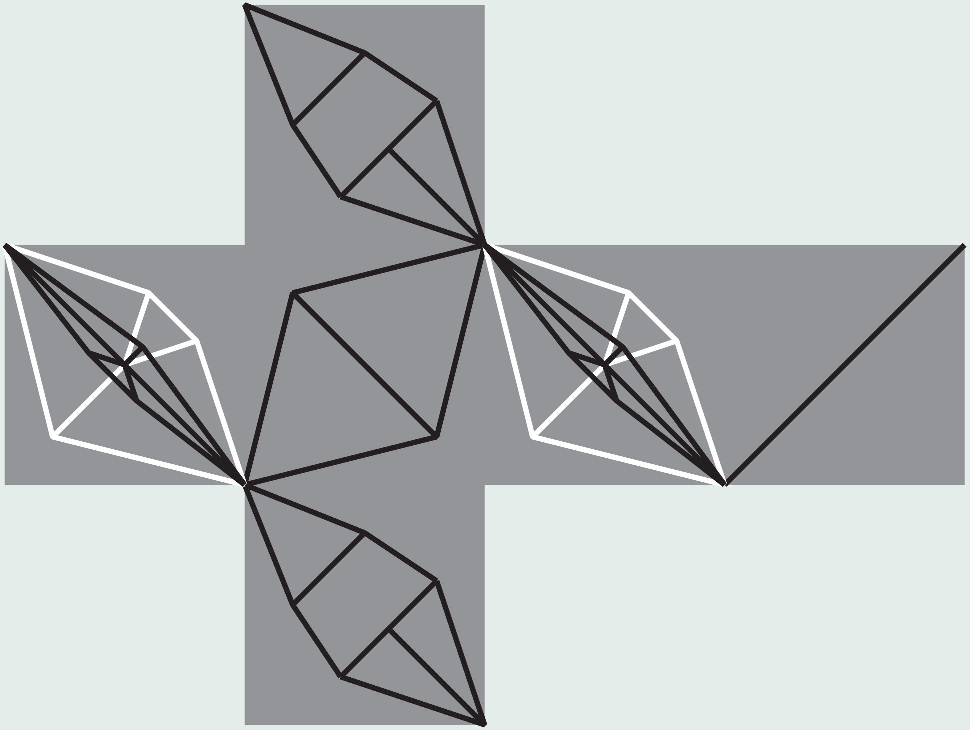


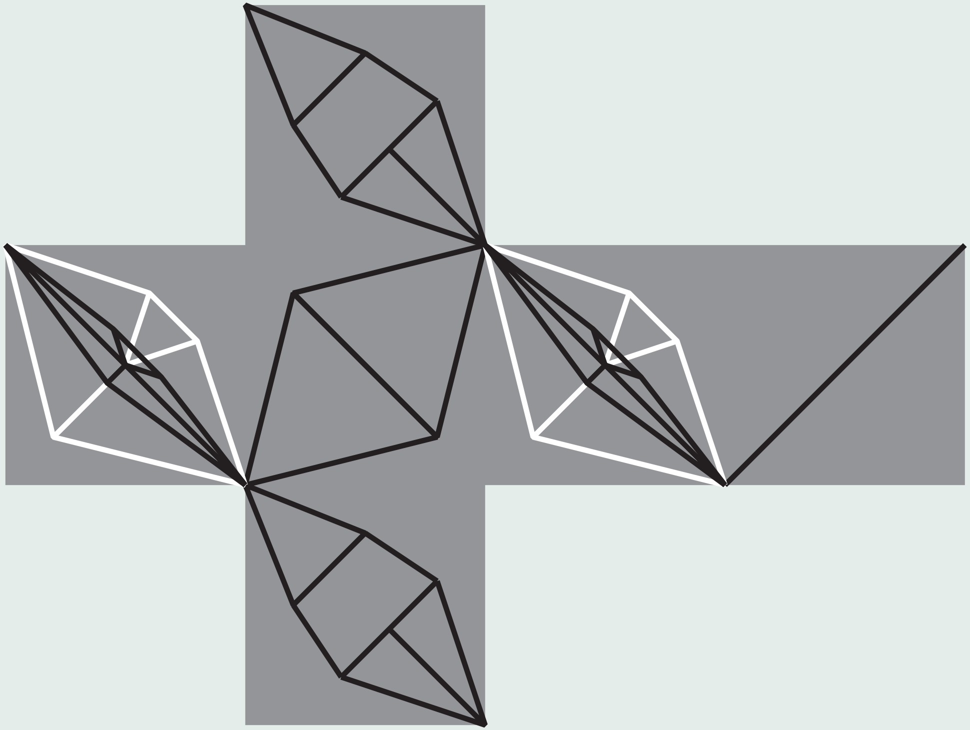


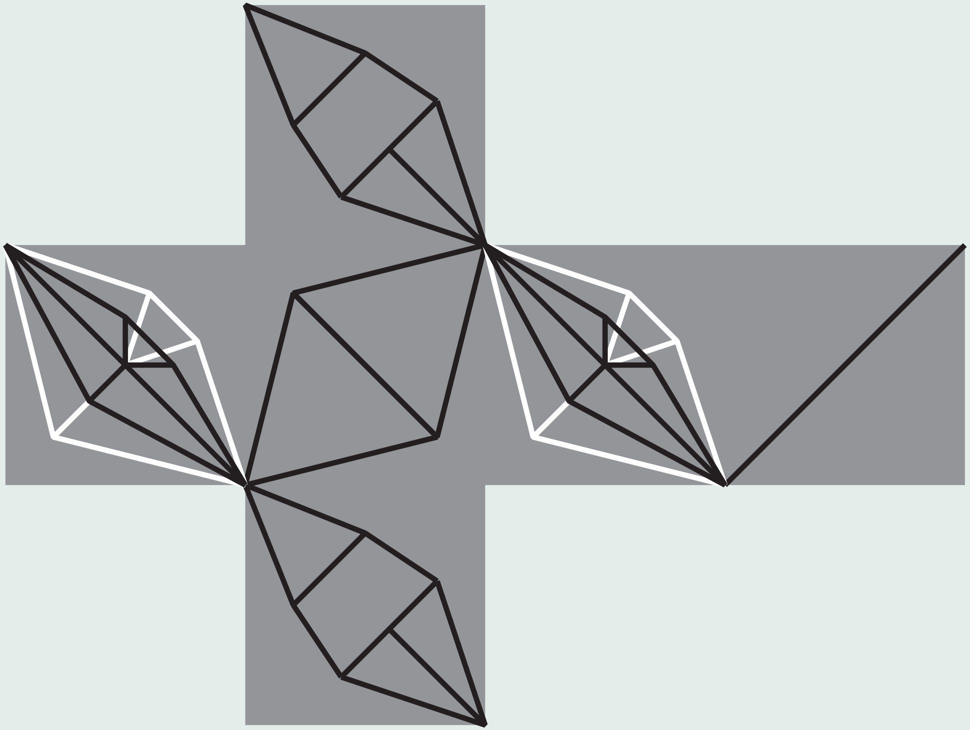


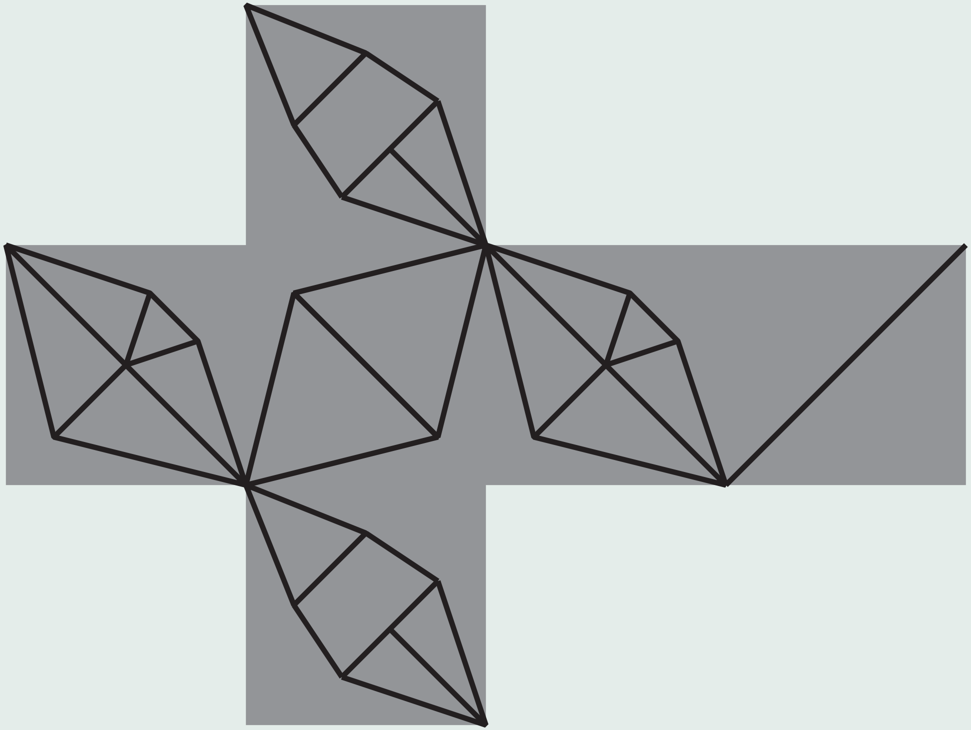




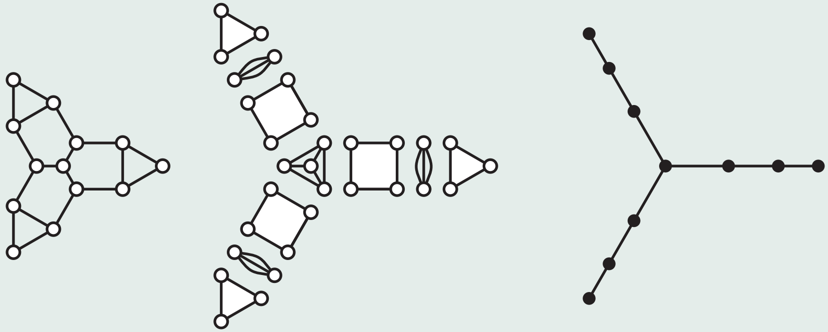








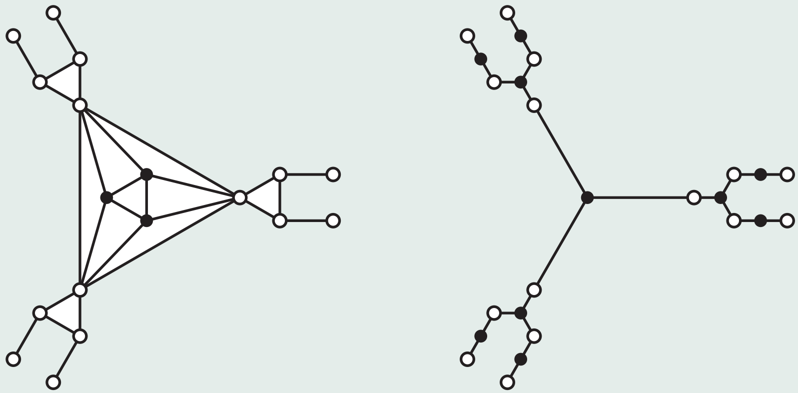
The 3-Block Tree



For a 2-connected graph G, the cycle matroid of G is the 2-sum over the cycle matroids of the 3-blocks of G:

$$M(G) = M(G_0) \bigoplus_{e_1} M(G_1) \ldots \bigoplus_{e_k} M(G_k)$$

The Block-Cutpoint Tree



For any graph G, the cycle matroid of G is the direct sum over the cycle matroids of the blocks of G:

$$M(G) = \sum M(G_i)$$

4. The Program

Program: Given a planar graph with whose automorphism group has finitely many vertex orbits.

1. If 3-connected – embed and straighten so that the automorphisms are represented by isometries (euclidean, spherical, or hyperbolic).

   Apply geometric methods.

2. Else if two-connected – form the 3–block tree and use the program on each block, and merge with data on the automorphisms of the tree.

3. Else if connected – from the block-cutpoint tree and apply the program to each block, merging with the tree automorphisms.

4. Else apply program to each connected component, and merge with permutations of isomorphic components.

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