The Cube and its Petrie Dual embedded in $\mathbb{R}^3$

Brigitte Servatius

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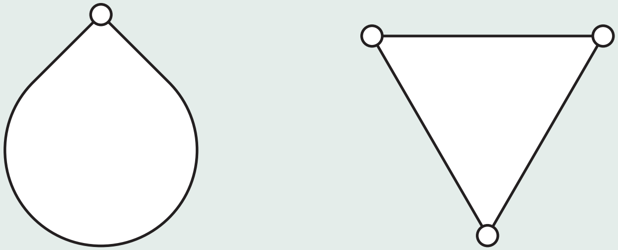
1. Polygonal maps

Start with a set of polygons.



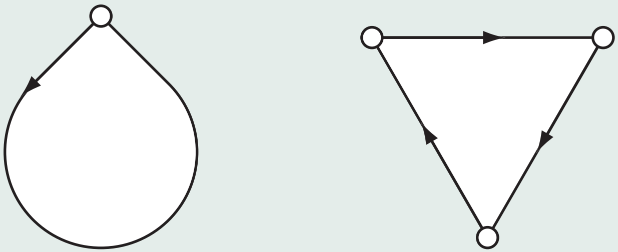
Consider them *oriented*

Assume: total # of edges is even.

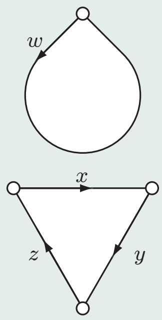
$\exists$ a perfect matching of edges: m
For each matched edge pair specify

> $+$ matched respecting edge orientation
> $-$ matched reversing edge orientation

$(m, \pm)$ is a *map* provided the resulting complex is connected.

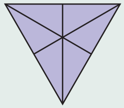


Example

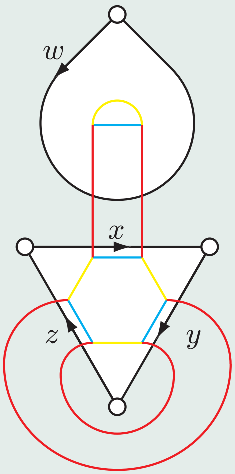


$(x, y; -)$
$(z, w; +)$

Idea
Barycentric Subdivision



This information is conveniently collected in the flag graph



Flag graph: Edges 3 colored.
01 cycles: faces
12 cycles: vertices
02 cycles: edges
of the resulting complex

$\mathfrak{M}(\tau_0, \tau_1, \tau_2)$

In [2] we find:
"The spirit of the present paper is probably best described by the desire to rid the theory of regular polyhedra of the psychologically motivated crutch of 'membranes' spanning the polygons used as building blocks."

Map $\mathfrak{M} = (\tau_0, \tau_1, \tau_2)$

Dual $\mathrm{Du}(\mathfrak{M}) = (\tau_2, \tau_1, \tau_0))$

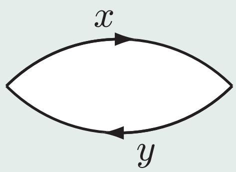
Antipodal Dual $\mathrm{A}(\mathfrak{M}) = (\tau_0, \tau_1, \tau_0\tau_2))$

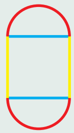
Petrie Dual $\mathrm{Pe}(\mathfrak{M}) = (\tau_0\tau_2, \tau_1, \tau_0))$

2. Orientability

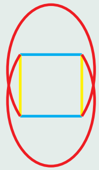
$\langle \tau_1\tau_2, \tau_2\tau_0, \tau_0\tau_1 \rangle = \langle \mathcal{V}, \mathcal{E}, \mathcal{F} \rangle$ has
either one flag orbit $\longrightarrow$ non-orientable map
or two flag orbits $\longrightarrow$ orientable map

Example



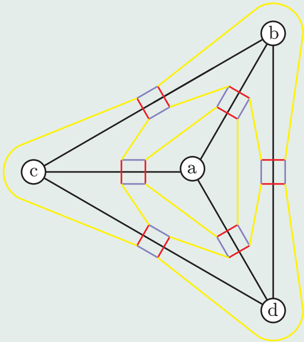


$\mathfrak{M}$ orientable

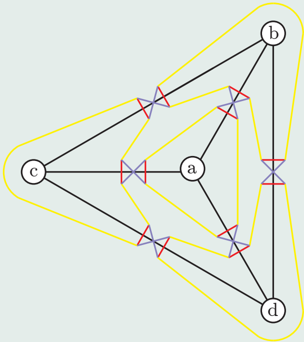


A($\mathfrak{M}$) non-orientable

Examples of Petrie duals - Tetrahedron



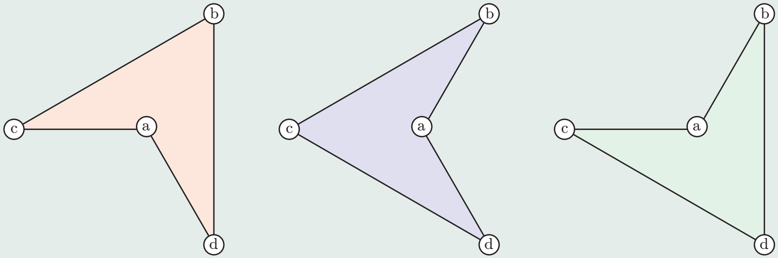
a)



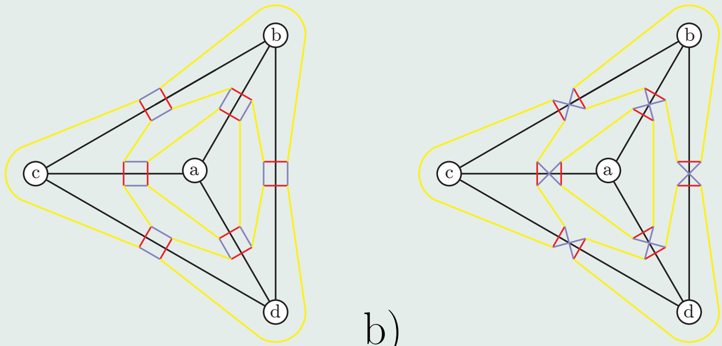
b)

a) The tetrahedron with superimposed flag graph, with the flag matchings τ_0 in blue, τ_1 in yellow and τ_2 in red. b) The flag graph for the Petrie Dual.

The three Petrie quadrilaterals,



Examples of Petrie duals - Tetrahedron



a) b)

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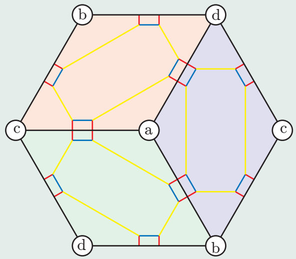
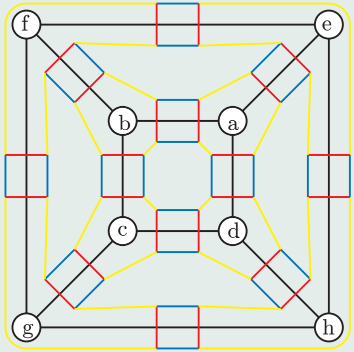
A hexagon with opposite sides identified with a twist – non-orientable. (projective plane)

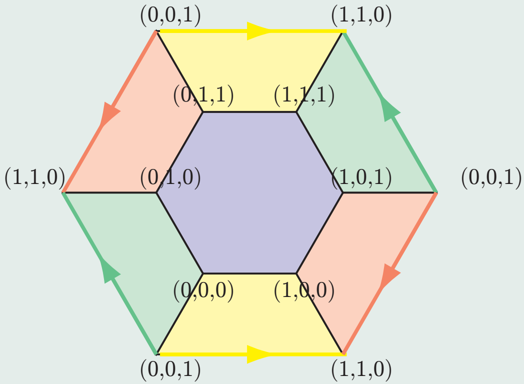


Examples of Petrie duals - Cube

The cube has four hexagonal Petrie cycles

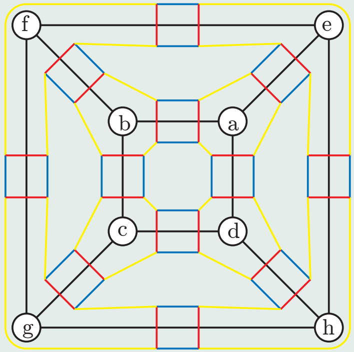


Four Petrie hexagons

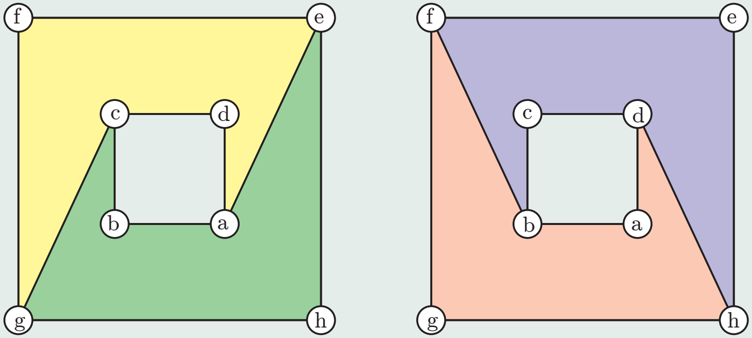


Examples of Petrie duals - Cube

The cube has four hexagonal Petrie cycles



Four Petrie hexagons arranged as the upper and lower half of a torus. which combine in pairs to form two annuli, which in turn join to form a torus – orientable.



Tetrahedron – Petrie dual not orientable

Cube – Petrie dual orientable

Theorem

Let $\mathfrak{M}(\tau_0, \tau_1, \tau_2)$ *be an orientable map.*
$\mathrm{Pe}(\mathfrak{M})$ *is orientable if and only if* $G(\mathcal{V}, \mathcal{E})$ *is bipartite.*

Theorem

Let $\mathfrak{M}(\tau_0, \tau_1, \tau_2)$ *be an non-orientable map.*
$\mathrm{Pe}(\mathfrak{M})$ *is non-orientable if* $G(\mathcal{V}, \mathcal{E})$ *is bipartite.*

The graph of a self-Petrie orientable map must be bipartite.

The graph of a self-Petrie non-orientable map need not be bipartite.

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3D Realizability
Ribbon Embeddings
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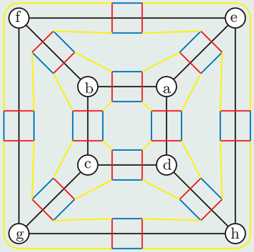
3. 3D Realizability

$\mathfrak{M}$ and $\mathrm{Du}(\mathfrak{M})$ can be drawn on the same surface in a nice way.
There is a 3D representation in the orientable case.

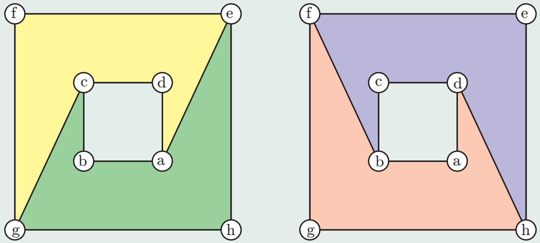
$\mathfrak{M}$ and $\mathrm{Pe}(\mathfrak{M})$ describe the same graph on two different surfaces.
In the orientable case, can we find two surfaces in 3D such that their intersection is their common graph?

Example: Cube graph



$\mathfrak{M}$

Sphere

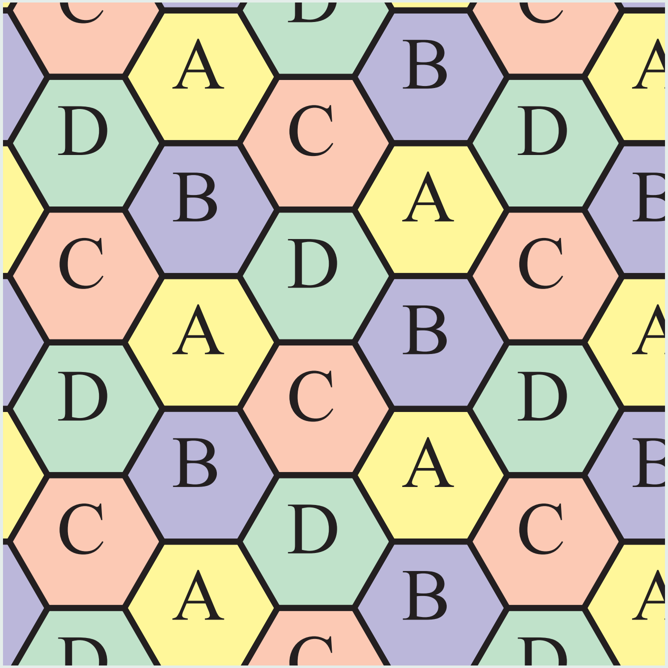


Pe($\mathfrak{M}$)

Torus

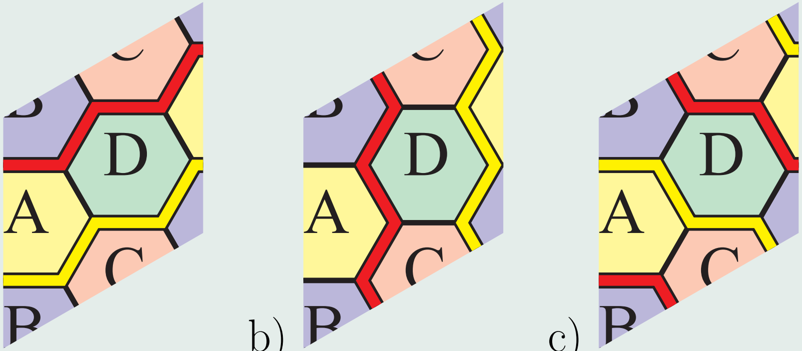
Fact: In any 3D representation of Pe($\mathfrak{M}$) at least one pair of quadrilaterals is linked.

A flat torus in the plane with 4 hexagons.

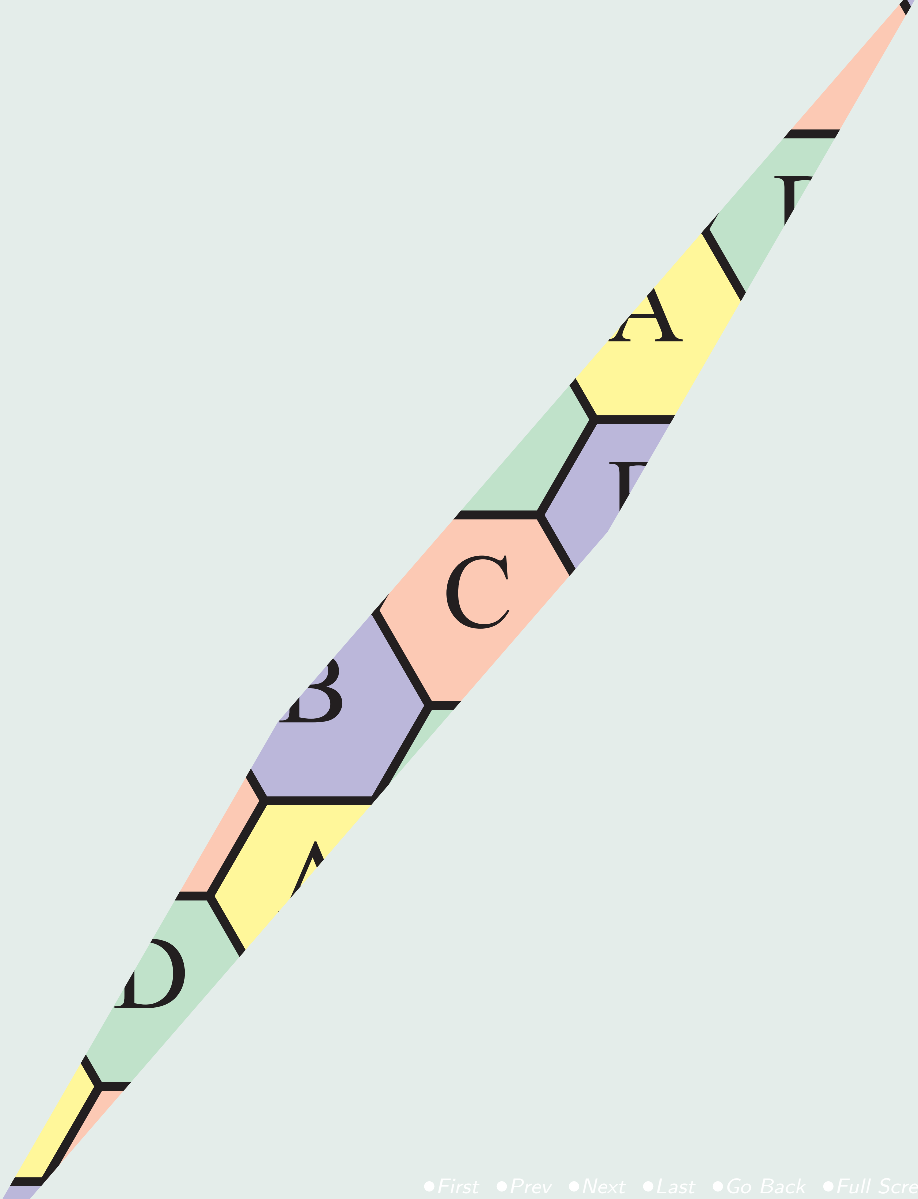


A framing of the torus in 3D

a) b) c)



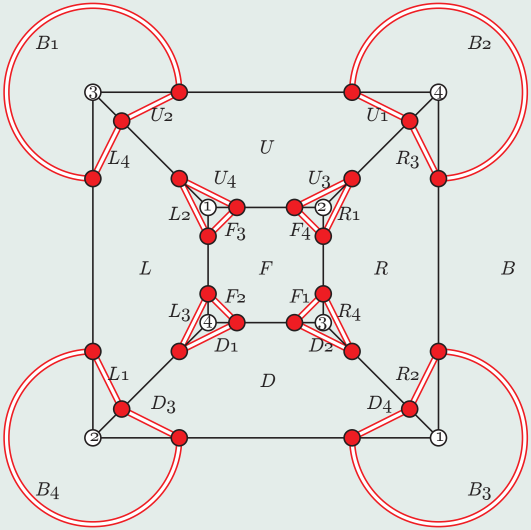
Other Framings


A
C
B
D

4. Ribbon Embeddings

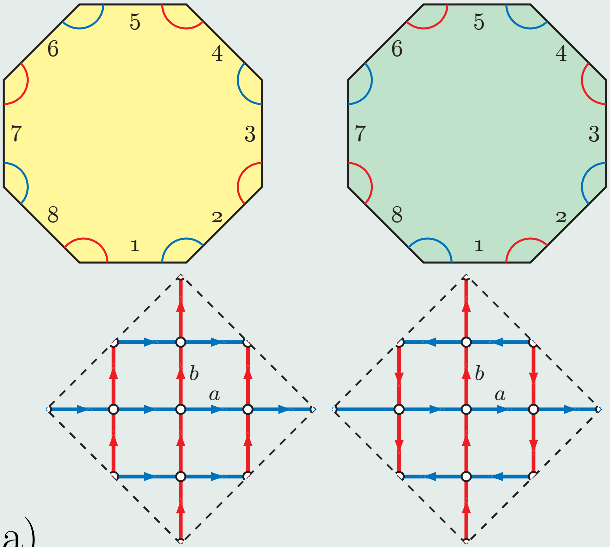
Example: Cube graph

Question: Is there at least a ribbon complex? The cube graph with ribbons sewn on for each four cycle and each Petrie six cycle?

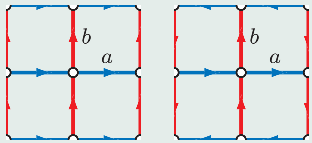


Answer: No (by studying the labeled graph [1] associated to the ribbon complex.)

For self-Petrie-dual maps it is possible to have ribbon complexes in $\mathbb{R}^3$
in fact
These examples are realizable as intersections of 2 spheres or two tori respectively



a)



b)

Non-orientable case: Example – Klein Bottle

Ribbon Complex? Yes!

And the moral is:
If we have a 3D representation of a graph drawing on a surface, and then forget about the membranes in the Grünbaum spirit, we cam still, from the existence of linked cycles exclude some surfaces by looking just at the skeleton.

5. Bibliography

References

[1] Dan Archdeacon, C. Paul Bonnington, R. Bruce Richter, and Jozef Širáň. Sewing ribbons on graphs in space. *J. Combin. Theory Ser. B*, 86(1):1–26, 2002.

[2] Branko Grünbaum. Regular polyhedra—old and new. *Aequationes Math.*, 16(1-2):1–20, 1977.