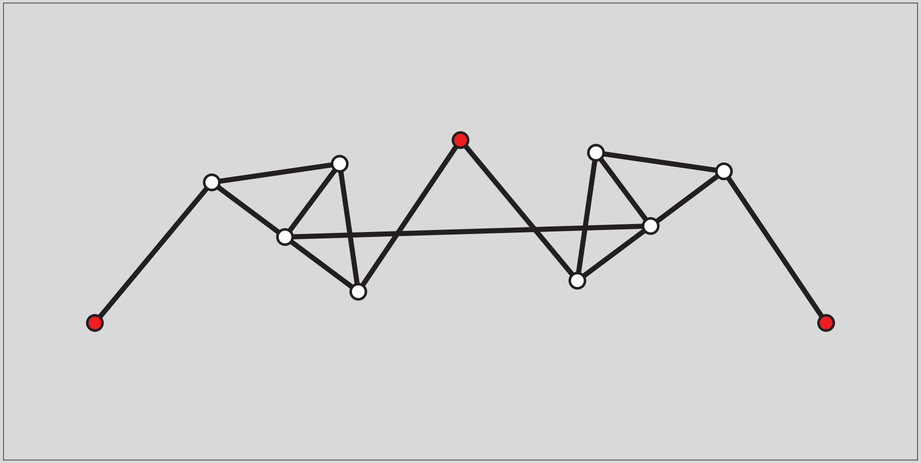
On configuration spaces of linkages

Brigitte Servatius



1. Introduction

framework (in m-space)

a triple $(V, E, \overrightarrow{p})$,

(V, E) is a graph

$$\overrightarrow{p} : V \longrightarrow \mathbb{R}^m$$

globally rigid framework

all solutions to the system of quadratic equations obtained from requiring all edge lengths to be fixed, with the coordinates of the vertices as variables, correspond to congruent frameworks

rigid framework

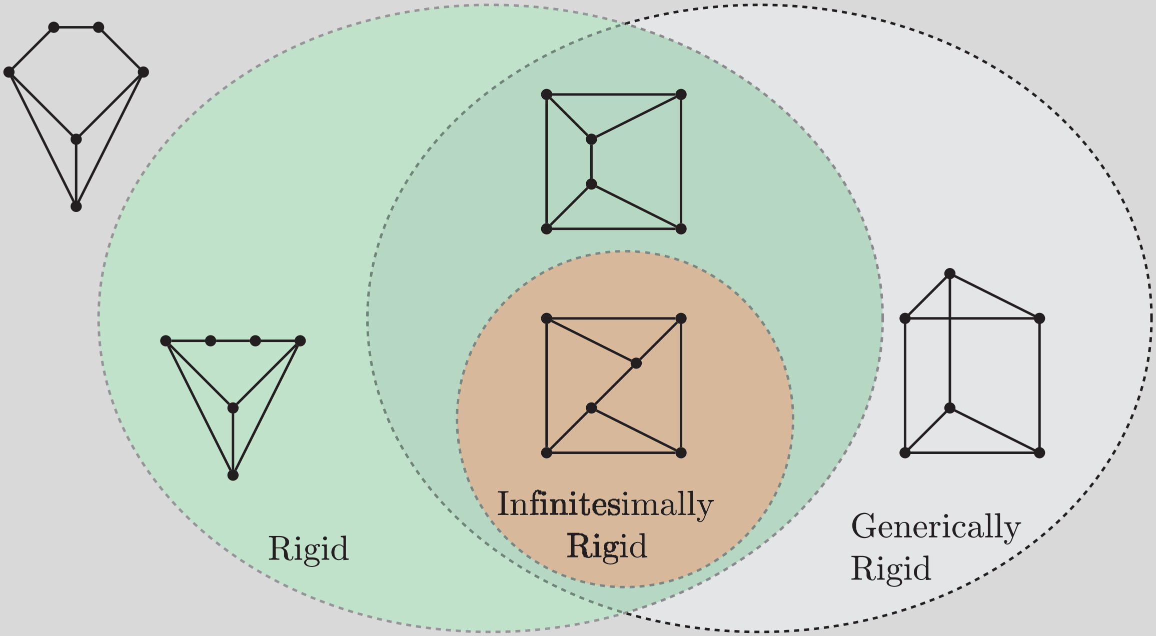
if all solutions to the corresponding system in some neighborhood of the original solution (as a point in mn-space) come from congruent frameworks.

generic framework

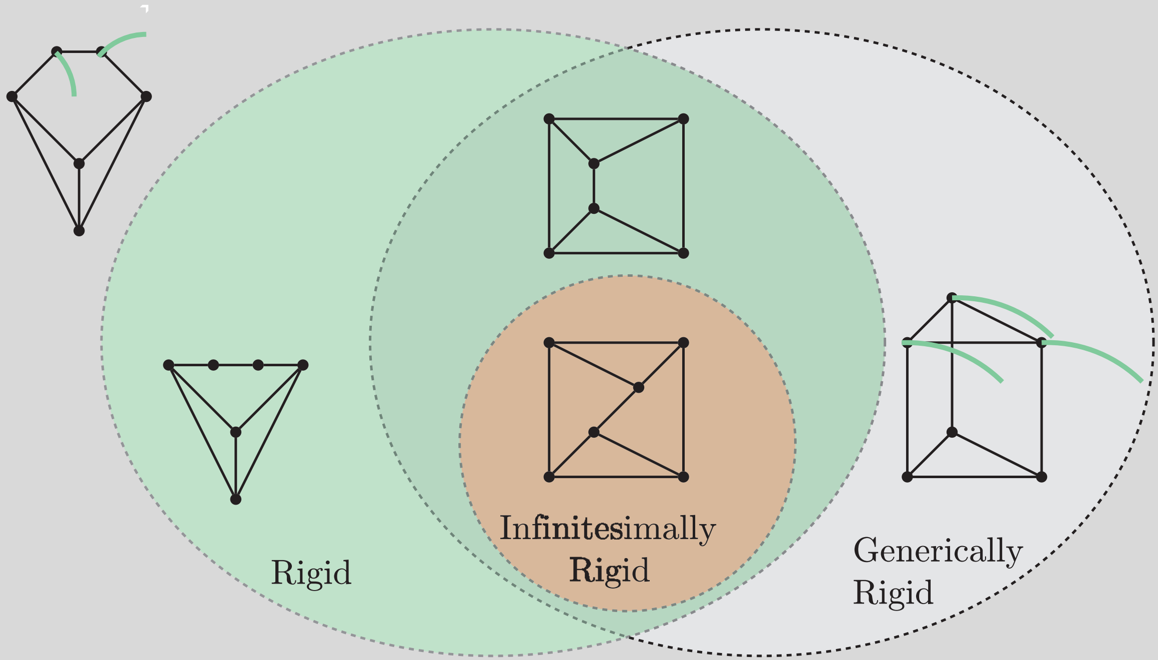
all frameworks corresponding to points in a neighborhood of $P = \overrightarrow{p}(V)$ in $\mathbb{R}^{nm}$ are rigid or not rigid as is $(V, E, \overrightarrow{p})$.

generic pointset

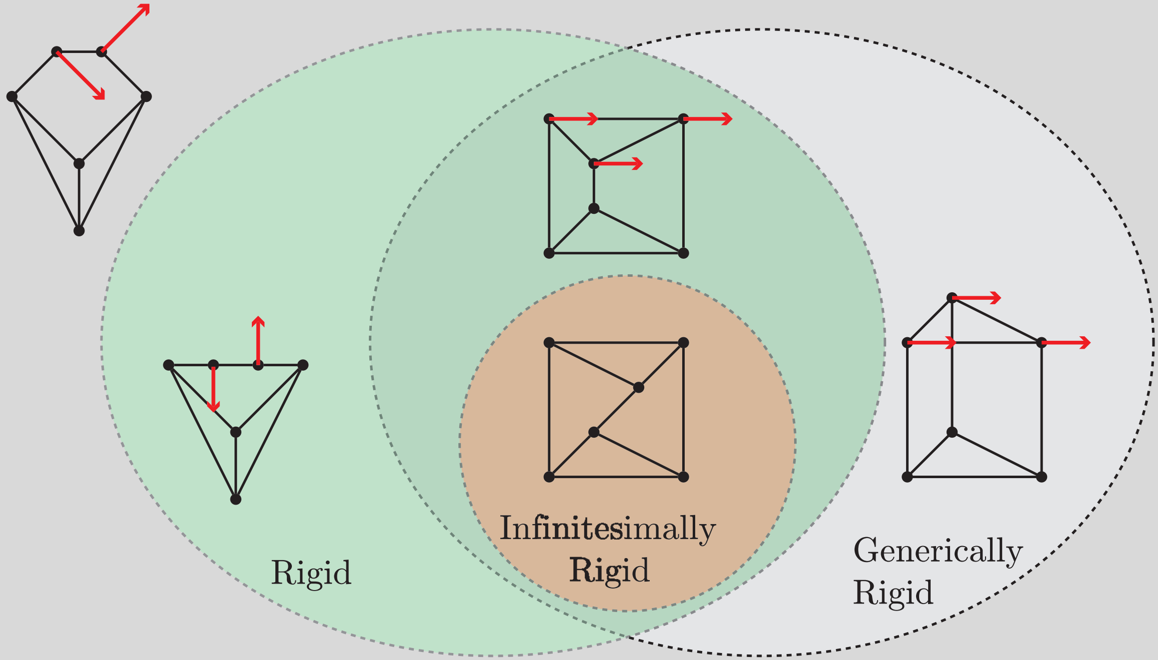
A set of points P in m-space is said to be *generic* if each frame-work $(V, E, \overrightarrow{p})$ with $\overrightarrow{p}(V) = P$ is generic.



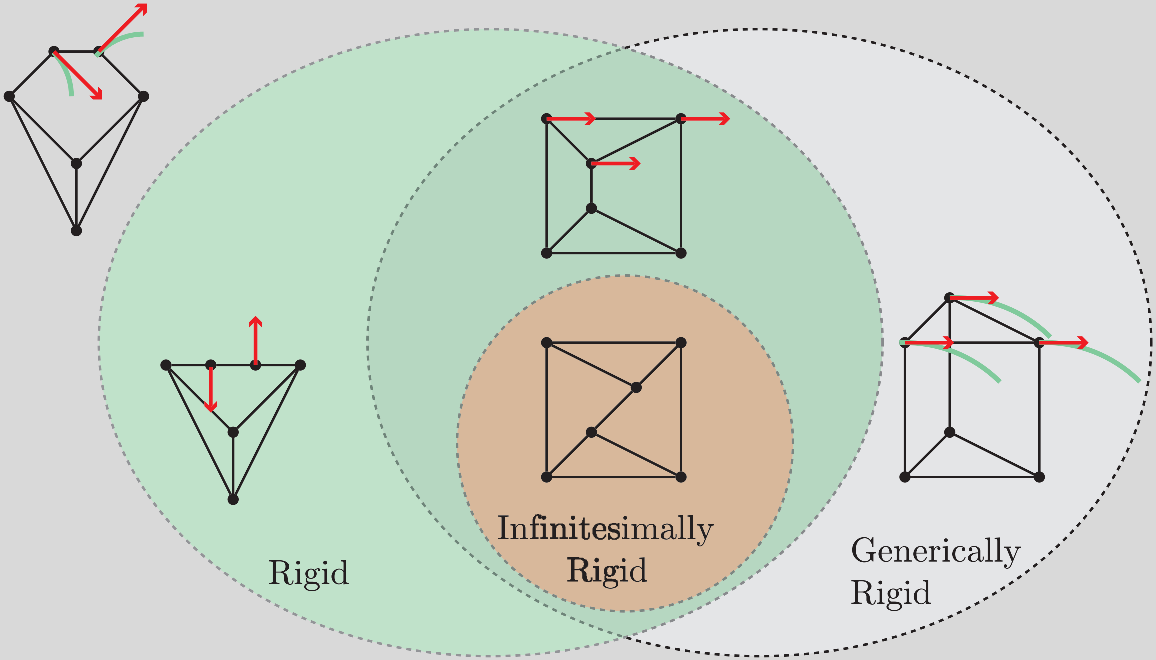
Different notions of rigidity.



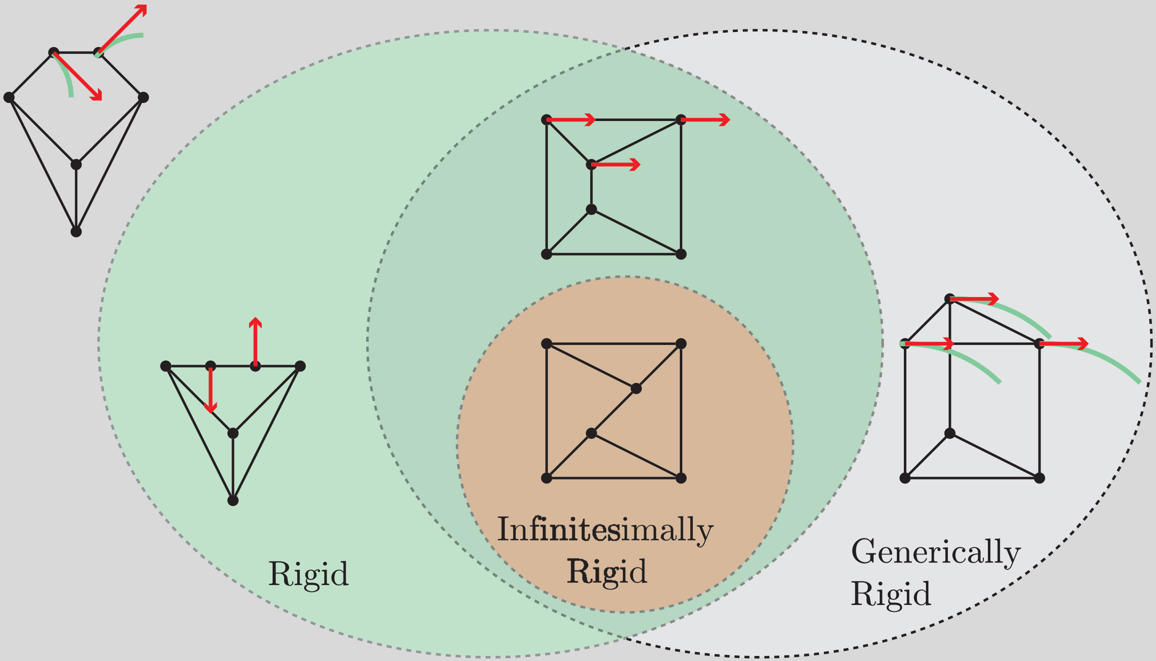
Rigidity – Non rigid graphs have a motion.



Non-infinitesimally rigid graphs have initial velocity candidates.



Generic rigidity is a property of the graph, *not* the embedding.



Generic rigidity of $G(V, E)$ is characterized in $\mathbb{R}^2$ by *Laman's Theorem* $\exists F \subseteq E$.

$$|F| = 2|V(F)| - 3$$
$$|F'| \leq 2|V(F')| - 3 \qquad \forall\ F' \subseteq F$$

rigid components

The maximal rigid subgraphs of G
– The rigid components partition E.
– $\mathcal{M}(G)$ is the direct sum over its restrictions on the rigid components.

The following theorem is equivalent to Laman's Theorem, it uses the rank function of $\mathcal{M}$ rather than independence to characterize rigidity.

Theorem 1 *[11] Let $G = (V, E)$ be a graph. Then G is rigid if and only if for all families of induced subgraphs $\{G_i = (V_i, E_i)\}_{i=1}^m$ such that $E = \cup_{i=1}^m E_i$ we have $\sum_{i=1}^m (2|V_i| - 3) \geq 2|V| - 3$.*

redundantly rigid graph $G(V, E)$

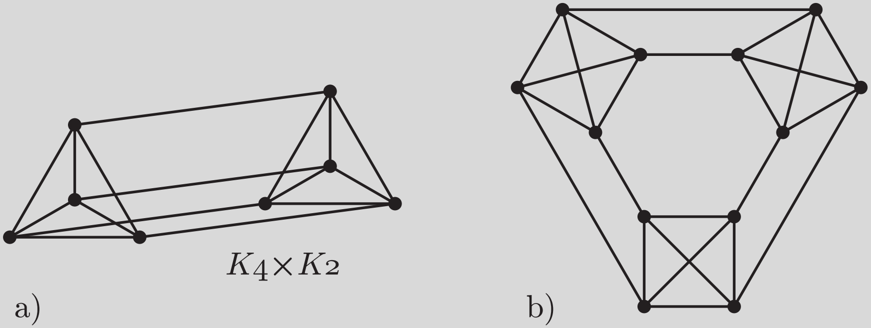
$G(V, E - e)$ is rigid for all $e \in E$
i.e. the removal of a single edge e from the rigid graph G does not destroy rigidity.

Redundant rigidity is a key to characterize global rigidity.

Theorem 2 *[5] Let G be a graph. Then G is globally rigid if and only if G is a complete graph on at most three vertices, or G is both 3-connected and redundantly rigid.*

2. Vertex transitive graphs

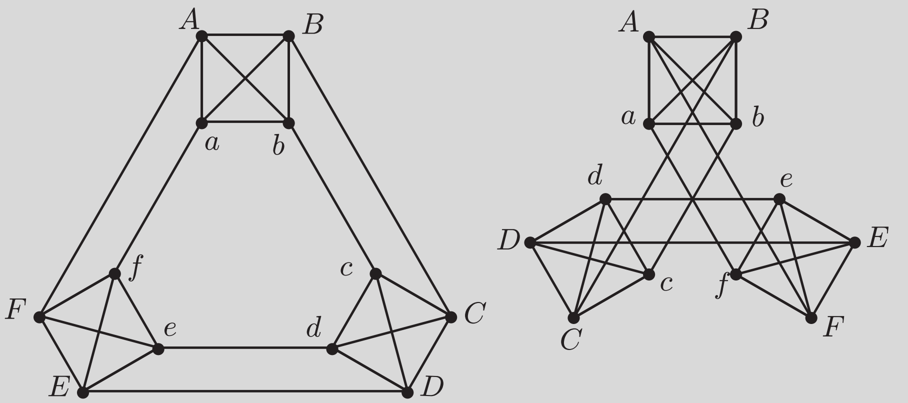
Theorem 3 *A four-regular vertex transitive graph is generically rigid in the plane if and only if it contains no subgraph isomorphic to* K_4*, or is* K_5 *or one of the graphs in the following figure.*



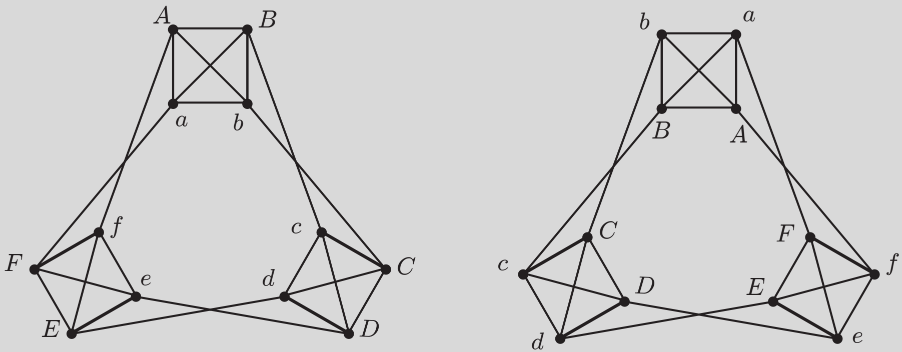
Vertex transitive rigid graphs containing K_4.

Theorem 4 *Let G be a vertex transitive non-rigid graph. Then G is k-regular with $k \leq 6$, and contracting the non-trivial rigid components of G produces a vertex transitive graph of regularity at most 5.*

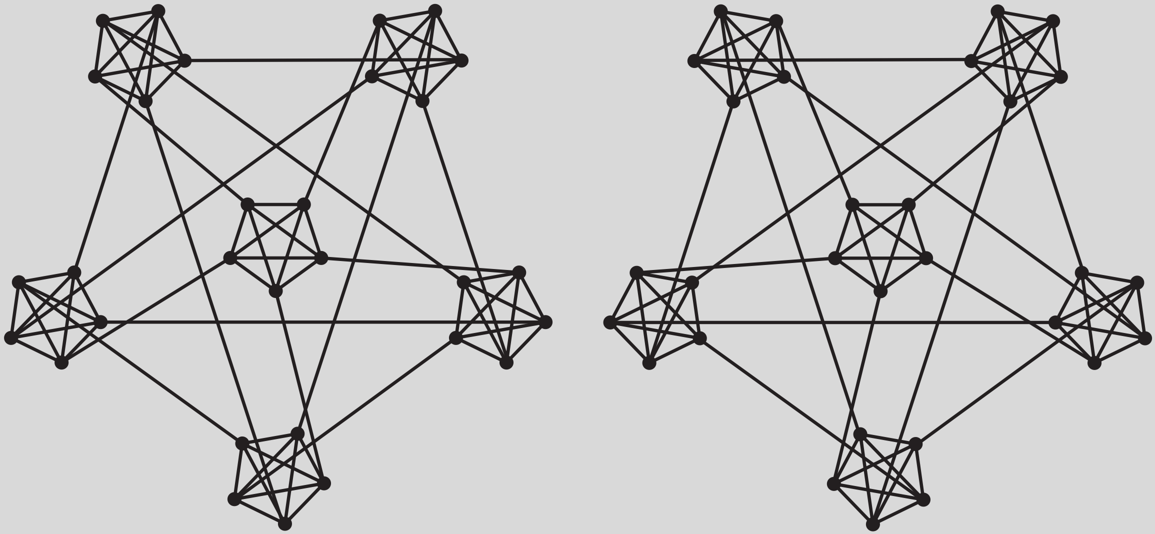
Theorem 5 *Let G be a connected k-regular vertex transitive graph on n vertices. Then G is not rigid if and only if either:*
(a) $k = 2$ and $n \geq 4$.
(b) $k = 3$ and $n \geq 8$.
(c) $k = 4$ and G has a factor consisting of s disjoint copies of K_4 where $s \geq 4$
(d) $k = 5$ and G has a factor consisting of t disjoint copies of K_5 where $t \geq 8$.



Two embeddings which are rigid, but neither infinitesimally rigid nor globally rigid.



Two embeddings which are rigid and infinitesimally rigid but not globally rigid.

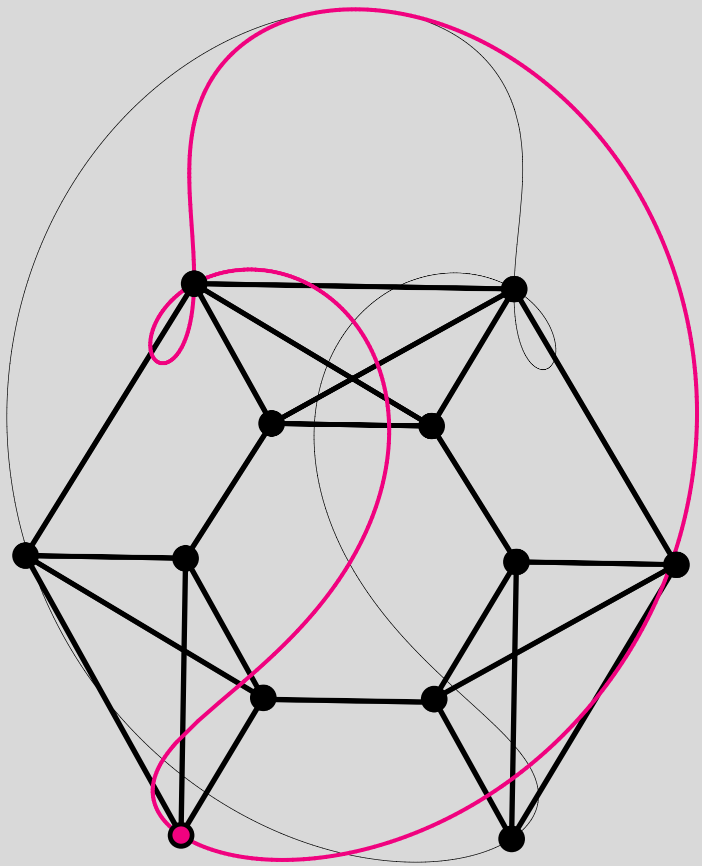


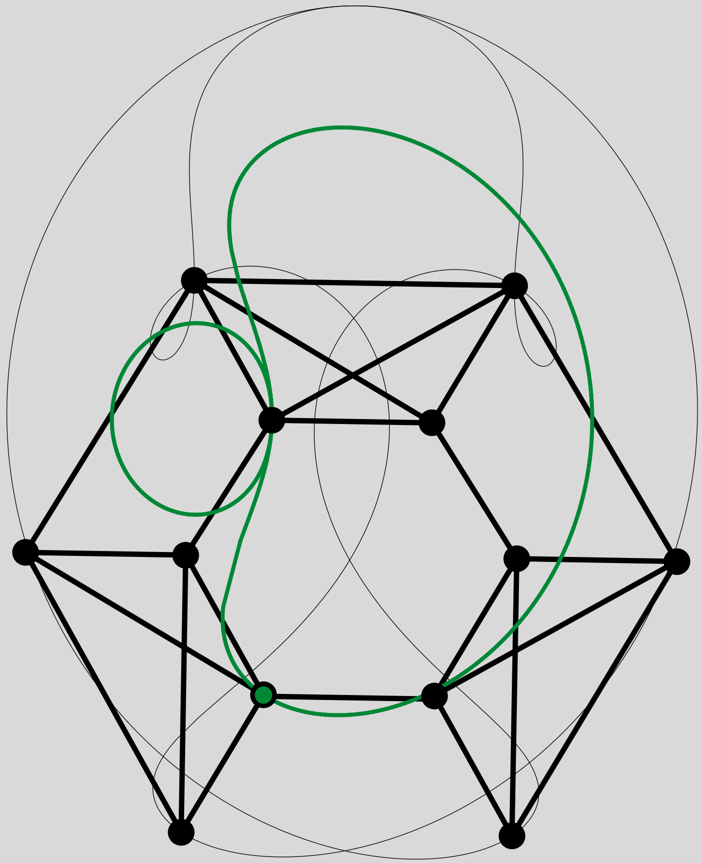
Two embeddings which are rigid, but not globally rigid.

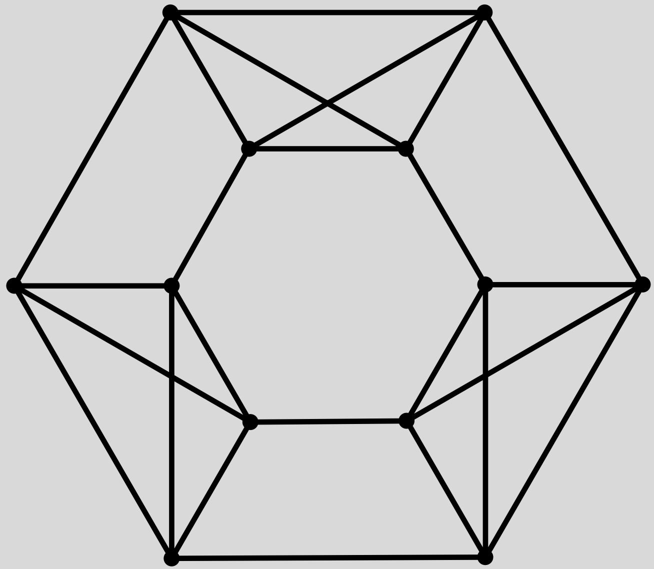
We observe that for a rigid G which is not redundantly rigid, $\mathcal{M}(G)$ is not connected. It is in fact the direct sum over the maximal redundantly rigid subgraphs (or singleton edges). The arguments in the preceding proofs are unaltered if we replace rigid components by redundantly rigid subgraphs and we obtain

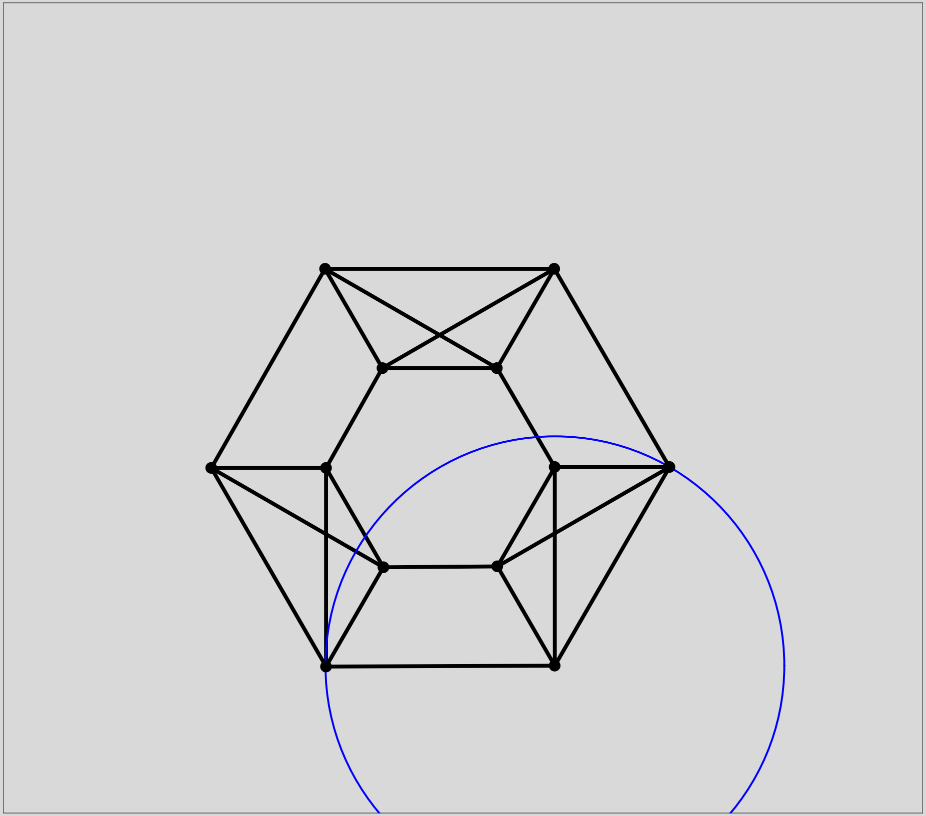
Theorem 6 *A vertex transitive rigid graph is also globally rigid unless it has a factor consisting of 3 copies of* K_4 *or 6 copies of* K_5*.*

3. Example

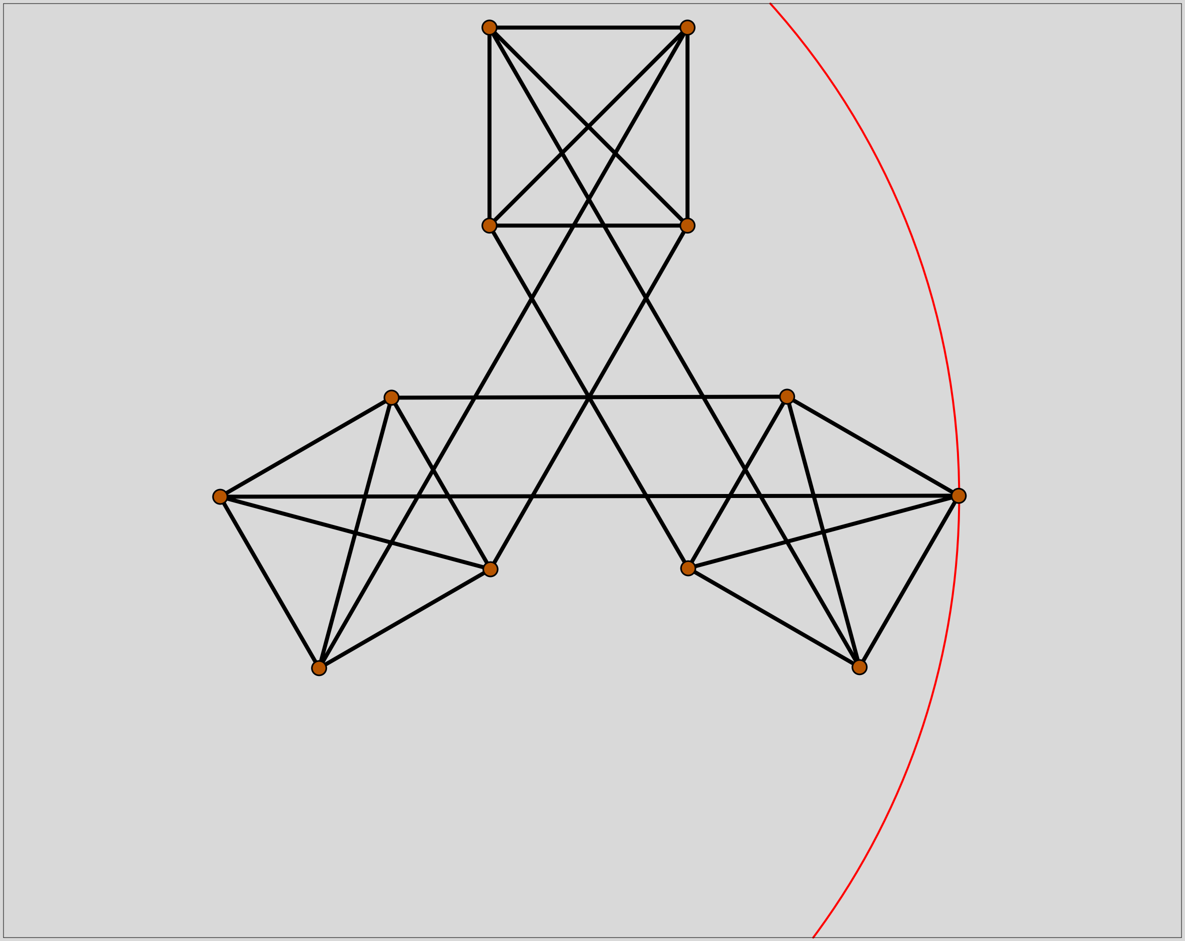


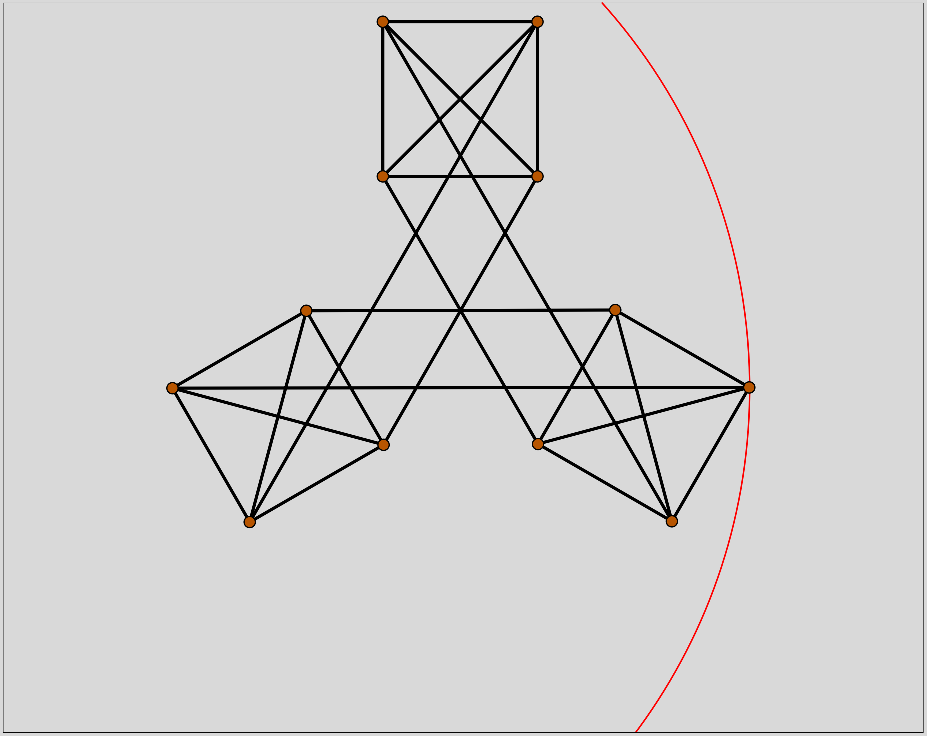






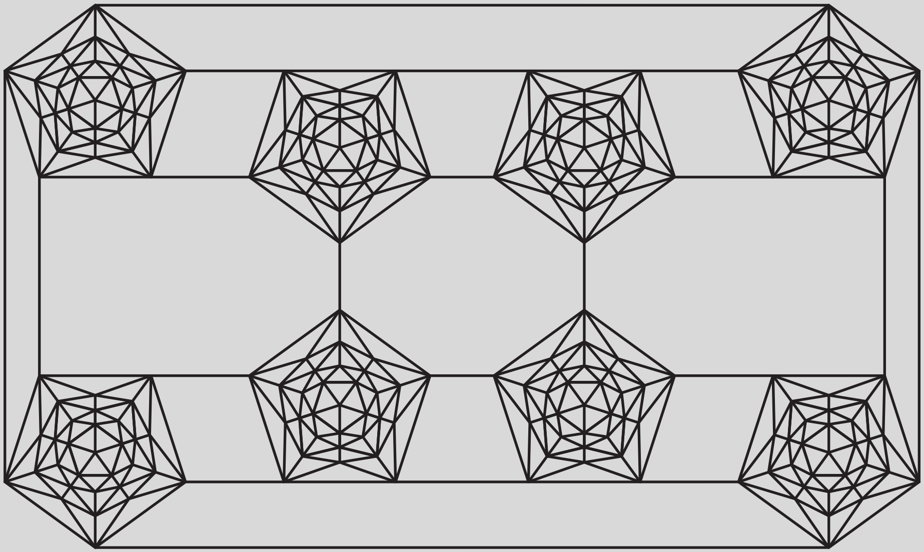
4. Example





5. Highly connected graphs

Let G be a graph and k be a positive integer. The graph G is *k-connected* if for all pairs of subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$, $|V(G_1) - V(G_2)| \geq 1$ and $|V(G_2) - V(G_1)| \geq 1$, we have $|V(G_1) \cap V(G_2)| \geq k$. Lovász and Yemini [11] showed that every 6-connected graph G is rigid. Planar graphs are at most 5-connected and one might wonder if the connectivity requirement can be lowered in order to imply rigidity of a planar graph. However, the following figure shows a 5connected planar nonrigid graph. The rigidity properties are easily checked using Theorem 1.



A planar 5-connected non-rigid graph.

However,Lovász and Yemini [11] note that their proof technique will show that $G - \{e_1, e_2, e_3\}$ is rigid for all $e_1, e_2, e_3 \in E$, and hence that G is redundantly rigid. This result was combined with Theorem 2 in [5] to deduce

Theorem 7 *Every (essentially)* 6*-connected graph is globally rigid.*

An even weaker connectivity condition is sufficient to imply that 4-regular graphs are globally rigid. A graph $G = (V, E)$ is said to be *cyclically k-edge-connected* if for all $X \subseteq V$ such that $G[X]$ and $G[V - X]$ both contain cycles, we have at least k edges from X to $V - X$.

Theorem 8 *Let $G = (V, E)$ be a cyclically 5-edge-connected 4-regular graph. Then G is globally rigid.*

6. Random graphs

Let $G_{n,d}$ denote the probability space of all d-regular graphs on n vertices with the uniform probability distribution. A sequence of graph properties A_n holds asymptotically almost surely, or a.a.s. for short, in $G_{n,d}$ if $\lim_{n\to\infty} \Pr_{G_{n,d}}(A_n) = 1$. Graphs in $G_{n,d}$ are known to be a.a.s. highly connected. It was shown by Bollobás [1] and Wormald [15] that if $G \in G_{n,d}$ for any fixed $d \geq 3$, then G is a.a.s. d-connected. This result was extended to all $3 \leq d \leq n-4$ by Cooper et al. [3] and Krivelevich et al. [8]. Stronger results hold if we discount 'trivial' cutsets. In [16], Wormald shows that if $G \in G_{n,d}$ for any fixed $d \geq 3$, then $G5$ is a.a.s. cyclically $(3d-6)$-edge-connected. Together with Theorem 8, this immediately gives:

Theorem 9 *If $G \in G_{n,4}$ then G is a.a.s. globally rigid.*

In fact this result holds for all $d \geq 4$.

Theorem 10 *If $G \in G_{n,d}$ and $d \geq 4$ then G is a.a.s. globally rigid.*

Theorem 11 *Let $G \in G(n,p)$, where $p = (\log n + k \log\log n + w(n))/n$, and $\lim_{n\to\infty} w(n) = \infty$.*
(a) If $k = 2$ then G is a.a.s. rigid.
(b) If $k = 3$ then G is a.a.s. globally rigid.

The bounds on p given in Theorem 11 are best possible since if $G \in G(n,p)$ and $p = (\log n + k \log\log n + c)/n$ for any constant c, then G does not a.a.s. have minimum degree at least k, see [2].

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Let $Geom(n, r)$ denote the probability space of all graphs on n vertices in which the vertices are distributed uniformly at random in the unit square and each pair of vertices of distance at most r are joined by an edge. Suppose $G \in Geom(n, r)$. Li, Wan and Wang [10] have shown that if $n\pi r^2 = \log n + (2k - 3) \log\log n + w(n)$ for $k \geq 2$ a fixed integer and $\lim_{n\to\infty} w(n) = \infty$, then G is a.a.s. k-connected. As noted by Eren et al. [4], this result can be combined with Theorem 7 to deduce that if $n\pi r^2 = \log n + 9 \log\log n + w(n)$ then G is a.a.s. globally rigid. On the other hand, it is also shown in [10] that if $n\pi r^2 = \log n + (k-1) \log\log n + c$ for any constant c, then G is not a.a.s. k-connected. It is still conceivable, however, that if $n\pi r^2 = \log n + \log\log n + w(n)$ then G is a.a.s. rigid, and that if $n\pi r^2 = \log n + 2 \log\log n + w(n)$ then G is a.a.s. globally rigid.

7. Configuration Spaces

Configuration Space

Parameter Space

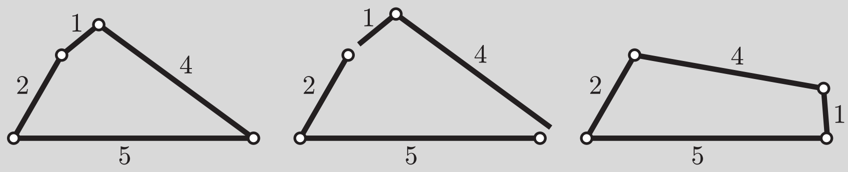
Moduli Space

8. Four Bar Mechanism

Normalize: Sum of the bar lengths is 12.

Cyclically order by length.

Linkage encoded by (a, b, c, d), with $a \leq b \leq c \leq d$.



Normalize: Sum of the bar lengths is 12.
Cyclically order by length.
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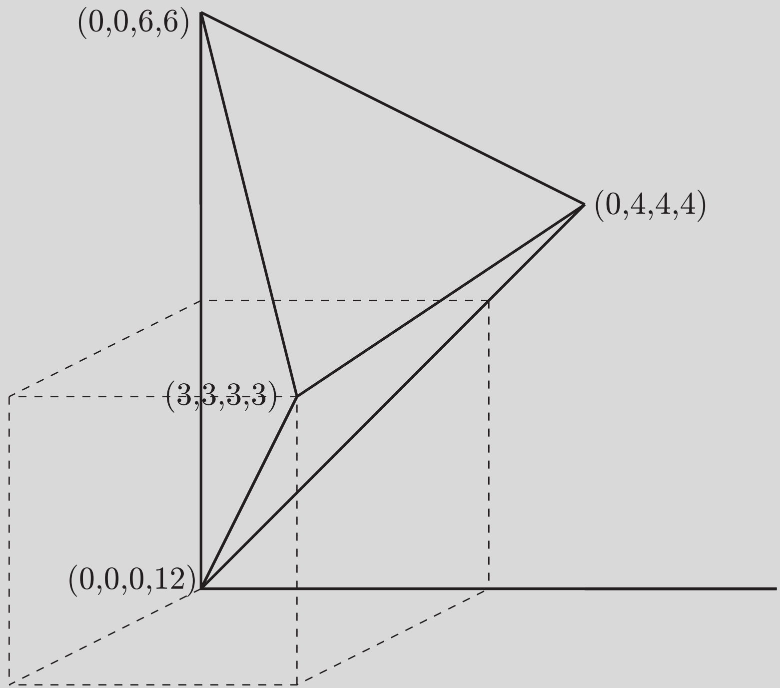

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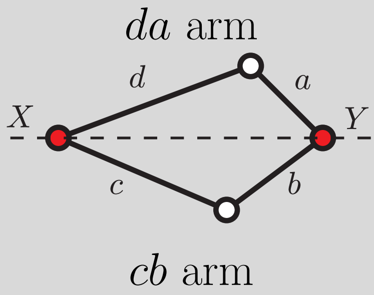
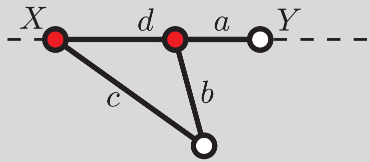
Consider the mechanism as two arms joining X and Y:

Normalize so that X and Y line on a horizontal line with Y to the right of X.



$$\max(d-a, c-b) \leq XY \leq \min(d+a, c+b)$$

If $c-b \leq d-a \leq c+b$ then the mechanism is realizable with the da arm fully closed.



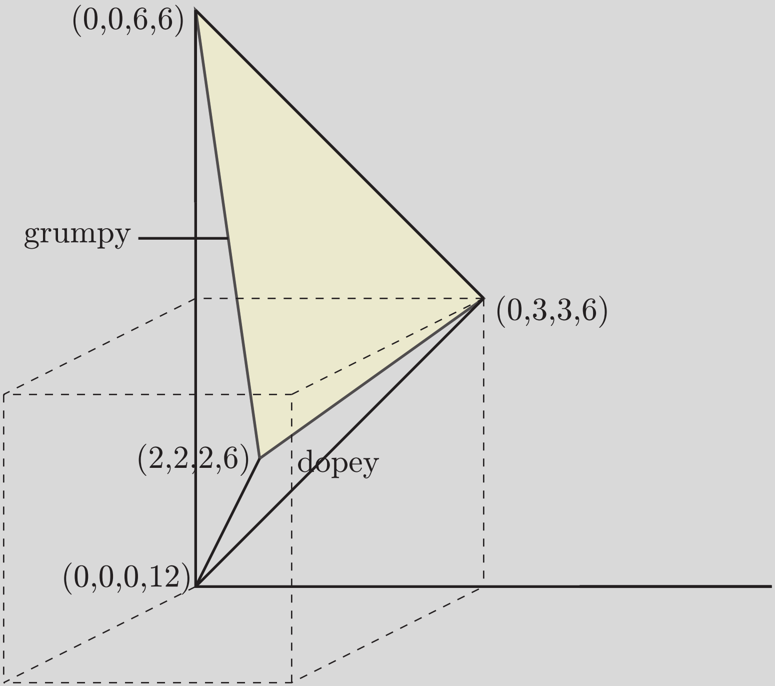
Otherwise $d-a > c+b$, or $d > a+b+c$ and the mechanism not is realizable at all.

Condition of Completion:

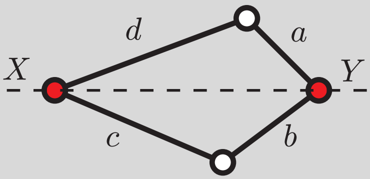
$a + b + c \geq d$

If $a + b + c > d$ linkage unconstructible in Euclidean space - Configuration Space empty - *Dopey* (Simplet)

If $a + b + c = d$ linkage uniquely constructible. It is rigid, and takes stress - Configuration Space a single point - *Grumpy* (Grincheux)



If $d < a + b + c$ then the configuration space is more than a point



and

$$\max(d - a, c - b) \leq XY \leq \min(d + a, c + b).$$

From the normalization, $\max(d - a, c - b) = d - a$.
So we have three cases

$\min(d + a, c + b) = c + b < d + a$ - The cb arm opens fully

$\min(d + a, c + b) = d + a < c + b$ - The da arm opens fully

$c + b = d + a$ - Both arms open fully

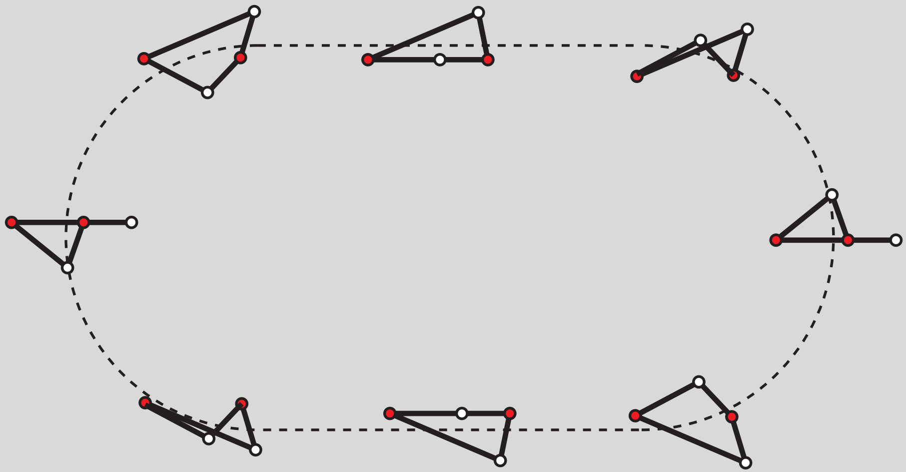
Sleepy: (Dormeur)

$d + a > b + c$

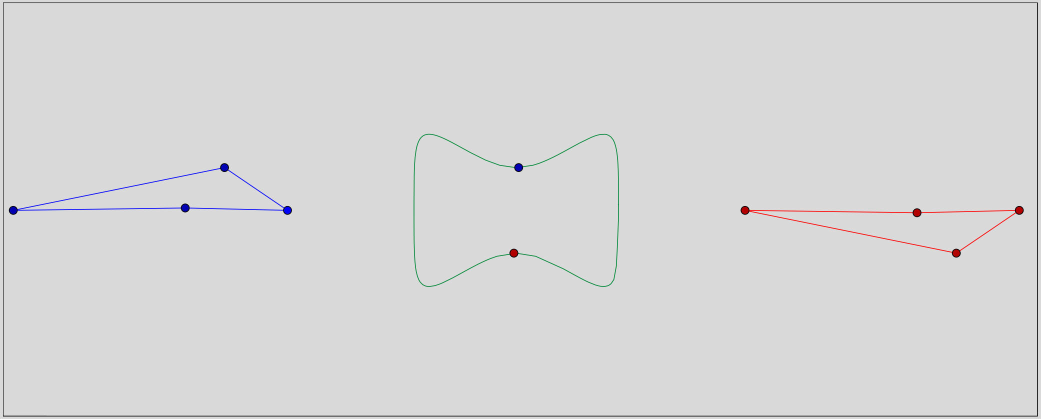
In a Sleepy mechanism the da arm cannot fully open.

The configuration space is S^1. There are no choices. You can traverse the whole configuration space while asleep.

Example$(2, 2, 3, 5)$ - The endpoints of the longest-shortest sub-linkage are shown in red on a horizontal line with the endpoint of the shorter rod to the right.



A sleepy mechanism



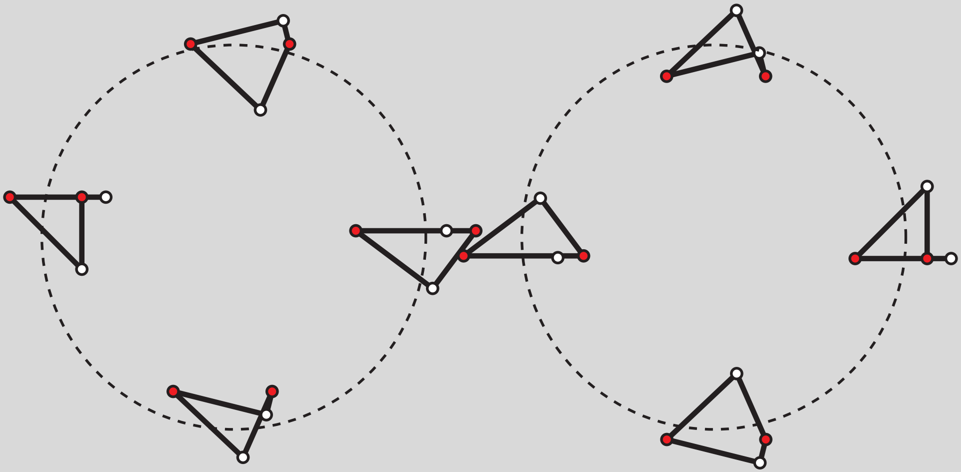
No bar makes a full turn.

There is a motion connecting the framework and its mirror image.

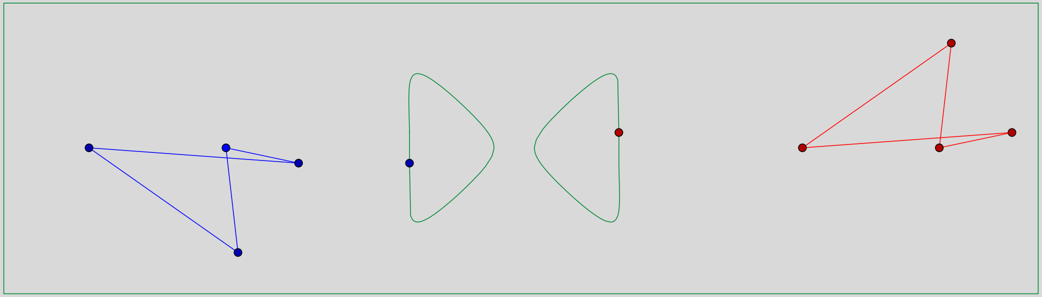
Bashful: (Timide)

$d + a < b + c$. The *cb* arm cannot fully open.
But, the *da* arm has its full range of motion, (*Grashof*) but the *cb* arm is forced to lie on one side of the line XY.
Configuration Space: *The disjoint union of two circles. The mechanism cannot achieve its full potential.*

Example: $(1, 3, 4, 4)$



A Bashful Mechanism



The smallest bar makes a full turn

The there is no motion from the framework to its mirror image.

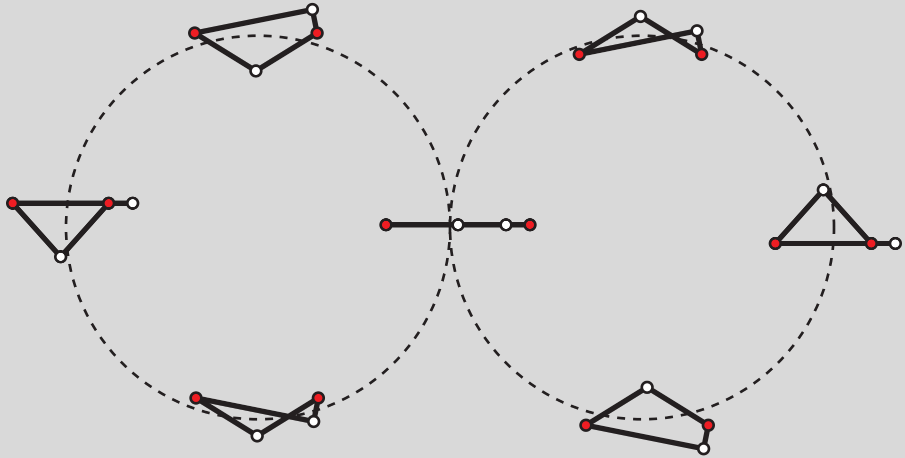
Sneezy: (Atchoum)

$d + a = b + c$

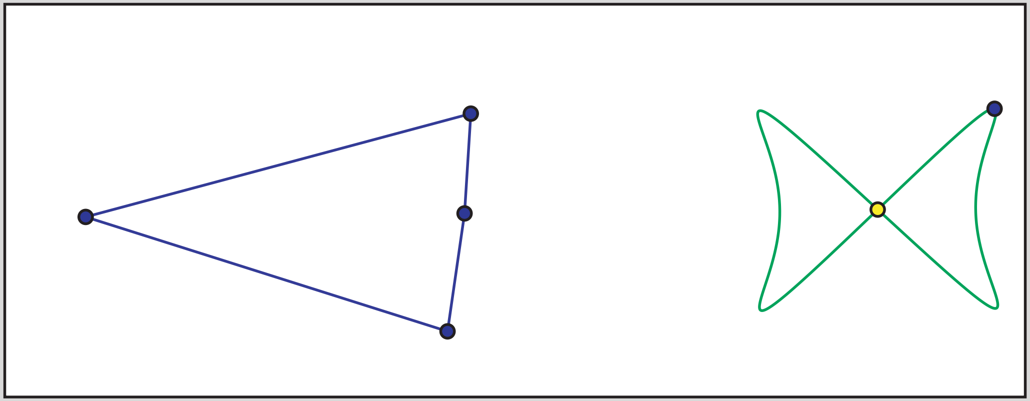
The da and the cb arm can simultaneously fully open.

Configuration Space: *The one point union of two circles. If you sneeze at the singular point then you may pop over to the other S^1.*

Example: $(1, 3, 3, 5)$



Note: Also require $d - a > c - b$



The shortest arm makes two full turns to traverse the space.

The configuration space is projected onto the y-coordinates of the middle points.

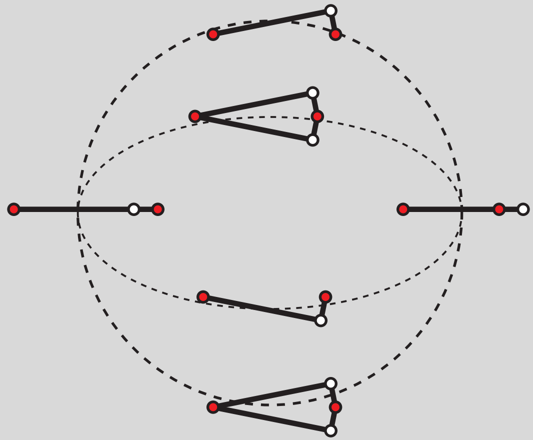
Happy: (Joyeaux)

$d + a = b + c \qquad d - a = c - b \; (d = c, \, a = b)$

The da and the cb arm can simultaneously fully open, and they can also fully close

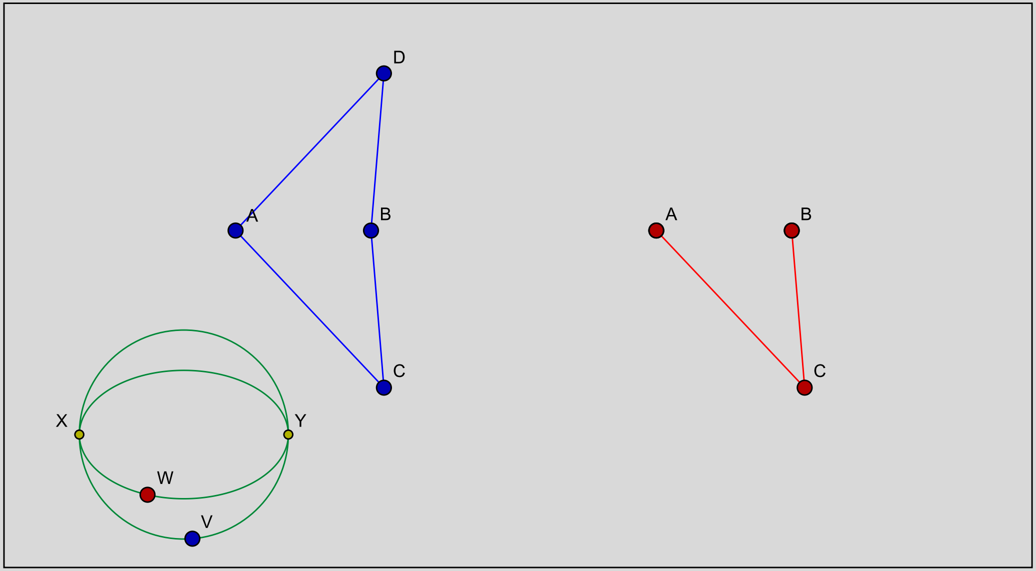
Configuration Space: *The two point union of two circles, much more fun to play with.*

Example: $(1, 1, 5, 5)$



Note: Also require $b < c$

A happy mechanism



The shortest arm makes two full turns to traverse the space.

The configuration space is projected onto the y-coordinates of the middle points.

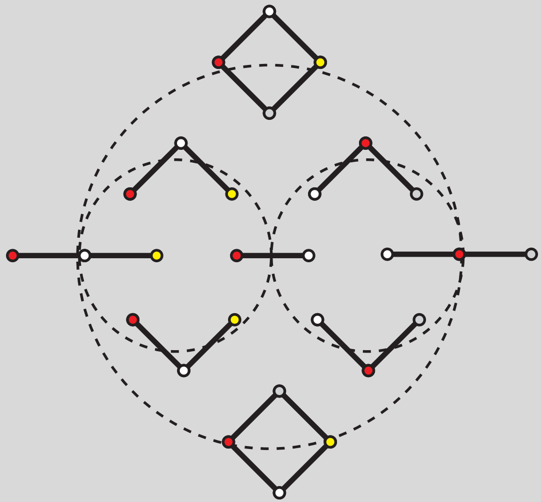
Doc: (Prof)

$a = b = c = d$

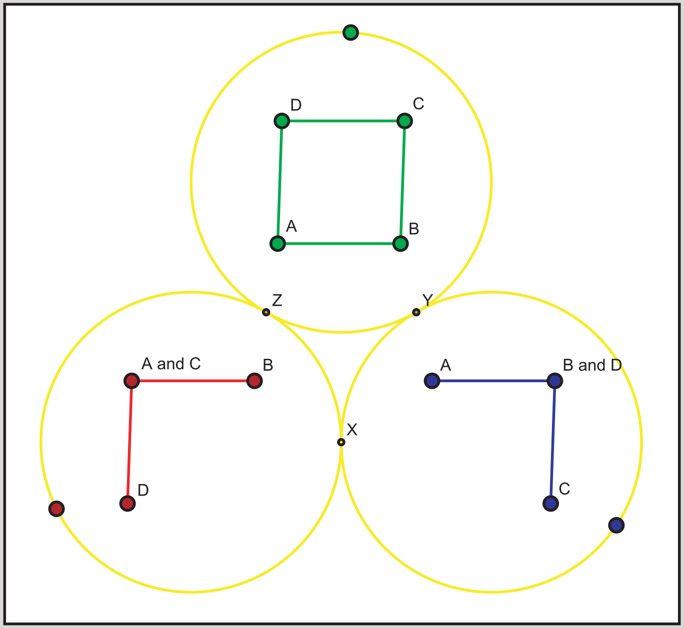
The *da* and the *cb* arm can simultaneously fully open, and they can also fully close, and X and Y can merge, at which point the arms can move independently.

Configuration Space: *A one point ring of three circles. The most complex.*

Example: $(3, 3, 3, 3)$



A doc mechanism

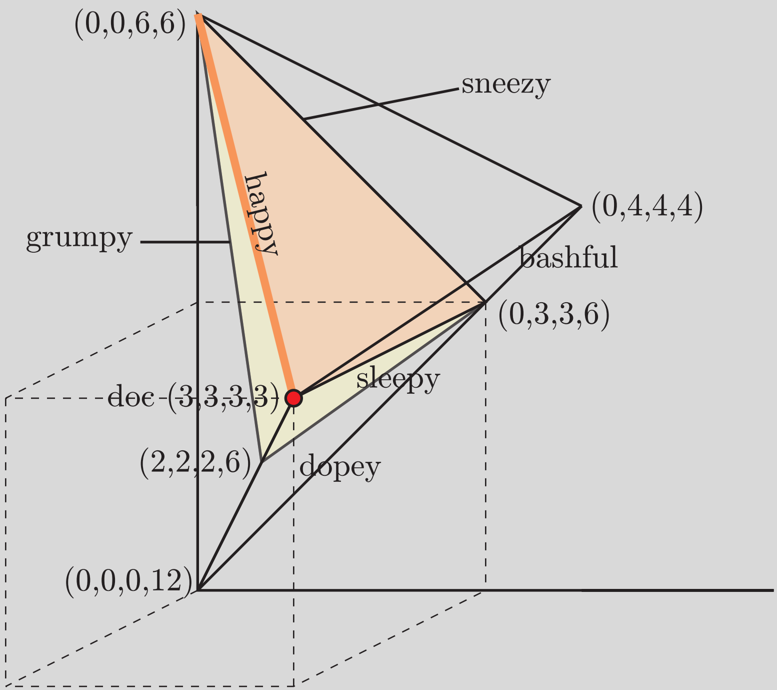


Each colored point on the configuration space correspond to the conformation of the mechanism of the same color.

8.1. Decomposition by configuration type

The Grumpy mechanisms are the boundary between the Dopey and Sleepy mechanisms.

The Sneezy and Happy mechanisms together with the Doc mechanisms form the boundary between the Sleepy and Bashful mechanisms.



8.2. General n-gon (Farber et al.)

The Configuration space of an n-gon in the plane is determined by the lattice of *short* subsets of the edge lengths.

The problem for a general graph in the plane is open.

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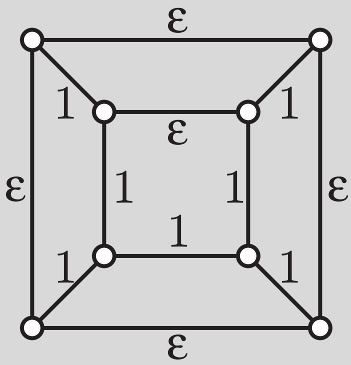
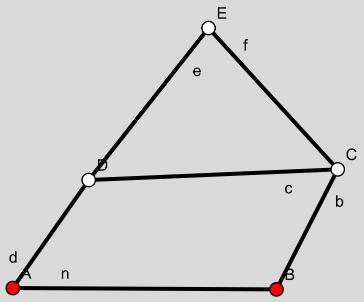
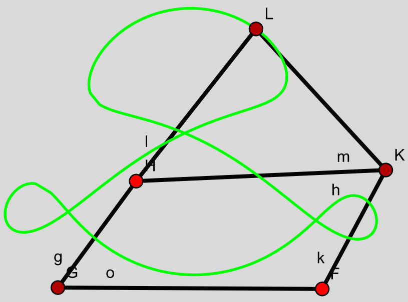
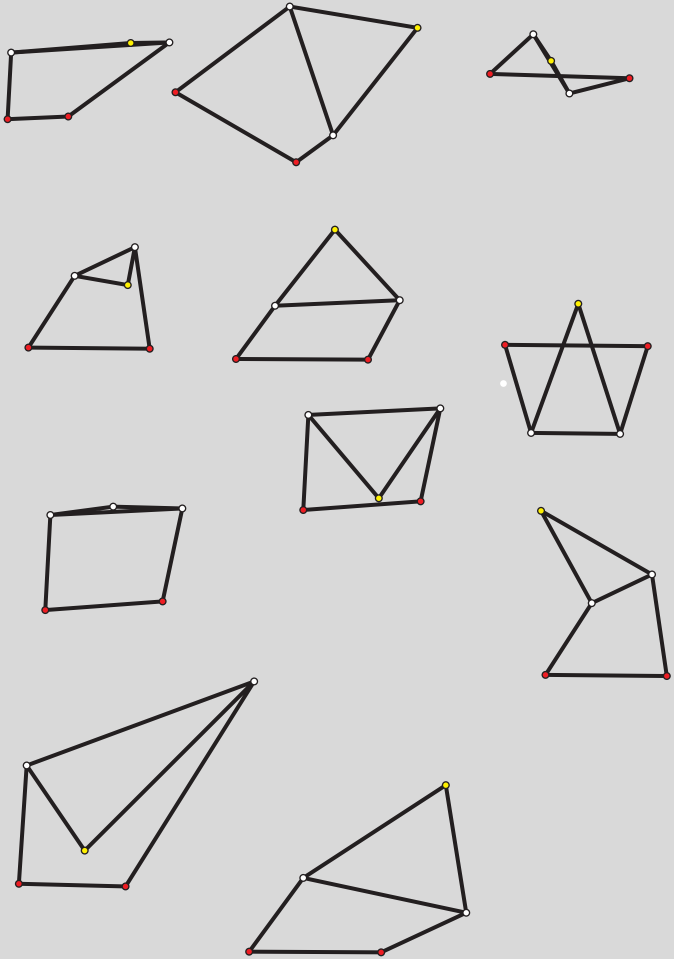
8.3. Dopey is not obvious

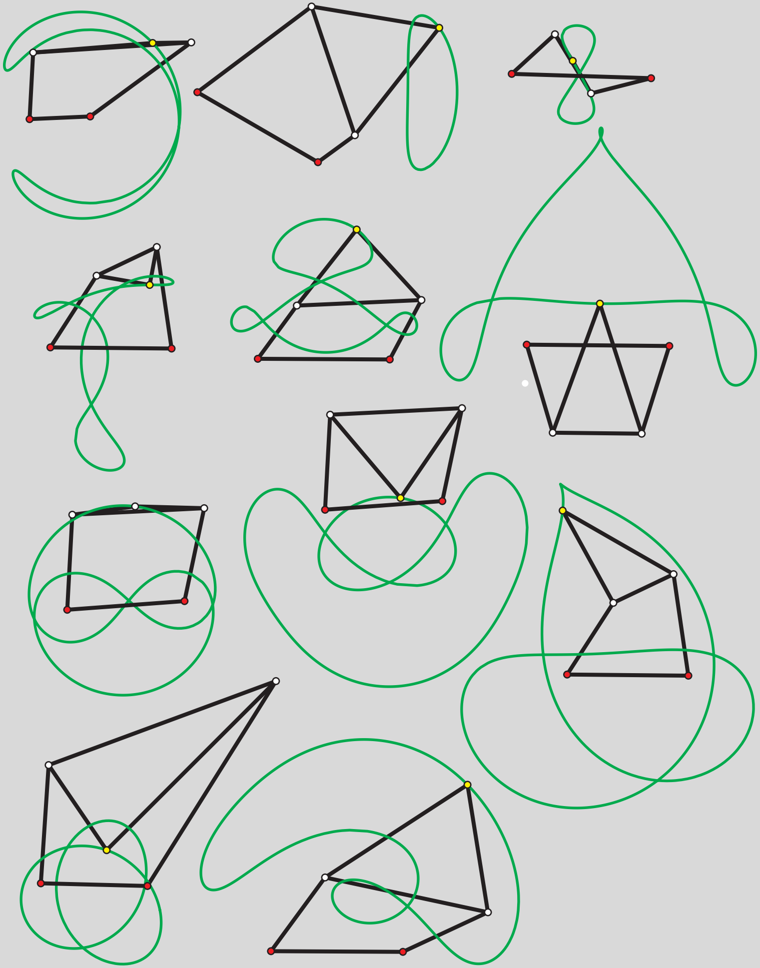


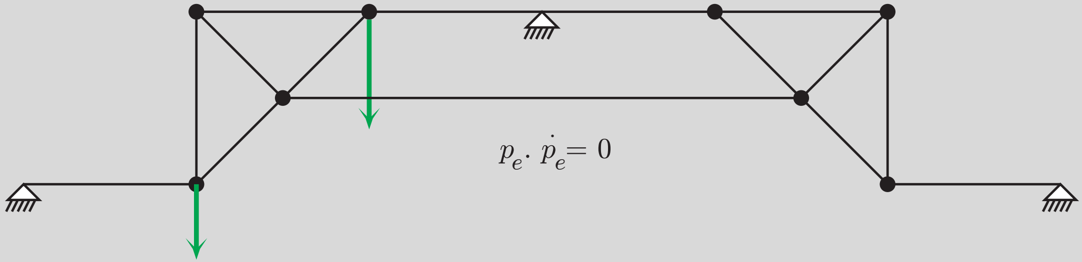
8.4. Coupler Curves











$$p_e \cdot \dot{p}_e = 0$$

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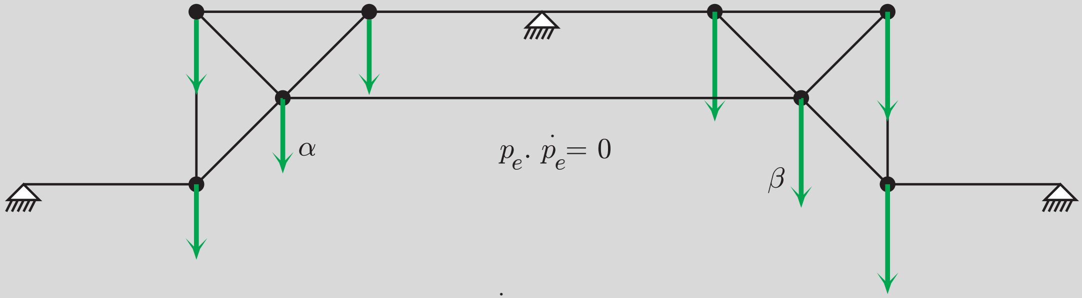

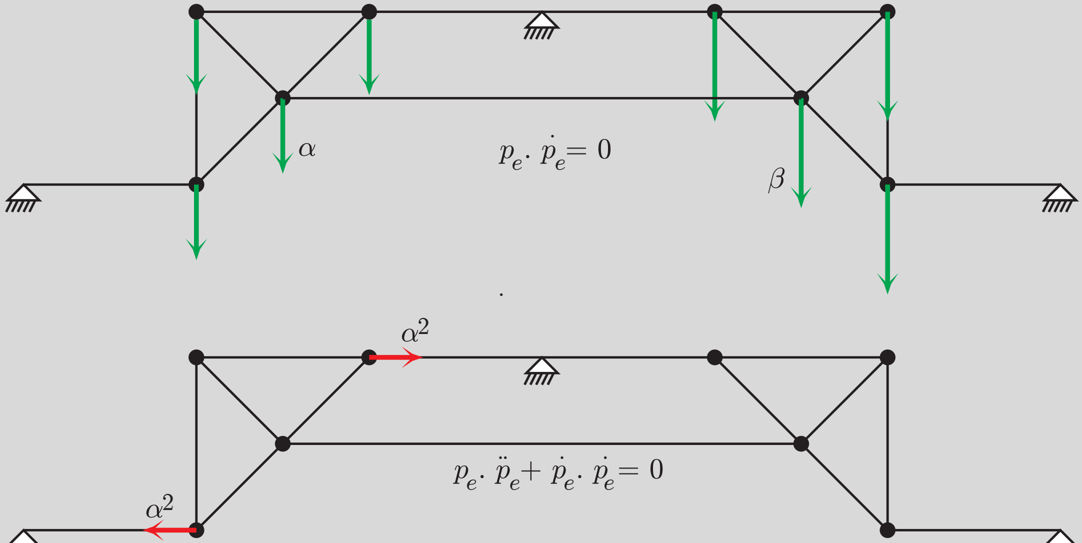
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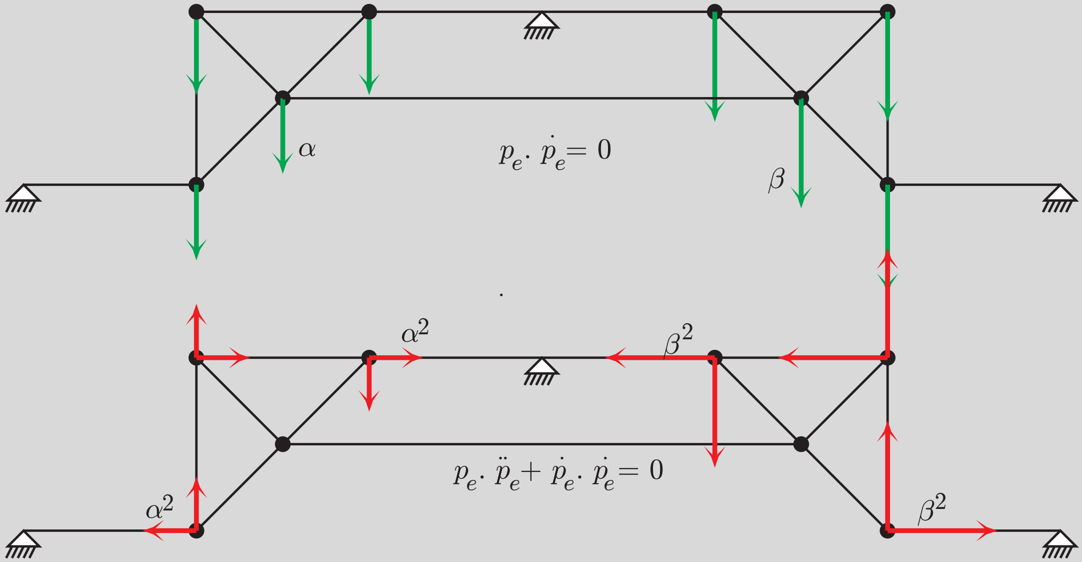
β
α²
$p_e \cdot \ddot{p}_e + \dot{p}_e \cdot \dot{p}_e = 0$
α²


α
$p_e \cdot \dot{p}_e = 0$
β
α^2
β^2
$p_e \cdot \ddot{p}_e + \dot{p}_e \cdot \dot{p}_e = 0$
α^2
β^2

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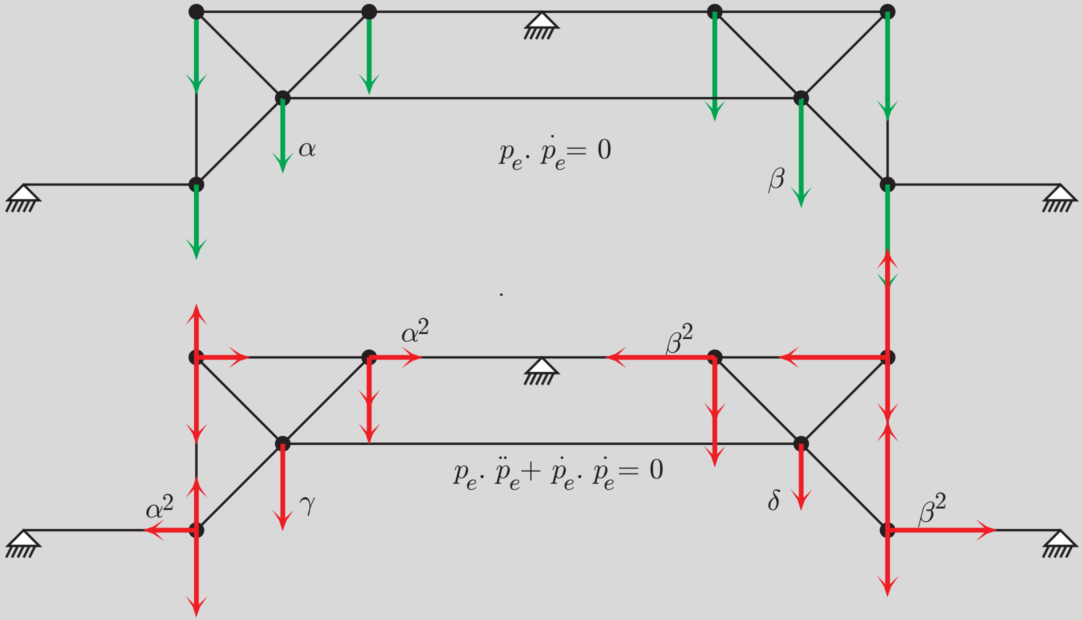
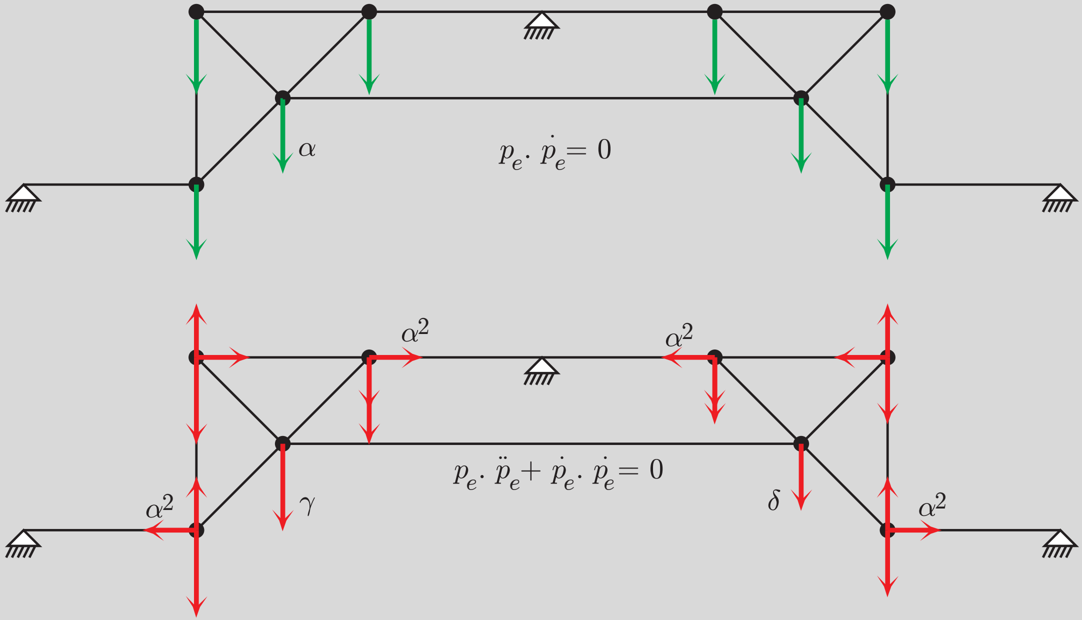
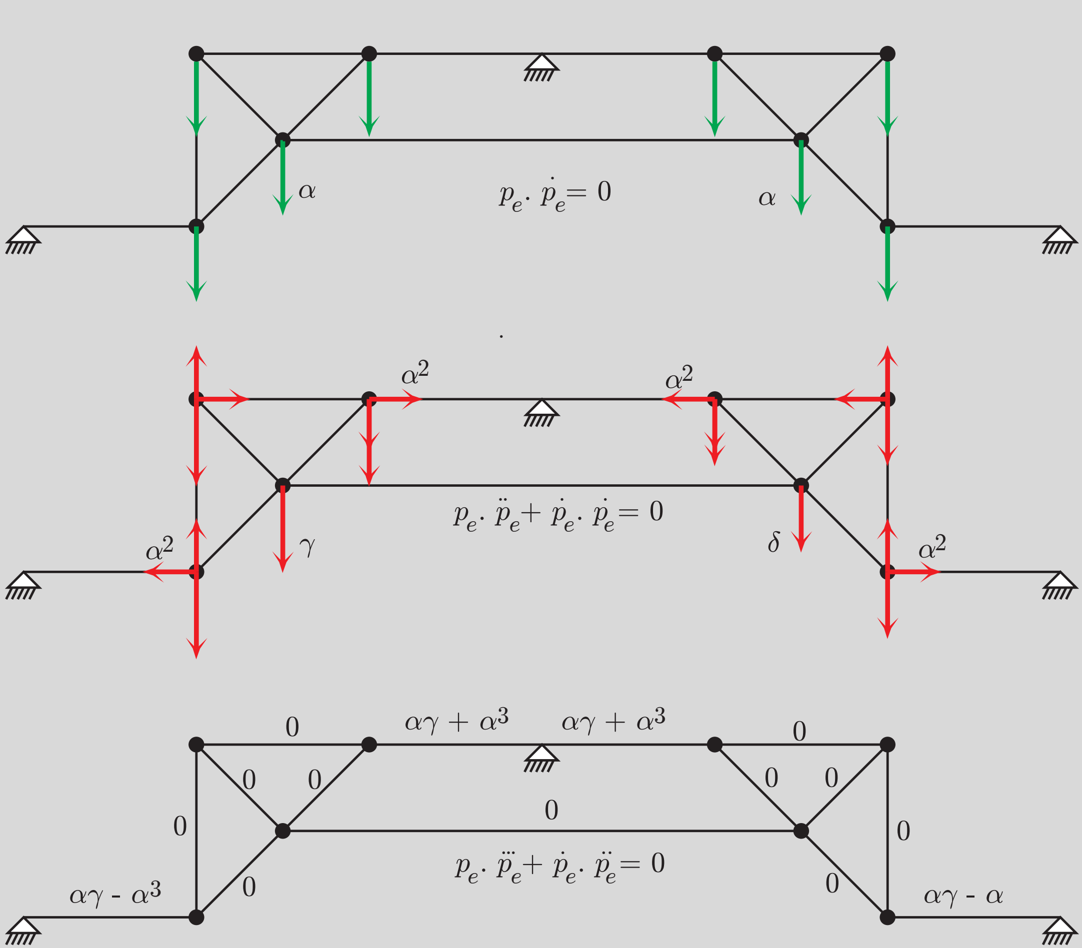



α
$p_e \cdot \dot{p}_e = 0$
α^2
α^2
$p_e \cdot \ddot{p}_e + \dot{p}_e \cdot \dot{p}_e = 0$
α^2
γ
δ
α^2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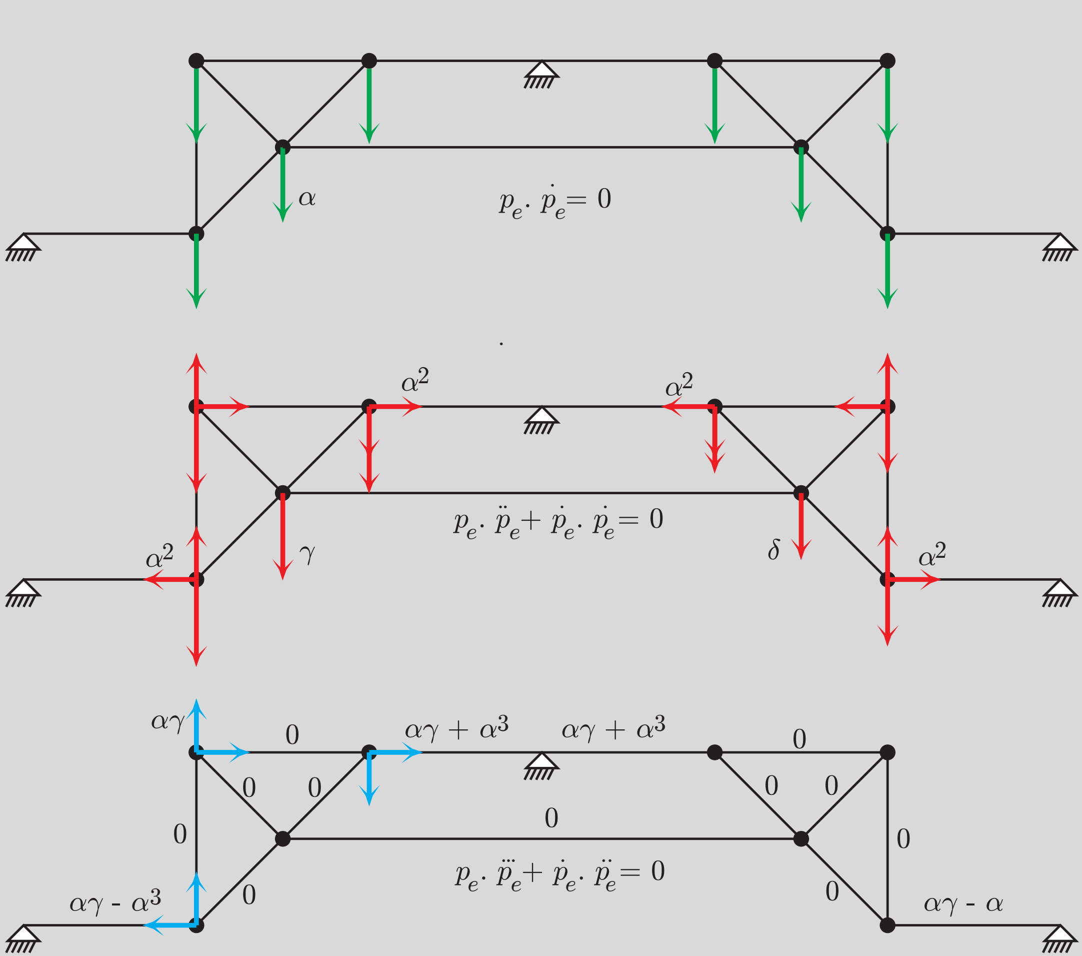
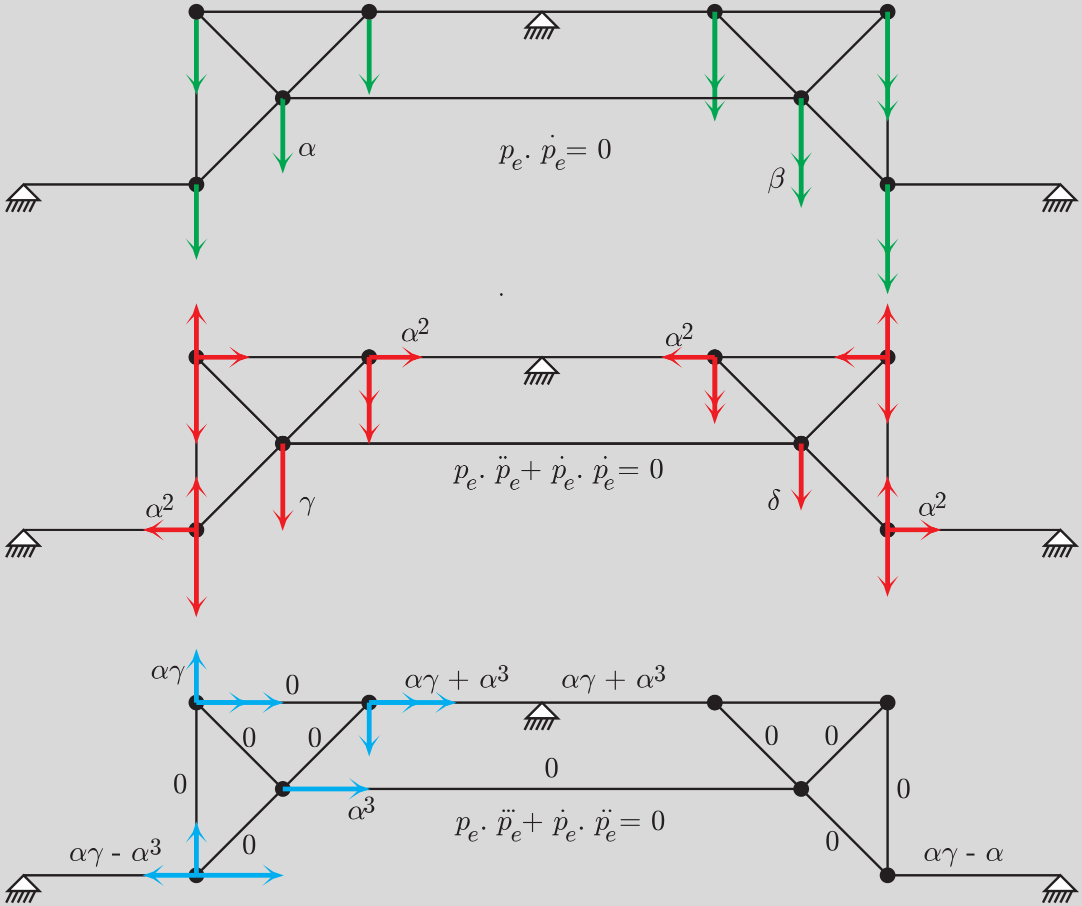
$p_e \cdot \dot{p}_e = 0$
α
α2
α2
$p_e \cdot \ddot{p}_e + \dot{p}_e \cdot \dot{p}_e = 0$
α2
γ
δ
α2
0
αγ + α3
αγ + α3
0
0
0
0
0
0
0
$p_e \cdot \dddot{p}_e + \dot{p}_e \cdot \ddot{p}_e = 0$
0
αγ - α3
0
0
αγ - α



α
$p_e \cdot \dot{p}_e = 0$
α^2
α^2
$p_e \cdot \ddot{p}_e + \dot{p}_e \cdot \dot{p}_e = 0$
α^2
γ
δ
α^2
αγ
0
$\alpha\gamma + \alpha^3$
$\alpha\gamma + \alpha^3$
0
0
0
0
0
0
0
0
$p_e \cdot \dddot{p}_e + \dot{p}_e \cdot \ddot{p}_e = 0$
$\alpha\gamma - \alpha^3$
0
0
αγ - α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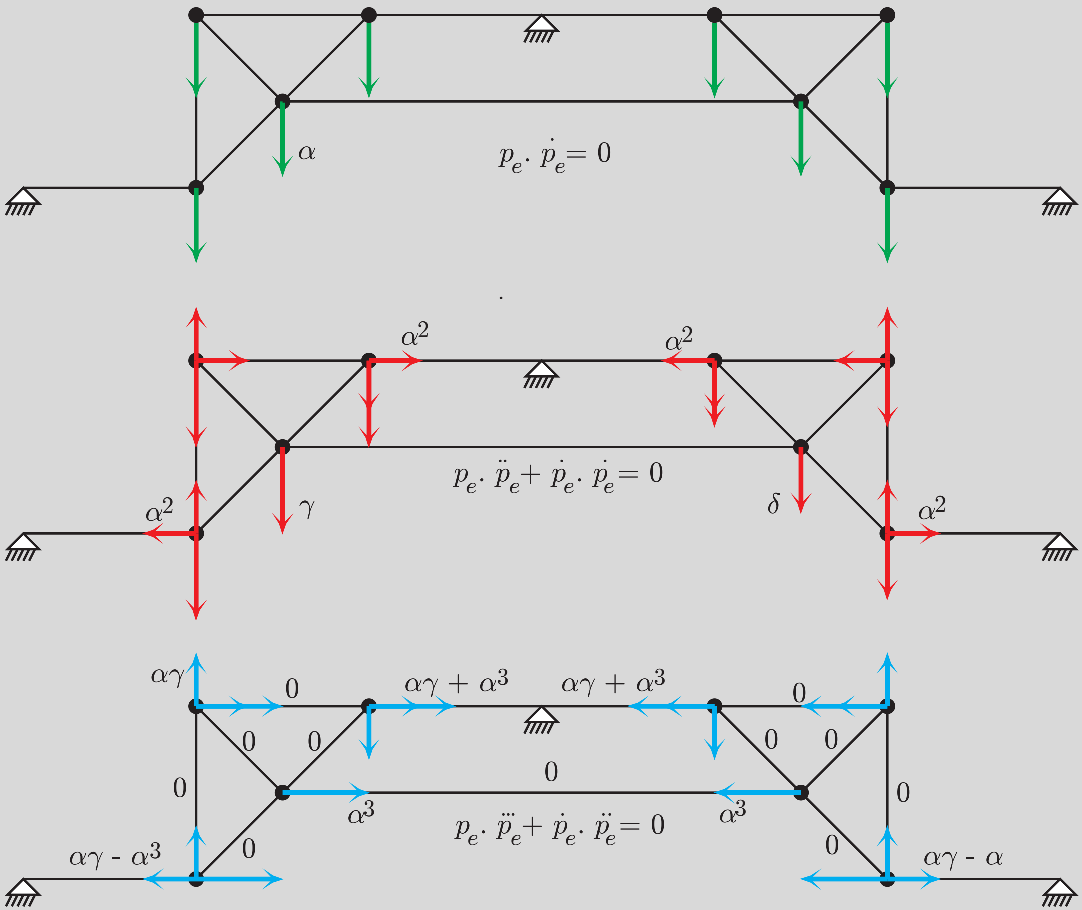
β
α^2
α^2
$p_e \cdot \ddot{p}_e + \dot{p}_e \cdot \dot{p}_e = 0$
α^2
γ
δ
α^2
$\alpha\gamma$
0
$\alpha\gamma + \alpha^3$
$\alpha\gamma + \alpha^3$
0 0
0 0
0
0
0
α^3
$p_e \cdot \dddot{p}_e + \dot{p}_e \cdot \ddot{p}_e = 0$
$\alpha\gamma$ - α^3
0
0
$\alpha\gamma$ - α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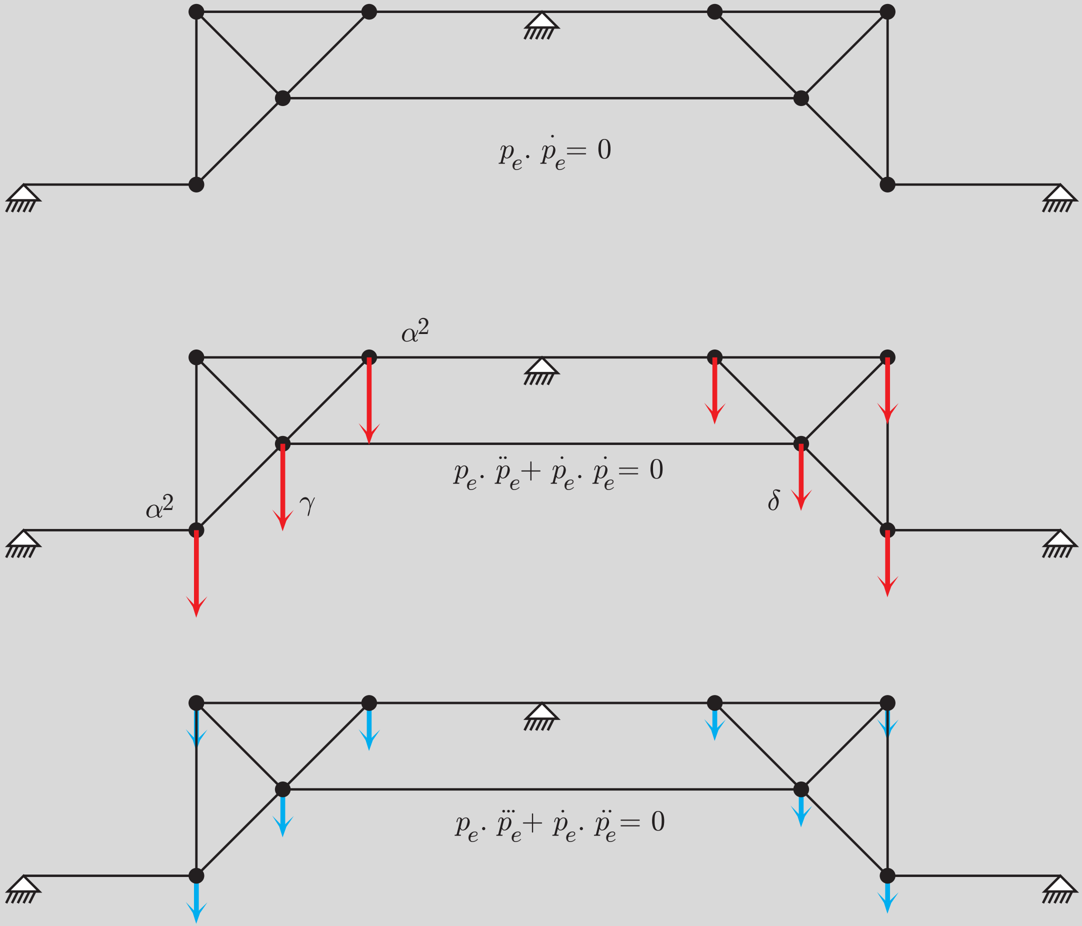
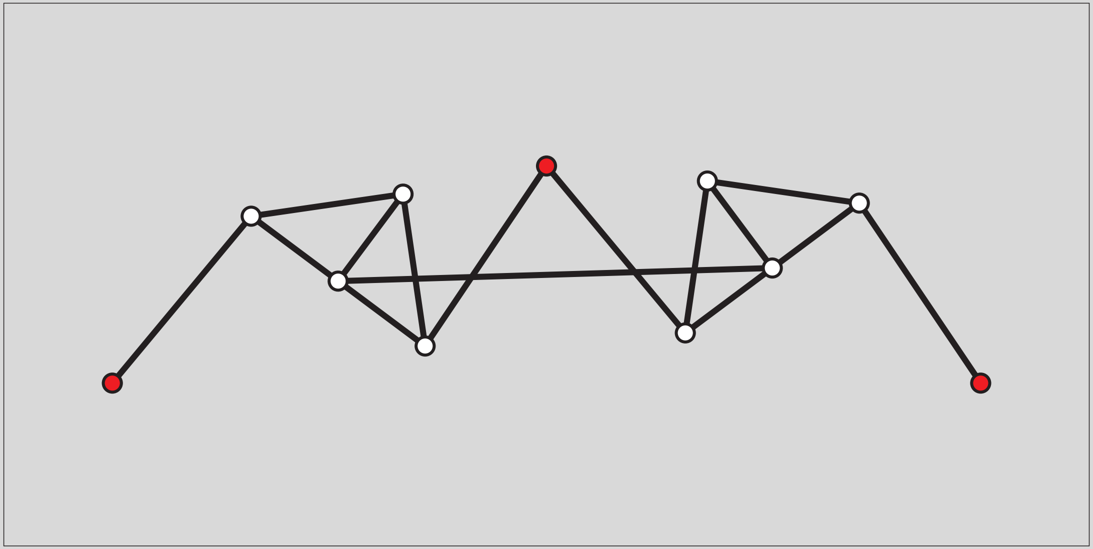
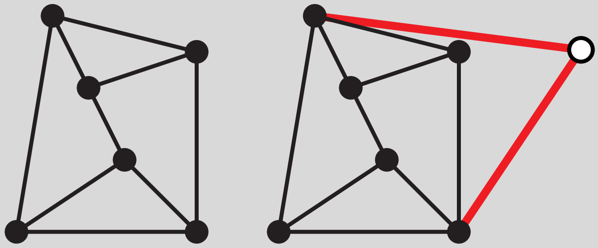
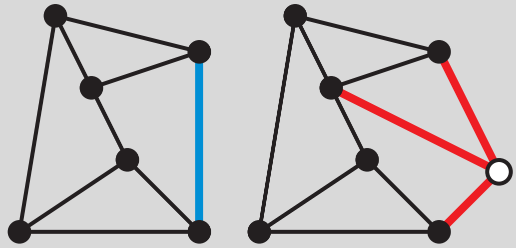
α^2
α^2
α^2
α^2
γ
δ
$p_e \cdot \ddot{p}_e + \dot{p}_e \cdot \dot{p}_e = 0$
αγ
0
$\alpha\gamma + \alpha^3$
$\alpha\gamma + \alpha^3$
0
0
0
0
0
0
0
0
α^3
α^3
$p_e \cdot \dddot{p}_e + \dot{p}_e \cdot \ddot{p}_e = 0$
$\alpha\gamma$ - α^3
0
0
$\alpha\gamma$ - α


$p_e \cdot \dot p_e = 0$
α^2
α^2
γ
δ
$p_e \cdot \ddot p_e + \dot p_e \cdot \dot p_e = 0$
$p_e \cdot \dddot p_e + \dot p_e \cdot \ddot p_e = 0$

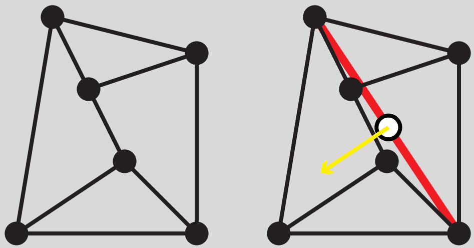


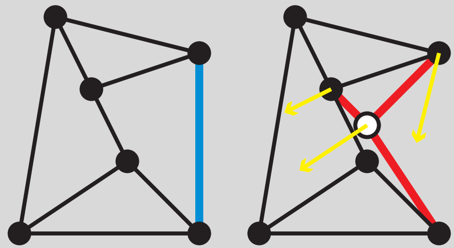
9. Extensions of Frameworks





- 0-extensions and 1-extensions of generically rigid graphs are generically rigid.
- For every infinitesimally rigid graph, and every pair of vertices, almost all 0-extensions are infinitesimally rigid.
- For every infinitesimally rigid graph, and every vertex edge pair, almost all 1-extensions are infinitesimally rigid.





Theorem 12 [7] *Given any generic mechanism containing an edge (a, b) and a vertex c whose motion $c(t)$ is non-circular and has non-trivial winding number about a, there exists a generic Henneberg-1 move at b such that every Henneberg 2 move, generic or not, involving (a, b) and c restricts the motion of c.*

10. Open problems

- Delete 4 edges at random from a random 4-regular graph. What is the expected number of rigid components?
- Are 6-regular random graphs rigid in 3-space?
- Describe the diffeomorphism types of the cube graph in the plane.
- Are random d-regular graphs unit distance graphs?
- Is the configuration space of a linkage most complex if all edge lengths are the same?

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