The Mountaineer's Equation

Herman Servatius

Definitions:
Mountaineer's . . .
Examples
Other Surfaces . . .
Linkages

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

1. Definitions:

The Earth: A (topological) sphere with a mountainous terrain satisfying:

1. The altitude function A is piecewise continuously differentiable.

2. All critical points of A are isolated points, which are either

Peaks Tops of mountains

Pits The opposite, usually occurring at the bottoms of seas or lakes, or glaciers.

Passes Points with two directions along which the tangent is horizontal, dividing the remaining directions into those directions along which the point is a local maximum, those (transverse) directions along which the point is a local minimum.

(So no mesas, or flat planes, or perfect ridges, or crazy passes which simultaneously separate multiple mountains)

Note: It is very difficult to measure A for the real earth. Much easier to use an approximate earth according to some source, say, National Geographic.

2. Mountaineer's Equation

Let the continuous altitude map be given.

Let P denote the total number of peaks.
Let p denote the total number of pits.
Let $\mathcal{P}$ denote the total number of passes.

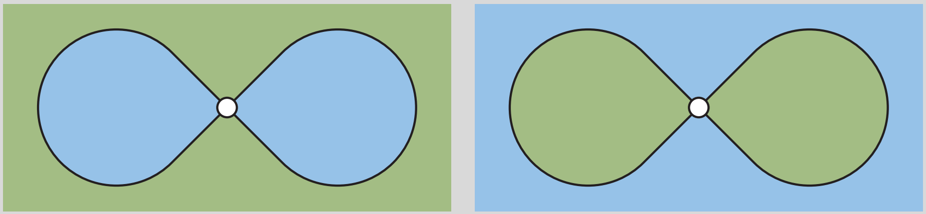
For any map of the earth:

$$P + p = \mathcal{P} + 2$$

Method of Proof:

1. Remove all water from the earth. (its only in the way)
2. Let it rain - a slow gentle drizzle, but world wide, until the whole earth is submerged.

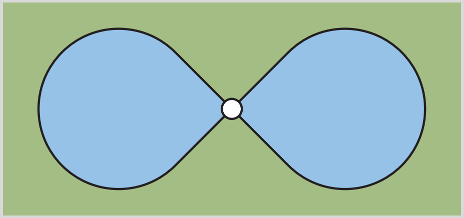
   Note: Let the earth be spongy enough so that the puddles only begin to appear when the water table reaches their level.
3. An ocean will (eventually) develop in every pit.
4. Sooner or later, the oceans will merge. There are two types of merge, either two oceans join, or an ocean joins with itself.



Strait　　　　Isthmus

Note: Not illustrated is the likely case that the seas have islands in them, and, likewise, the islands have seas within them.

If two ocean's merge the point of contact will form a *strait*, (Like the Strait of Gibraltar)

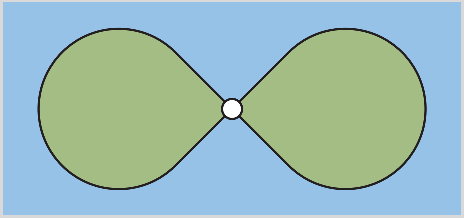


Strait

The the main part, *for us the one one with the deepest pit*, is usually thought of as the ocean, and the part of it across the strait, is regarded as merely a sea in that ocean.

We associate with the strait, the deepest pit in the sea.

As an ocean contour merges with itself, the point of contact will form a *isthmus*, (Like the Isthmus of Panama) At the point of contact, we have a figure-eight, each section of which, by the **Jordan Curve Theorem**, separates the sphere into two discs. So the land is about to be separated.



Isthmus

The the main part, *for us the one one with the tallest mountain*, is usually thought of as the mainland, and the part of it across the isthmus, is regarded as merely a future island.

We associate with each isthmus the tallest peak on the island.

At any generic moment, there are continents and seas.

Since every pit starts isolated, it will be the deepest point of some ocean at some times, and as things merge, the only way it can avoid being associated with some strait is if it is the deepest point of the whole earth (the Aleutian Trench).

Conversely, every peak sooner or later finds itself isolated, the only special point on its own continent (island), and and the only way it can avoid be associated with some isthmus, if it is the tallest point of the whole earth, (Mount Everest).

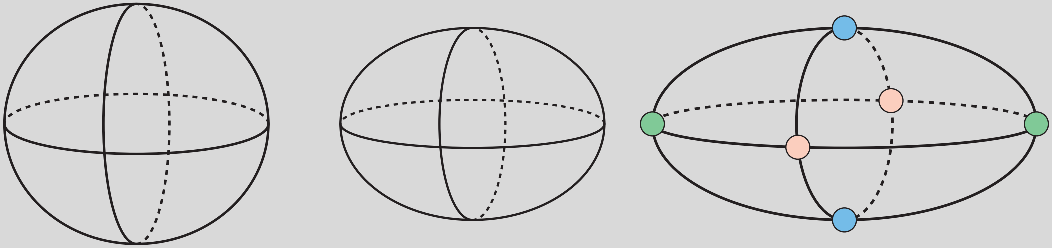
Since the merging happens at different times, the correspondence between the isthmuses and straits with the sub-optimal peaks and pits is one-to-one and onto. So we have, letting I be the number of isthmuses and S be the number of straits:

$$\mathcal{P} = I + S = (P - 1) + (p - 1)$$

So the equation is proved.

3. Examples

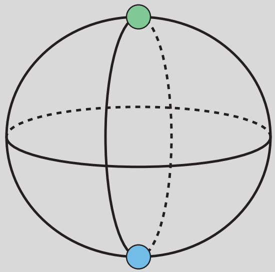
The first two Earths are unacceptable. Why?



Sphere Oblate Spheroid Elipsoid

$$P + p = 2 + 2, \qquad \mathcal{P} + 2 = 2 + 2$$

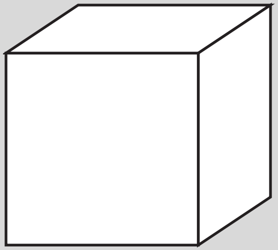
Well, on second thought, just suspend the spherical earth just above the surface of the sun, so that the earth's gravitational field is negligible.



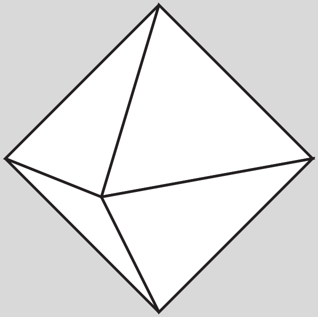
Sphere

$$P + p = 1 + 1, \qquad \mathcal{P} + 2 = 0 + 2$$

Let's try a polyhedral earth: (Put the center of mass pretty close to, but not exactly at, the centroid. – Why?)
Can you tell where the pits, peaks and passes are?



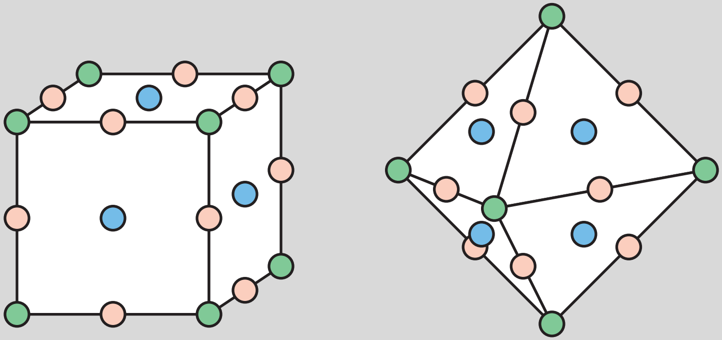
Cube



Octahedron

Let's try a polyhedral earth: (Put the center mass pretty close to the center.)

Can you tell where the pits, peaks and passes are?



Cube Octahedron

Cube: $P - \mathcal{P} + p = V - E + F = 8 - 12 + 6 = 2$

Octahedron: $P - \mathcal{P} + p = V - E + F = 6 - 12 + 8 = 2$

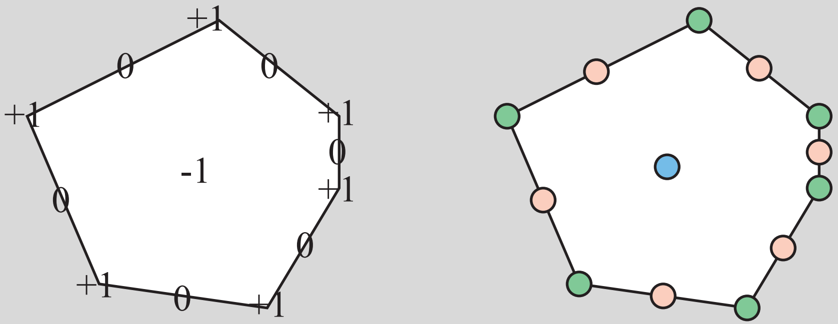
Definitions:
Mountaineer's . . .
Examples
Other Surfaces . . .
Linkages

Home Page
Title Page
Page 13 of 28
Go Back
Full Screen
Close
Quit

4. Other Surfaces get Rained on

In fact, on any polygonal surface, it is easy to engineer, say piecewise linear interpolation, a height function so that the mountaineers count $P - \mathcal{P} + p$ will give the Euler Characteristic:

$$V - E + F$$

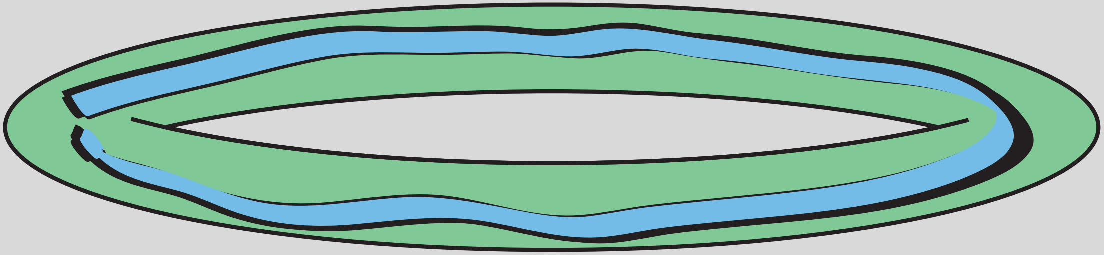


Does this happen for ALL height functions on orientable surfaces?

From the above argument, we know some height functions give $P - \mathcal{P} + p = V - E + F = \chi(S)$. If our original argument goes through unchanged, we would always get 2.

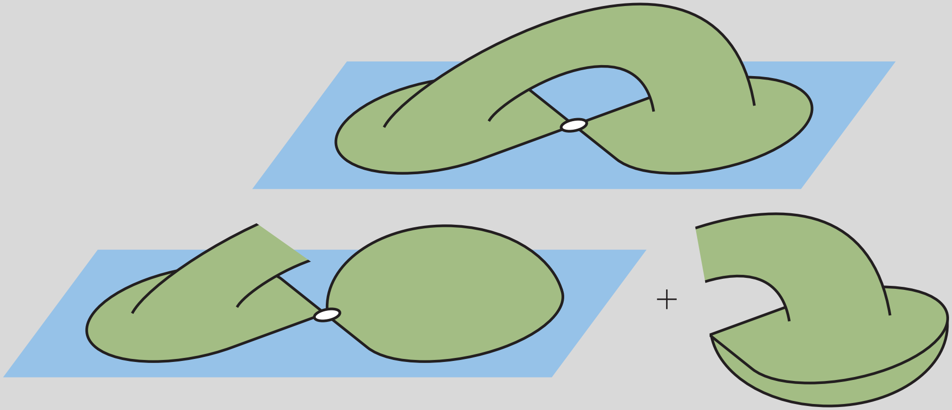
Look back and try to find the what must be changed:

The Jordan Curve Theorem is only for the sphere.
An ocean meeting itself may form an isthmus separating the land, or, it might not. . .



The curves of the figure eight may be *non-separating simple closed curves.*
Each time this happens we get no island, so we have no peak to assign.

Instead, we cut the surface along the curve, a topological circle, and patch it with two disks, continuing the height function the one where it is decreasing to a pit, and on the other, where it is increasing, with a peak.



The new surface is simpler, having one fewer non-separating curve.
It has one new *Pseudo-Peak*, one new *Pseudo-Pit*, and the isthmus will now match with the shallow new pseudo-peak.
Note: The new surface seems to interpenetrate itself, but that is only in the picture. The surface itself is abstract. If are imagining it in 3-space, just nudge the part that seems intersecting a little into the fourth dimension.

Continue in this way to get a count on the new surface: with pseudo-peaks pP and pseudo-pits pp satisfying the old Mountaineer's equation:

$$\mathcal{P} = I + S = (P + PP - 1) + (p + pp - 1) = P + p - 2 + 2g$$

where g, the *genus* is the number of non-separating simple closed curves of the original surface.
So, for the original surface we have

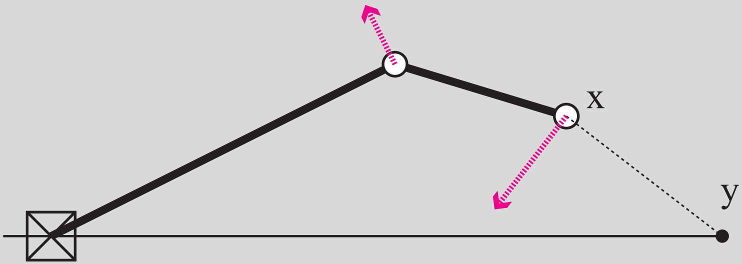
$$P - \mathcal{P} + p = 2 - 2g = \chi(S)$$

as we suspected.
and with a little more work, we could prove that every every simple closed orientable surface is represented as a *handlebody*, that is, an g-holed torus.

5. Linkages

A simple robot arm in $\mathbb{R}^2$:

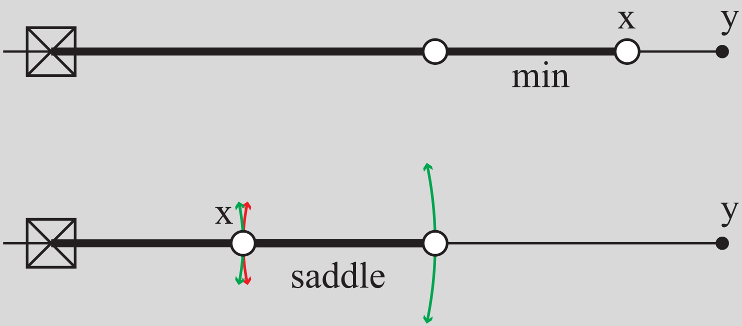
Two effectors. What is the *Configuration Space.*



The configuration space is a surface. Why?
What is the surface? You should be able to tell before we compute it.
Height Function: (squared) distance from x to y. Is the given position a critical point?

If either point has a velocity which projects non-trivially on the end effector, there is only a one dimensional space of velocities which leave the height from function (infinitesimally) fixed.

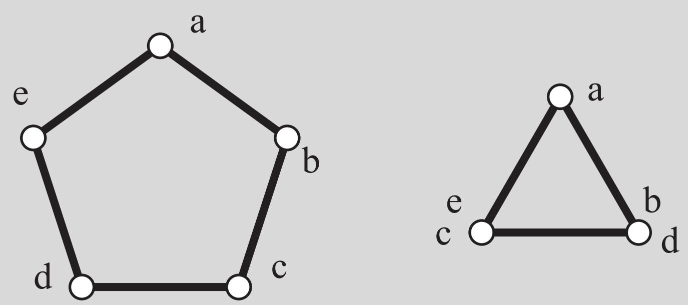
The only critical positions have the arm entirely along the x-axis.
The critical configurations on the positive side, one min and one saddle point.
If the mechanism turns as a whole, *in either direction*, x moves away from y. If the end effector moves alone, it moves toward y.



Other ones are the other side are a max and a saddle. So the mountaineer's equation gives $P - \mathcal{P} + p = 1 - 2 + 1 = 0$.
So the configuration space is a torus, as we expected.

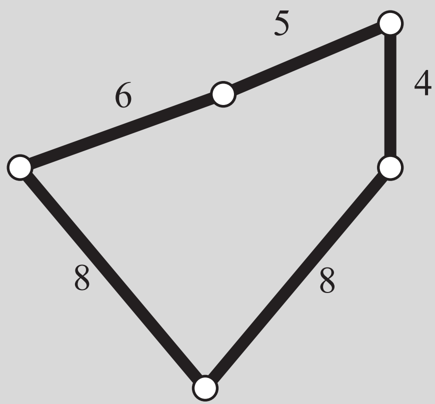
Polygonal Linkages

A simple count tells us to expect a surface.
Pin two vertices to eliminate translations and rotations.
The remaining 3 points have 6 degrees of freedom in the plane.
There are 4 non-pinned linked which constrain them.
6 - 4 = 2, a surface.



But a regular pentagon has jackknifed positions which make the analysis tricky.
We will work on a more generic example.

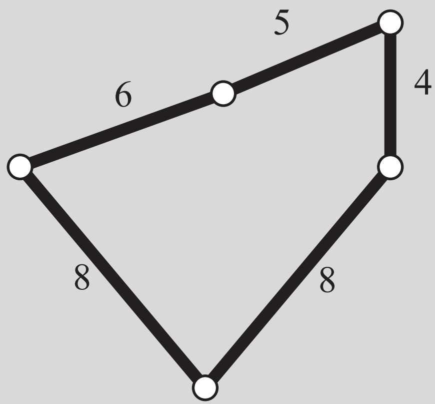
Take a pentagon with sides $\{4, 5, 6, 8, 8\}$ in that order.



We normalize so that the endpoints of the $8, 8$ segment always lie on the x-axis, with the endpoint incident to the bar of length 6 at the origin, and their joining link lies below.
The distance between this points, as bounded by those links, is between 0 and 16, we can continue any configuration of the $4, 5, 6$ linkage with endpoints on the x-axis, uniquely via the normalization.

Definitions:
Mountaineer's . . .
Examples
Other Surfaces . . .
Linkages
Home Page
Title Page
Page 22 of 28
Go Back
Full Screen
Close
Quit

So we need only concern ourselves with those three bars.



For a height function, we take the squared distance to the point 16 on the x-axis.

Alice: But we know the surface is a sphere because at the extremes of height there is a single point, and in between we have a circle for each 4-bar mechanisms. q.e.d.

Bob: Sorry, Alice, it is a torus, because the bars 4 and 5 make a full circle, and for each angle they make, the remaining four-bar has one degree of freedom, so another circle, and $S^1 \times S^1$ is a torus. Q.E.D!

Neither argument is valid. In each case, the critical configurations must be carefully considered: they are critical.

The critical configurations are all along the x axis with the following displacement vectors, whose type we will determine:

$+6+5+4$:	Pit	$-6+5+4$:	Pass
$+6+5-4$:	Pass	$-6+5-4$:	Pass
$+6-5+4$:	Pass	$-6-5+4$:	Pass
$+6-5-4$:	Pass	$-6-5-4$:	Peak

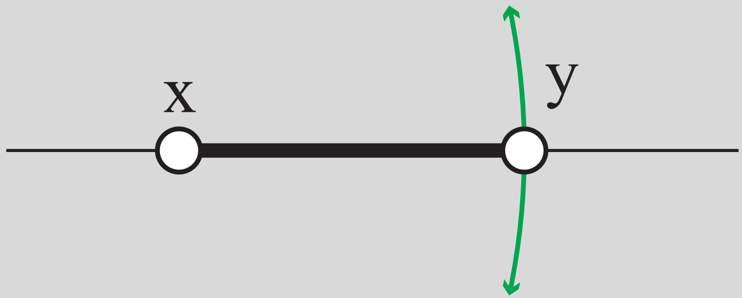
Which would give that the configuration space is a 3-holed torus.

The pit and peak are obvious.

To determine the passes, we need to show that each configuration has two directions, one of which represents a local max and one of which represents a local min.

The observation we need is that if we have a rigid bar along a line, and x is constrained to the line and y moves initially on a circle centered on that line:

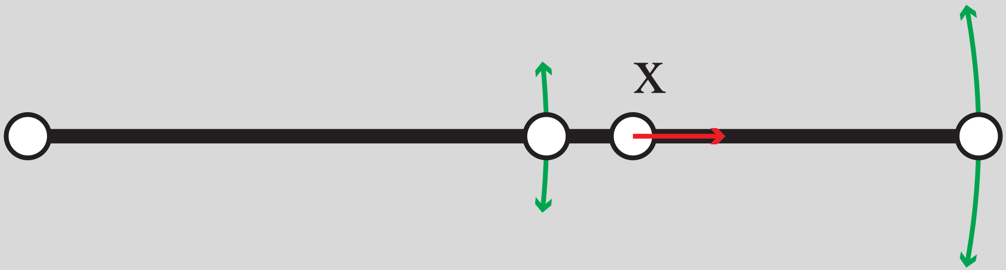
If the center is between x and y, then x is moved to the left, and if the center is to the left of x, then x moves to the right.

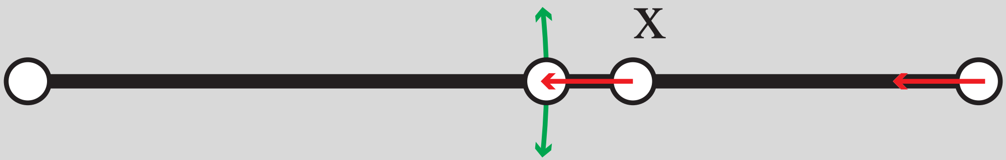


Definitions:
Mountaineer's . . .
Examples
Other Surfaces . . .
Linkages
Home Page
Title Page
Page 26 of 28
Go Back
Full Screen
Close
Quit

$$+6 + 5 - 4$$

You can the rotate the two longer bars as one, and contrast with rotating the longest bar while keeping the other two points on the x-axis.





Pass

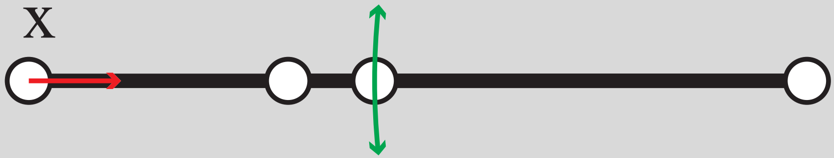
$$+6 - 5 + 4$$

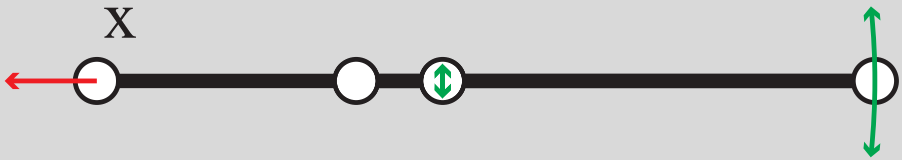
You rotate bar 5 alone, contrasted with rotating bar 6 alone.

Pass.

$$+6-5-4$$

You can the rotate bar 5 alone, and contrast with rotating the jackknifed bar 5 and 6 together.





Pass.
And the other three cases are mirror images, so all give but two gives passes, and the configurations space is a surface of genus 3, a 3-holed torus.