1. Regular Objects

–To a geometer - Platonic Solids
-To a topologist - Platonic Maps
To a graph theorist - Platonic Graphs
2. Combinatorics

\( \mathcal{M} \) is a regular map on the sphere:
vertices: \( m \)-valent
faces: \( n \)-valent.

\[ 2|E| = m|V| = n|F| \]

Euler characteristic: \( \chi(\mathcal{M}) = 2 \)
Thus
\[ \frac{2|E|}{m} - |E| + \frac{2|E|}{n} = 2. \]

Integer solutions: \((m, n > 1)\)

\[ \frac{1}{n} + \frac{1}{m} > \frac{1}{2}, \]

\( m = 2, n \geq 2, \) (an \( n \)-cycle separating the sphere into two \( n \)-gonal faces)
\[ m = 3, \, m = 3 \text{ (tetrahedron)} \]

\[ m = 3, \, n = 4 \text{ (octahedron)}, \]

\[ m = 3 \text{ and } n = 5 \text{ (icosahedron)}, \]

\[ m = 4 \text{ and } n = 3 \text{ (cube)}, \]

\[ m = 5 \text{ and } n = 3 \text{ (dodecahedron)}, \]

\[ m \geq 6 \text{ and } n = 2 \text{ (two vertices connected by } n \text{ edges forming } n \text{ 2-gons)}. \]
3. How many automorphisms?
4. Fundamental Cayley
5. Wallpaper
6. Pattern counting

M. C. Escher laboriously examined multitudes of sketches to determine how many different patterns would result by repeatedly translating a $2 \times 2$ square having its four unit squares filled with copies of an asymmetric motif in any of four aspects.
7. Boarder Patterns

- The $1 \times 2$ case

\[
b = \begin{array}{c}
\end{array} \quad p = \begin{array}{c}
\end{array} \quad q = \begin{array}{c}
\end{array} \quad d = \begin{array}{c}
\end{array}
\]

\[
bb : \begin{array}{c}
\end{array} \quad bq : \begin{array}{c}
\end{array} \quad qb : \begin{array}{c}
\end{array} \quad qq : \begin{array}{c}
\end{array}
\]

\[
bb^* = \cdots bbbbbbb\cdots = \text{\[Diagram: bb^*\]}
\]
\[
bq^* = \cdots bqqbqbq\cdots = \text{\[Diagram: bq^*\]}
\]
\[
qb^* = \cdots qbqbqb\cdots = \text{\[Diagram: qb^*\]}
\]
\[
qq^* = \cdots qqqqqqq\cdots = \text{\[Diagram: qq^*\]}
\]

\[
bq^* = qb^*
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\[
bb^* = qq^*
\]
Rotations only

\[ bb^* = qq^* = \cdots bbbbbb \cdots = \]
\[ bq^* = qb^* = \cdots bqbqbq \cdots = \]
\[ bd^* = pq^* = db^* = qp^* = \cdots bdbdbd \cdots = \]
\[ bp^* = dq^* = pb^* = qd^* = \cdots bpbpbp \cdots = \]
\[ dd^* = pp^* = \cdots dddd \cdots = \]
\[ dp^* = pd^* = \cdots dpdpdp \cdots = \]

Rotations and mirrors

\[ bb^* = qq^* = dd^* = pp^* = \cdots bbbbbb \cdots = \]
\[ bq^* = qb^* = dp^* = pd^* = \cdots bqbqbq \cdots = \]
\[ bd^* = pq^* = db^* = qp^* = \cdots bdbdbd \cdots = \]
\[ bp^* = dq^* = pb^* = qd^* = \cdots bpbpbp \cdots = \]
$S = \{w_1, \ldots\} –$ signatures

$P –$ permutations

$\langle P \rangle –$ permutation group

$\# \text{ different strip patterns } = \# \text{ orbits}$

Burnside’s Lemma: The number of orbits equals the average number of points fixed by the permutations in the group.

$N –$ number of orbits

$\text{fix}(p) –$ the set of signatures fixed by the permutation $p$.

$$N = \frac{1}{|\langle P \rangle|} \sum_{p \in \langle P \rangle} |\text{fix}(p)|$$
\[ T(XY) = YX, \]
\[ R(XY) = R(Y)R(X) \]

\[(R(b) = q, R(q) = b, R(p) = d \text{ and } R(d) = p)\]

\[ \langle P \rangle \text{ and its action on four signatures} \]

Burnside’s Lemma: \(8/4 = 2\).
\[ M(\mathbf{XY}) = M(\mathbf{X})M(\mathbf{Y}) \]

Without reflections - first 4 columns: \( 24/4 = 6 \).

With reflections - \( 32/8 = 4 \).
8. The $1 \times 12$ case

The signature $w = bbdbbpbppqqpp$ on a ring, and $T^4 M(w)$, and their pattern. $T^6 M(w) = w$, so $w \in \text{fix}(T^6 M)$. 
\[ T(a_1 a_2 \ldots a_{12}) = a_2 a_3 \ldots a_{12} a_1. \]

\[ R(a_1 a_2 \ldots a_{11} a_{12}) = R(a_{12}) R(a_{11}) \ldots R(a_2) R(a_1). \]

\[ \langle P \rangle = \langle T, R \mid R^2 = T^{12}, RTR = T^{-1} \rangle \]

\[ M(a_1 \ldots a_{12}) = M(a_1) \ldots M(a_{12}), \]

\[ \langle T, R, M \rangle = \langle T, R \rangle \oplus \langle M \rangle \]
12 rotations of $180^\circ$ around axis in the horizontal plane through the center of the ring.
6 have axes passing through the centers of two motifs
6 have axes on the midpoints of motif boundaries, with the motifs being divided into 6 pairs of orbits.
$6 \cdot 4^6$ fixed signatures.
12 rotations about the vertical axis of \( \frac{i}{12} \cdot 360^\circ = i \cdot 30^\circ \), \( i = 1 \ldots 12 \).
i and 12 have a common divisor: \( k \),

\[
i = pk, \quad 12 = qk
\]

\( q \cdot (i \cdot 30^\circ) \) is a multiple of \( 360^\circ \)

\[
|\text{orbit}(i \cdot 30^\circ)| = \frac{12}{\gcd(i, 12)} - \text{There are } \gcd(i, 12) \text{ of them.}
\]
For each divisor $k$ of 12 there are rotations with orbit size $k$.

Each of these will have $4^{12/k}$ fixed signatures, since we are free to choose any of the four aspects for each orbit.

Twelve has divisors 12, 6, 4, 3, and 2.

$12 : i = 1, 5, 7, 11,$

$6 : i = 2, 10 = 1 \cdot \frac{12}{6}, 5 \cdot \frac{12}{6},$

$4 : i = 3, 9 = 1 \cdot \frac{12}{4}, 3 \cdot \frac{12}{4},$

$3 : i = 4, 8 = 1 \cdot \frac{12}{3}, 2 \cdot \frac{12}{3},$

$2 : i = 6 = 1 \cdot \frac{12}{2}$

$$\varphi(12) \cdot 4^{12/12} + \varphi(6) \cdot 4^{12/6} + \varphi(4) \cdot 4^{12/4} + \varphi(3) \cdot 4^{12/3}$$

$$+ \varphi(2) \cdot 4^{12/2} + \varphi(1) \cdot 4^{12/1}$$

fixed signatures.
twelve reflections in vertical mirrors,

6 of which pass through the center of a motif,
6 of which pass through the boundaries of the aspects,

$6 \cdot 4^6$ fixed signatures
Rotary Reflections

\[ \varphi(12) \cdot 4^{12/12} + \varphi(6) \cdot 4^{12/6} + \varphi(4) \cdot 4^{12/4} + \varphi(2) \cdot 4^{12/2} \]

fixed signatures
9. **The $1 \times 15$ case**

Rotations:

$$\varphi(15) \cdot 4^{15/15} + \varphi(5) \cdot 4^{15/5} + \varphi(3) \cdot 4^{15/3} + \varphi(1) \cdot 4^{15/1}$$

Rotary Reflections: 0  
Vertical Mirror Reflections: 0  
Horizontal Axis Rotations: 0
10. The $1 \times n$ case

In the general case, the group $\langle T, R, M \rangle$ has elements:
- Vertical axis rotations $T^i$
- Horizontal axis rotations $T^i R$
- Vertical reflections $T^i RM$
- Rotary reflections $T^i M$.

and acts on a signature $w = Q_1 Q_2 \cdots Q_n$, $Q_i \in \{b, q, d, p\}$ via:
- Translation: $T(Q_1 Q_2 \cdots Q_n) = Q_2 Q_3 \cdots Q_n Q_1$.
- Rotation: $R(Q_1 Q_2 \cdots Q_n) = R(Q_n) \cdots R(Q_2) R(Q_1)$.
- Mirror: $M(Q_1 Q_2 \cdots Q_n) = M(Q_1) M(Q_2) \cdots M(Q_n)$. 
vertical axis rotations:

2 aspects – \( p(n) = \sum_{k|n} \varphi(k)2^{n/k} \)

4 aspects – \( P(n) = \sum_{k|n} \varphi(k)4^{n/k} \)

rotary reflections

4 aspects – \( R(n) = \sum_{k|n,2|k} \varphi(k)4^{n/k} \)

horizontal axis rotations/vertical mirror reflections:

2 aspects \( q(n) = \begin{cases} (n/2)^{2^{n/2}} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \)

4 aspects \( Q(n) = \begin{cases} (n/2)^{r^{n/2}} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \)
Two aspects and rotational symmetry

\[ f(n) = \frac{p(n) + q(n)}{2n} \]

Four aspects and rotational symmetry

\[ F(n) = \frac{P(n) + Q(n)}{2n} \]

Four aspects and general symmetry

\[ G(n) = \frac{P(n) + 2Q(n) + R(n)}{4n} \]
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11. The $1 \times 4$ case: Escher revisited

1. $\text{bbbb}$
2. $\text{bb bq}$
3. $\text{bb b p}$
4. $\text{bb bd}$
5. $\text{bb qq}$
6. $\text{bb qp}$
7. $\text{bb q d}$
8. $\text{bb pq}$
9. $\text{bb pp}$
10. $\text{bb pd}$
1. $f(n) - \text{A053656 in Sloan’s On-Line Encyclopedia of integer sequences}$

2. $G(n) \approx 2F(n)$

3. $G(n) \approx 4^n/(4n)$ ($G(p) = \lceil 4^n/(4n) \rceil$ for $n = p$ prime)

4. Symmetric motifs

5. Over/Under weave motifs

6. Multiple motifs
12. Self-Duality
13. Maps are not enough

A self-dual graph with no corresponding self-dual map.
14. The Block-Cutpoint Tree

For any graph $G$, the cycle matroid of $G$ is the direct sum over the cycle matroids of the blocks of $G$:

$$M(G) = \sum M(G_i)$$
15. The 3-Block Tree

For a 2-connected graph $G$, the cycle matroid of $G$ is the 2-sum over the cycle matroids of the 3-blocks of $G$:

$$M(G) = M(G_0) \bigoplus_{e_1} M(G_1) \ldots \bigoplus_{e_k} M(G_k)$$
16. The Program

Program: Given a planar graph with whose automorphism group has finitely many vertex orbits.

1. If 3-connected – embed and straighten so that the automorphisms are represented by isometries (euclidean, spherical, or hyperbolic).
   Apply geometric methods.

2. Else if two-connected – form the 3–block tree and use the program on each block, and merge with data on the automorphisms of the tree.

3. Else if connected – from the block-cutpoint tree and apply the program to each block, merging with the tree automorphisms.

4. Else apply program to each connected component, and merge with permutations of isomorphic components.
17. Bibliography


