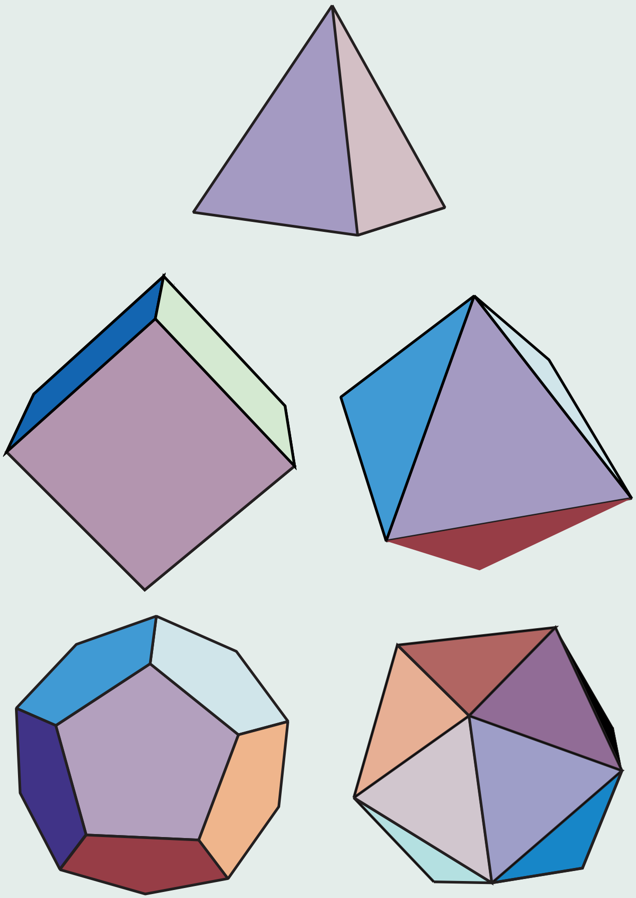
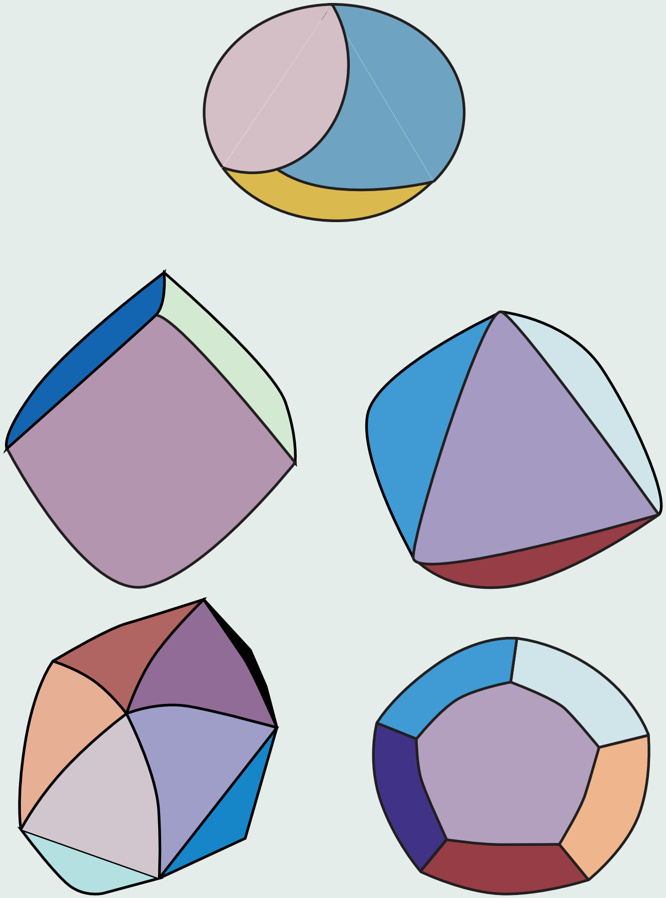
1. Regular Objects

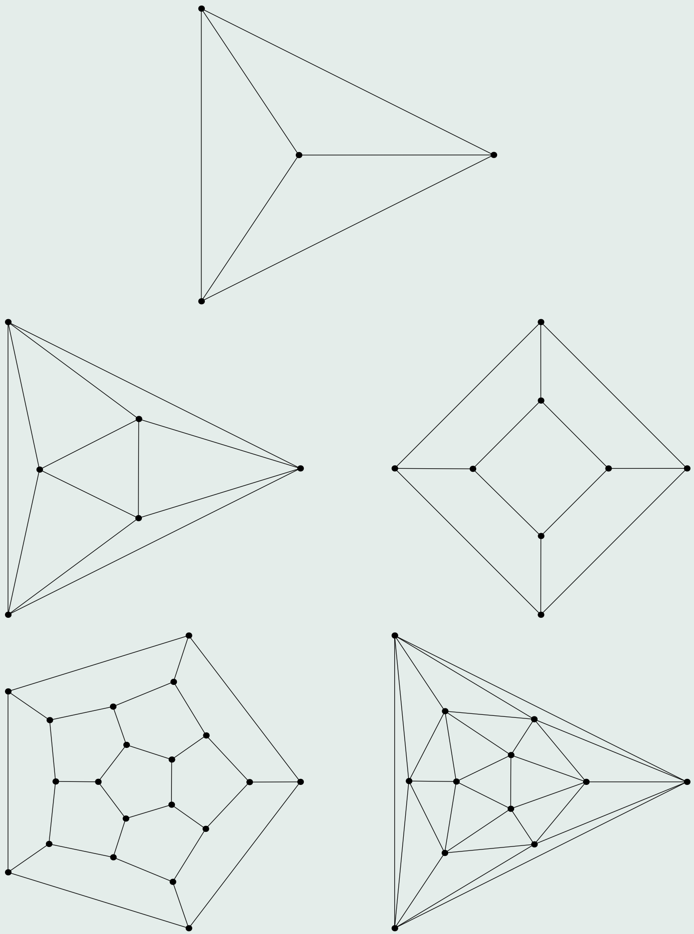
–To a geometer - Platonic Solids



–To a topologist - Platonic Maps



–To a graph theorist - Platonic Graphs



2. Combinatorics

$\mathcal{M}$ is a regular map on the sphere:
vertices: m-valent
faces: n-valent.

$$2|E| = m|V| = n|F|$$

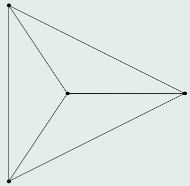
Euler characteristic: $\chi(\mathcal{M}) = 2$
Thus

$$\frac{2|E|}{m} - |E| + \frac{2|E|}{n} = 2.$$

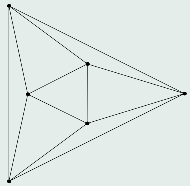
Integer solutions: $(m, n > 1)$

$$1/n + 1/m > 1/2,$$

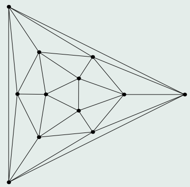
$m = 2$, $n \geq 2$, (an n-cycle separating the sphere into two n-gonal faces)



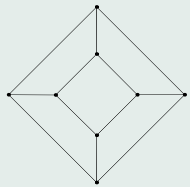
$m = 3$, $m = 3$ (tetrahedron)



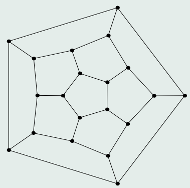
$m = 3$, $n = 4$ (octahedron),



$m = 3$ and $n = 5$ (icosahedron),

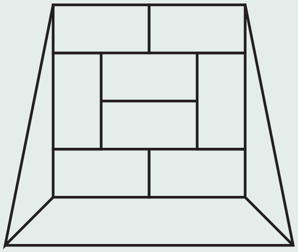


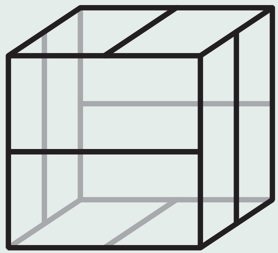
$m = 4$ and $n = 3$ (cube),

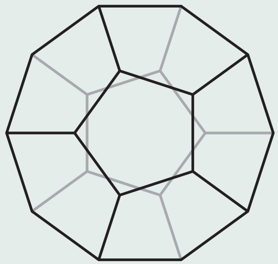


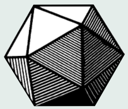
$m = 5$ and $n = 3$ (dodecahedron),
$m \geq 6$ and $n = 2$ (two vertices connected by n edges forming n 2-gons).

3. How many automorphisms?









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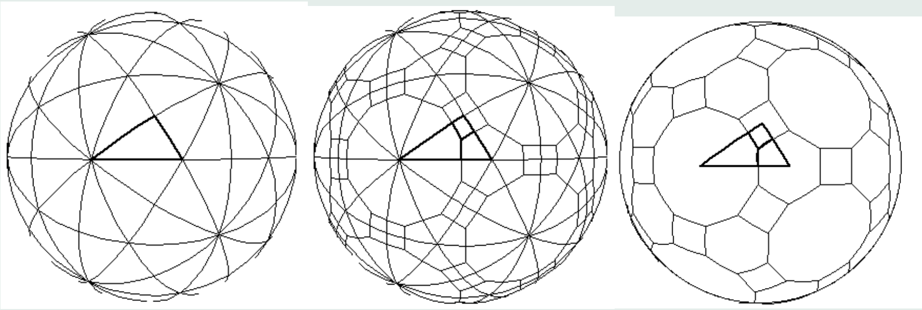
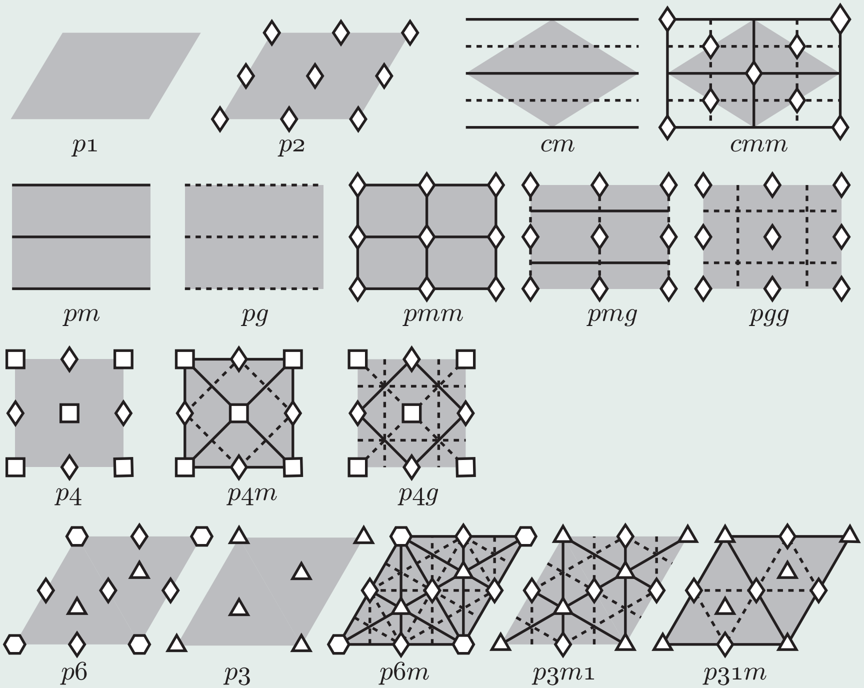
4. Fundamental Cayley

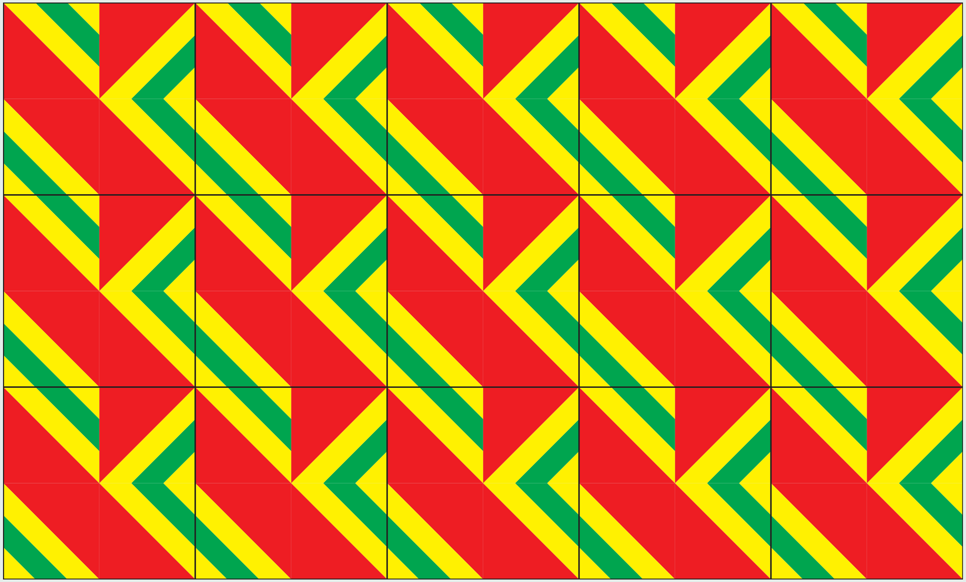


5. Wallpaper



6. Pattern counting

M. C. Escher laboriously examined multitudes of sketches to determine how many different patterns would result by repeatedly translating a 2×2 square having its four unit squares filled with copies of an asymmetric motif in any of four aspects.



7. Boarder Patterns

– The 1 × 2 case

$b =$  $p =$  $q =$  $d =$ 

$bb:$  $bq:$  $qb:$  $qq:$ .

$bb^* = \cdots bbbbbb \cdots =$ 

$bq^* = \cdots bqbqbq \cdots =$ 

$qb^* = \cdots qbqbqb \cdots =$ 

$qq^* = \cdots qqqqqq \cdots =$ 

$$bq^* = qb^*$$
$$bb^* = qq^*$$

Rotations only

$$bb^* = qq^* = \cdots bbbbbb \cdots =$$


$$bq^* = qb^* = \cdots bqbqbq \cdots =$$


$$bd^* = pq^* = db^* = qp^* = \cdots bdbdbd \cdots =$$


$$bp^* = dq^* = pb^* = qd^* = \cdots bpbpbp \cdots =$$


$$dd^* = pp^* = \cdots dddddd \cdots =$$


$$dp^* = pd^* = \cdots dpdpdp \cdots =$$


Rotations and mirrors

$$bb^* = qq^* = dd^* = pp^* = \cdots bbbbbb \cdots =$$


$$bq^* = qb^* = dp^* = pd^* = \cdots bqbqbq \cdots =$$


$$bd^* = pq^* = db^* = qp^* = \cdots bdbdbd \cdots =$$


$$bp^* = dq^* = pb^* = qd^* = \cdots bpbpbp \cdots =$$


$S = \{w_1, \ldots\}$ – signatures
P – permutations
$\langle P \rangle$ – permutation group

different strip patterns = # orbits

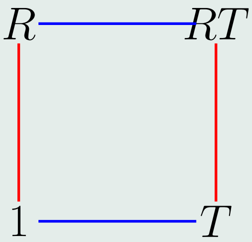
Burnside's Lemma: The number of orbits equals the average number of points fixed by the permutations in the group.

N – number of orbits
$\mathrm{fix}(p)$ – the set of signatures fixed by the permutation p.

$$N = \frac{1}{|\langle P \rangle|} \sum_{p \in \langle P \rangle} |\mathrm{fix}(p)|$$

$$T(XY) = YX,$$
$$R(XY) = R(Y)R(X)$$

$$(R(b) = q,\ R(q) = b,\ R(p) = d \text{ and } R(d) = p)$$

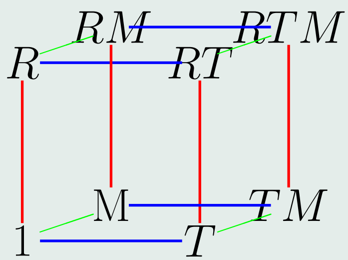


	I	R	T	RT
bb	$\boxed{bb}$	qq	$\boxed{bb}$	qq
bq	$\boxed{bq}$	$\boxed{bq}$	qb	qb
qb	$\boxed{qb}$	$\boxed{qb}$	bq	bq
qq	$\boxed{qq}$	bb	$\boxed{qq}$	bb

$\langle P \rangle$ and its action on four signatures

Burnside's Lemma: $8/4 = 2$.

$$M(XY) = M(X)M(Y)$$

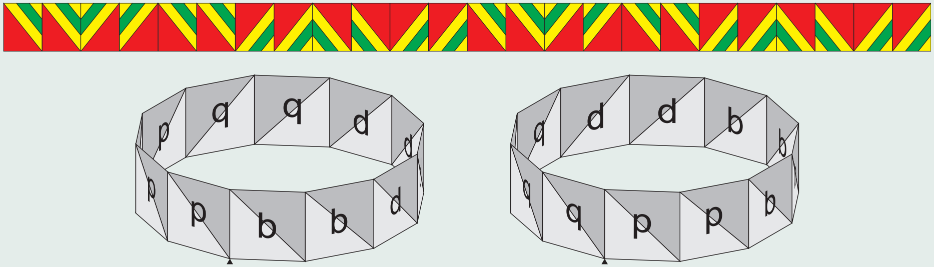


	I	R	T	RT	M	RM	TM	RTM
bb	bb	qq	bb	qq	pp	dd	pp	dd
bq	bq	bq	qb	qb	pd	pd	dp	dp
qb	qb	qb	bq	bq	dp	dp	pd	pd
qq	qq	bb	qq	bb	dd	pp	dd	pp
bp	bp	dq	pb	qd	pb	qd	bp	dq
bd	bd	pq	db	qp	pq	bd	qp	db
qp	qp	db	pq	bd	db	qp	bd	pq
qd	qd	pb	dq	bp	dq	bp	qd	pb
pb	pb	qd	bp	dq	bp	dq	pb	qd
pq	pq	bd	qp	db	bd	pq	db	qp
db	db	qp	bd	pq	qp	dq	pq	bd
dq	dq	bp	qd	pb	qd	pb	dq	bp
pp	pp	dd	pp	dd	bb	qq	bb	qq
pd	pd	pd	dp	dp	bq	bq	qb	qb
dp	dp	dp	pd	pd	qb	qb	bq	bq
dd	dd	pp	dd	pp	qq	bb	qq	bb

Without reflections - first 4 columns: $24/4 = 6$.

With reflections - $32/8 = 4$

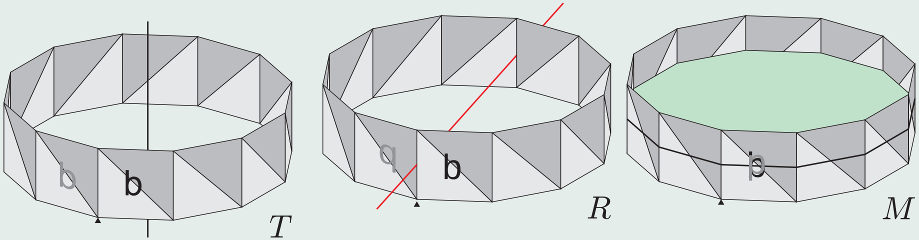
8. The 1×12 case



The signature $w = bbddbbppqqpp$ on a ring, and $T^4M(w)$, and their pattern. $T^6M(w) = w$, so $w \in \text{fix}(T^6M)$.

$$T(a_1a_2\dots a_{12}) = a_2a_3\dots a_{12}a_1.$$

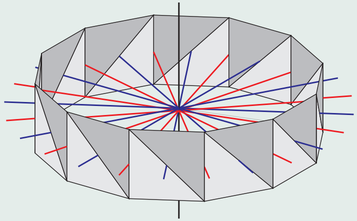
$$R(a_1a_2\dots a_{11}a_{12}) = R(a_{12})R(a_{11})\dots R(a_2)R(a_1).$$



$$\langle P\rangle = \langle T, R \mid R^2 = T^{12}, RTR = T^{-1}\rangle$$

$$M(a_1\dots a_{12}) = M(a_1)\dots M(a_{12}),$$

$$\langle T, R, M\rangle = \langle T, R\rangle \oplus \langle M\rangle$$

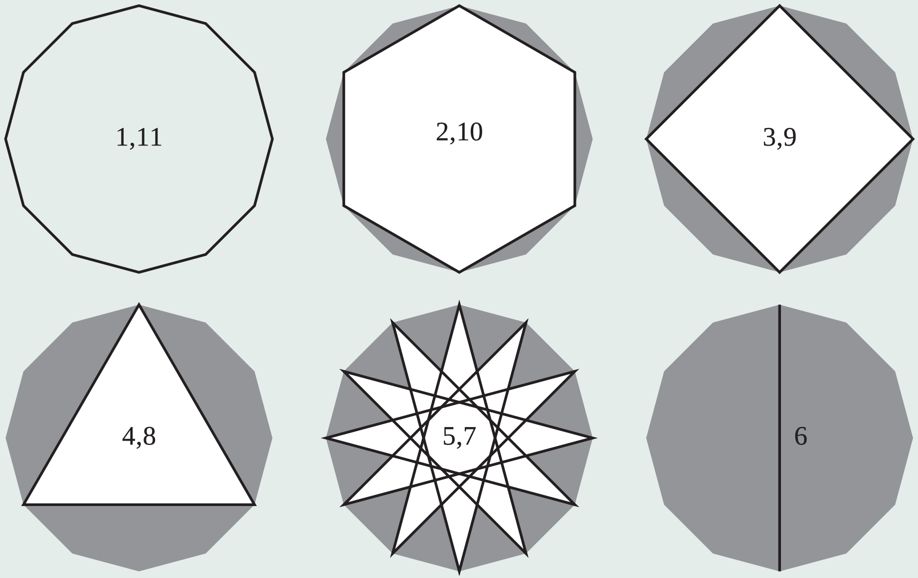


12 rotations of 180° around axis in the horizontal plane through the center of the ring.
6 have axes passing through the centers of two motifs
6 have axes on the midpoints of motif boundaries, with the motifs being divided into 6 pairs of orbits.
$6 \cdot 4^6$ fixed signatures.

12 rotations about the vertical axis of $\frac{i}{12}360° = i \cdot 30°$, $i = 1 \ldots 12$.
i and 12 have a common divisor: k,

$$i = pk, \qquad 12 = qk$$

$q \cdot (i \cdot 30°)$ is a multiple of $360°$



$|\text{orbit}(i \cdot 30°| = 12/\gcd(i, 12)$ – There are $\gcd(i, 12)$ of them.
$4^{12/\gcd(i,12)}$

For each divisor k of 12 there are rotations with orbit size k.

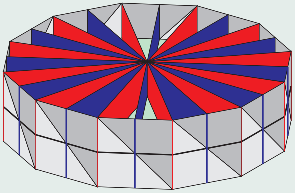
Each of these will have $4^{12/k}$ fixed signatures, since we are free to choose any of the four aspects for each orbit.

Twelve has divisors 12, 6, 4, 3, and 2.

$$\begin{aligned}
12 : i &= 1, 5, 7, 11, \\
6 : i &= 2, 10 && = 1 \cdot \frac{12}{6}, 5 \cdot \frac{12}{6}, \\
4 : i &= 3, 9 && = 1 \cdot \frac{12}{4}, 3 \cdot \frac{12}{4}, \\
3 : i &= 4, 8 && = 1 \cdot \frac{12}{3}, 2 \cdot \frac{12}{3}, \\
2 : i &= 6 && = 1 \cdot \frac{12}{2}
\end{aligned}$$

$$\begin{aligned}
\varphi(12) \cdot 4^{12/12} &+ \varphi(6) \cdot 4^{12/6} + \varphi(4) \cdot 4^{12/4} + \varphi(3) \cdot 4^{12/3} \\
&+ \varphi(2) \cdot 4^{12/2} + \varphi(1) \cdot 4^{12/1}
\end{aligned}$$

fixed signatures.

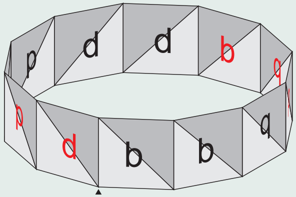


twelve reflections in vertical mirrors,

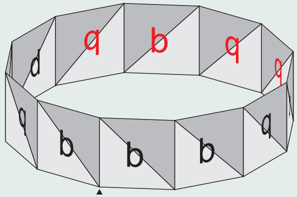
6 of which pass through the center of a motif,
6 of which pass through the boundaries of the aspects,

$6 \cdot 4^6$ fixed signatures

Rotary Reflections



a)

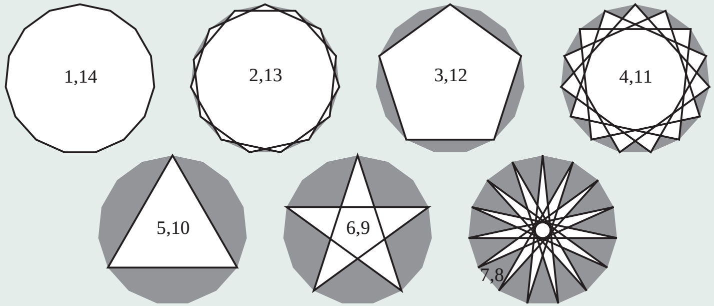


b)

$$\varphi(12)\cdot 4^{12/12}+\varphi(6)\cdot 4^{12/6}+\varphi(4)\cdot 4^{12/4}+\varphi(2)\cdot 4^{12/2}$$

fixed signatures

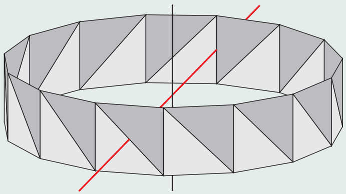
9. The 1×15 case

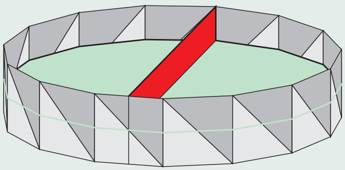


Rotations:

$$\varphi(15) \cdot 4^{15/15} + \varphi(5) \cdot 4^{15/5} + \varphi(3) \cdot 4^{15/3} + \varphi(1) \cdot 4^{15/1}$$

Rotary Reflections: 0
Vertical Mirror Reflections: 0
Horizontal Axis Rotations: 0





10. The $1 \times n$ case

In the general case, the group $\langle T, R, M\rangle$ has elements:

Vertical axis rotations	T^i
Horizonal axis rotations	T^iR
Vertical reflections	T^iRM
Rotary reflections	T^iM.

and acts on a signature $w = Q_1Q_2\cdots Q_n$, $Q_i \in \{b, q, d, p\}$ via

Translation:	$T(Q_1Q_2\cdots Q_n)$	$=$	$Q_2Q_3\cdots Q_nQ_1$.
Rotation:	$R(Q_1Q_2\cdots Q_n)$	$=$	$R(Q_n)\cdots R(Q_2)R(Q_1)$.
Mirror:	$M(Q_1Q_2\cdots Q_n)$	$=$	$M(Q_1)M(Q_2)\cdots M(Q_n)$.

vertical axis rotations:

$$2 \text{ aspects} - p(n) = \sum_{k|n} \varphi(k) 2^{n/k}$$

$$4 \text{ aspects} - P(n) = \sum_{k|n} \varphi(k) 4^{n/k}$$

rotary reflections

$$4 \text{ aspects} - R(n) = \sum_{k|n, 2|k} \varphi(k) 4^{n/k}$$

horizontal axis rotations/vertical mirror reflections:

$$2 \text{ aspects } q(n) = \begin{cases} (n/2)2^{n/2} & \text{for } n \text{ even and} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$4 \text{ aspects } Q(n) = \begin{cases} (n/2)r^{n/2} & \text{for } n \text{ even and} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

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Two aspects and rotational symmetry

$$f(n) = \frac{p(n) + q(n)}{2n}$$

Four aspects and rotational symmetry

$$F(n) = \frac{P(n) + Q(n)}{2n}$$

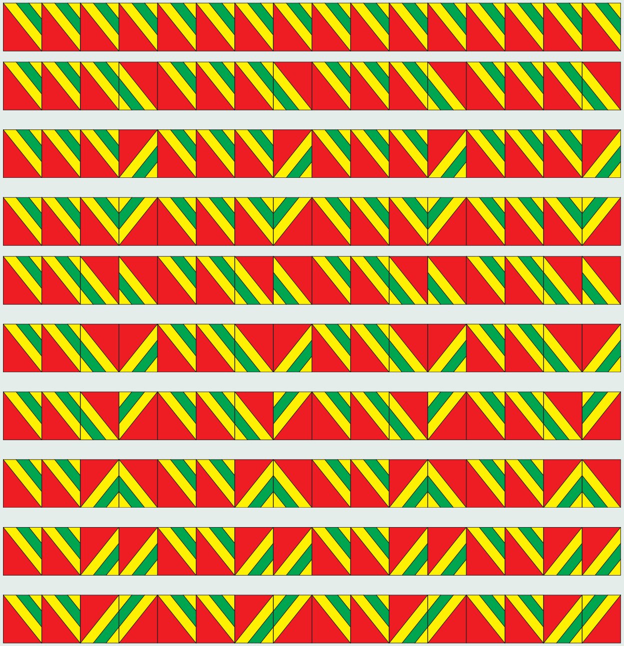
Four aspects and general symmetry

$$G(n) = \frac{P(n) + 2Q(n) + R(n)}{4n}$$

n	2 motifs f(n)	4 motifs, oriented F(n)	4 motifs, unoriented G(n)
1	1	2	1
2	2	6	4
3	2	12	6
4	4	39	23
5	4	104	52
6	9	366	194
7	10	1172	586
8	22	4179	2131
9	30	14572	7286
10	62	52740	26524
11	94	190652	95326
12	192	700274	350738
13	316	2581112	1290556
14	623	9591666	4798174
15	1096	35791472	17895736
16	2122	134236179	67127315
17	3856	505290272	252645136
18	7429	1908947406	954510114
19	13798	7233629132	3616814566
20	26500	27488079132	13744183772

11. The 1×4 case: Escher revisited

1 *bbbb*
2 *bbbq*
3 *bbbp*
4 *bbbd*
5 *bbqq*
6 *bbqp*
7 *bbqd*
8 *bbpq*
9 *bbpp*
10 *bbpd*



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11 *bbdq*


12 *bbdp*


13 *bbdd*


14 *bqbq*


15 *bqbp*


16 *bqbd*


17 *bqpd*


18 *bqdp*


19 *bpbp*


20 *bdbp*

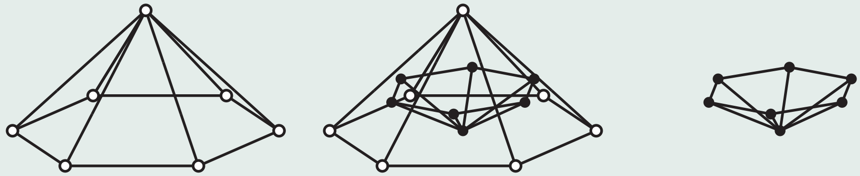

21 *bdbd*

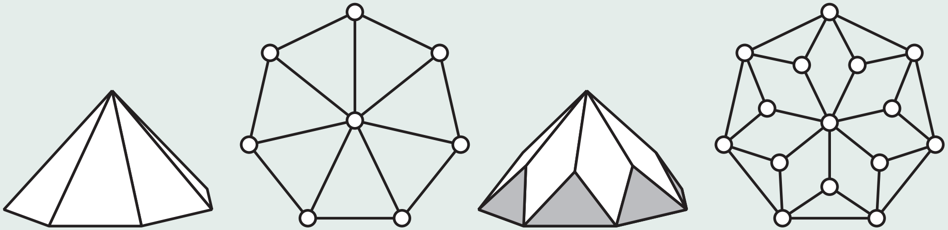

22 *bdqp*

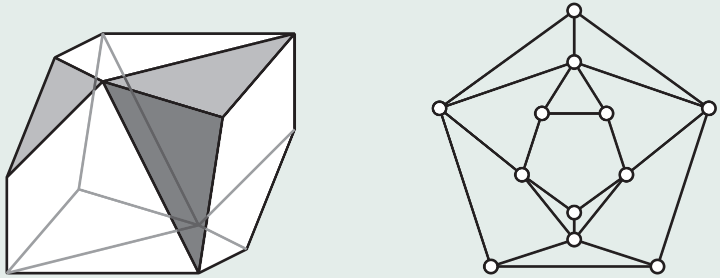

23 *bdpq*

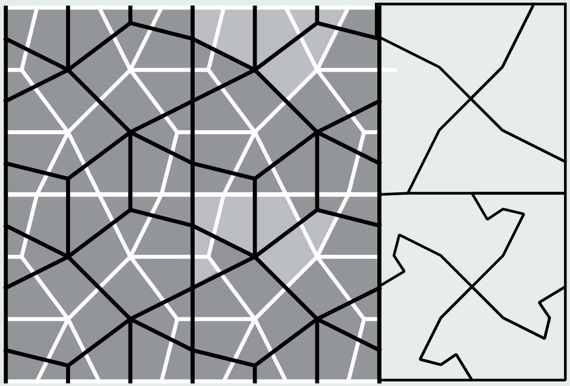

1. $f(n)$ – A053656 in Sloan's On-Line Encyclopedia of integer sequences
2. $G(n) \approx 2F(n)$
3. $G(n) \approx 4^n/(4n)$ $(G(p) = \lceil 4^n/(4n) \rceil$ for $n = p$ prime)
4. Symmetric motifs
5. Over/Under weave motifs
6. Multiple motifs

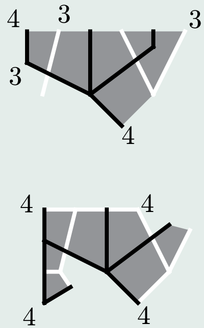
12. Self-Duality










4
3
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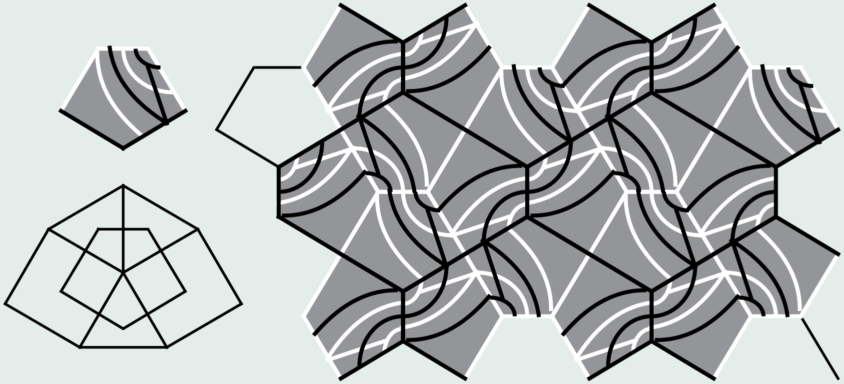
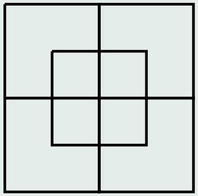
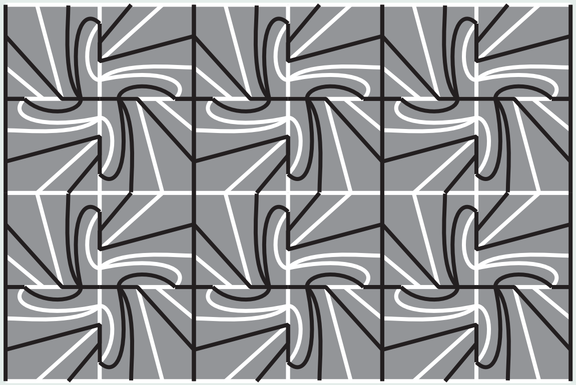
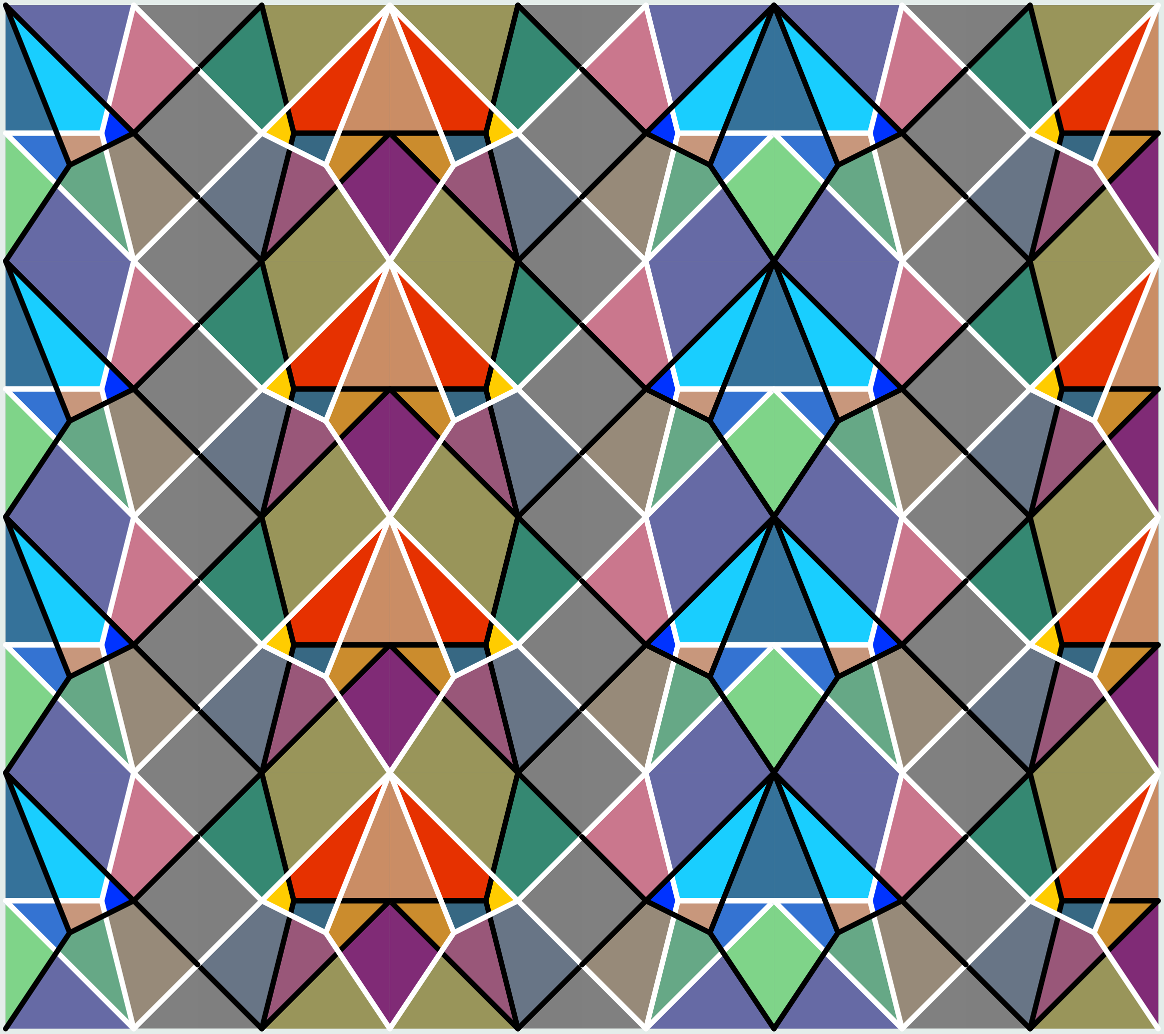
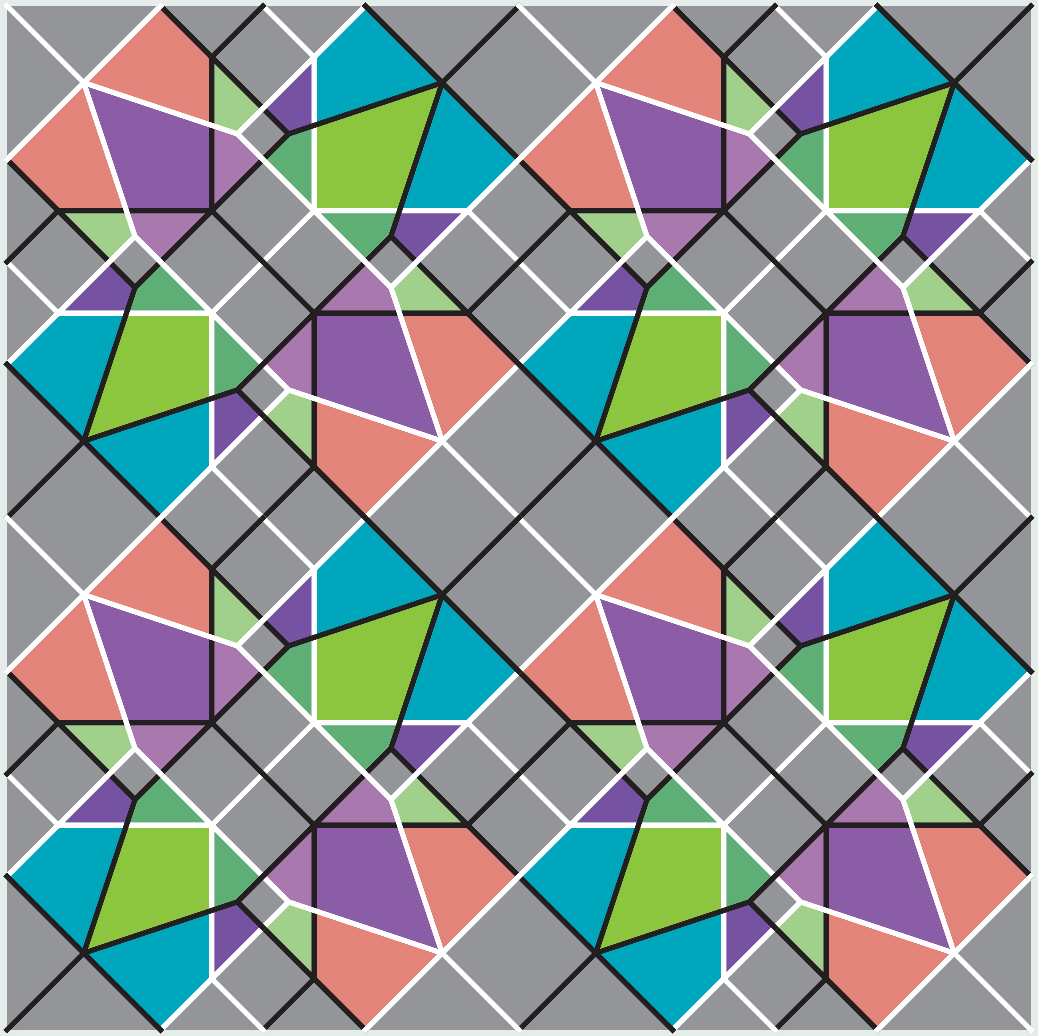
4











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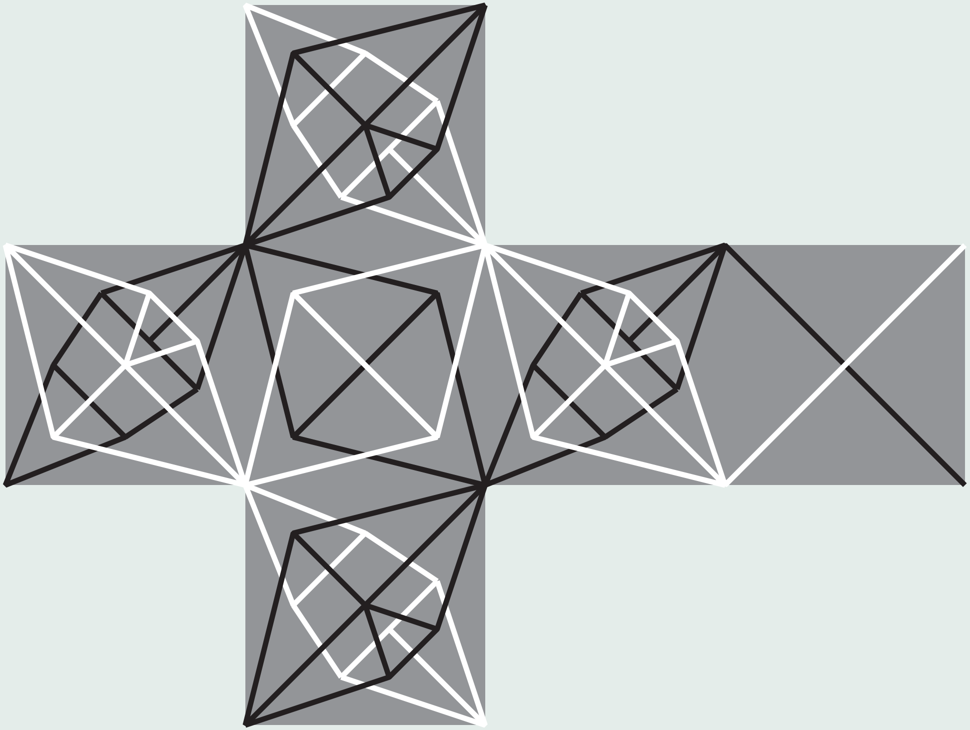
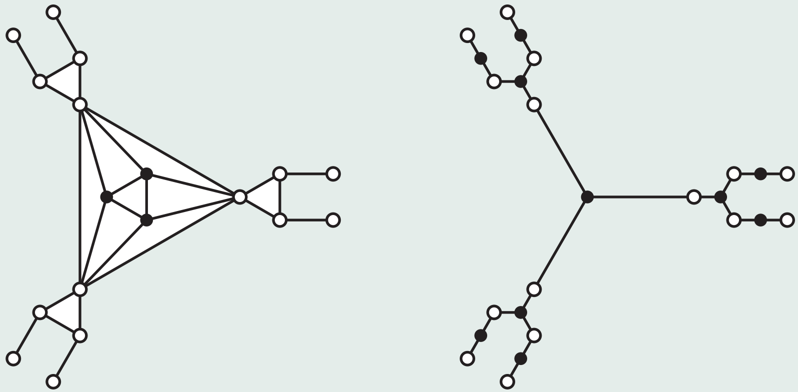
13. Maps are not enough



A self-dual graph with no corresponding self-dual map.

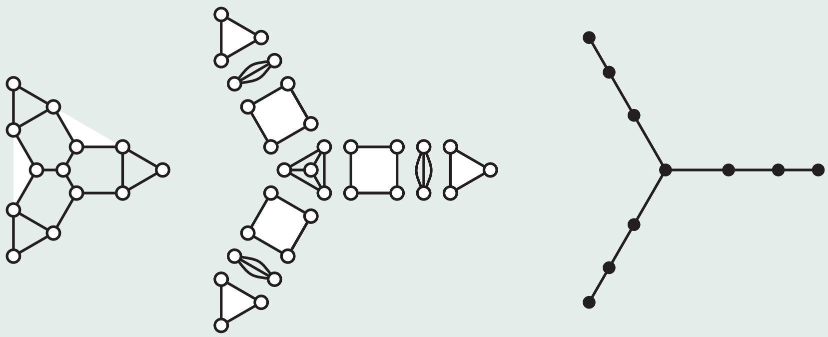
14. The Block-Cutpoint Tree



For any graph G, the cycle matroid of G is the direct sum over the cycle matroids of the blocks of G:

$$M(G) = \sum M(G_i)$$

15. The 3-Block Tree



For a 2-connected graph G, the cycle matroid of G is the 2-sum over the cycle matroids of the 3-blocks of G:

$$M(G) = M(G_0) \bigoplus_{e_1} M(G_1) \ldots \bigoplus_{e_k} M(G_k)$$

16. The Program

Program: Given a planar graph with whose automorphism group has finitely many vertex orbits.

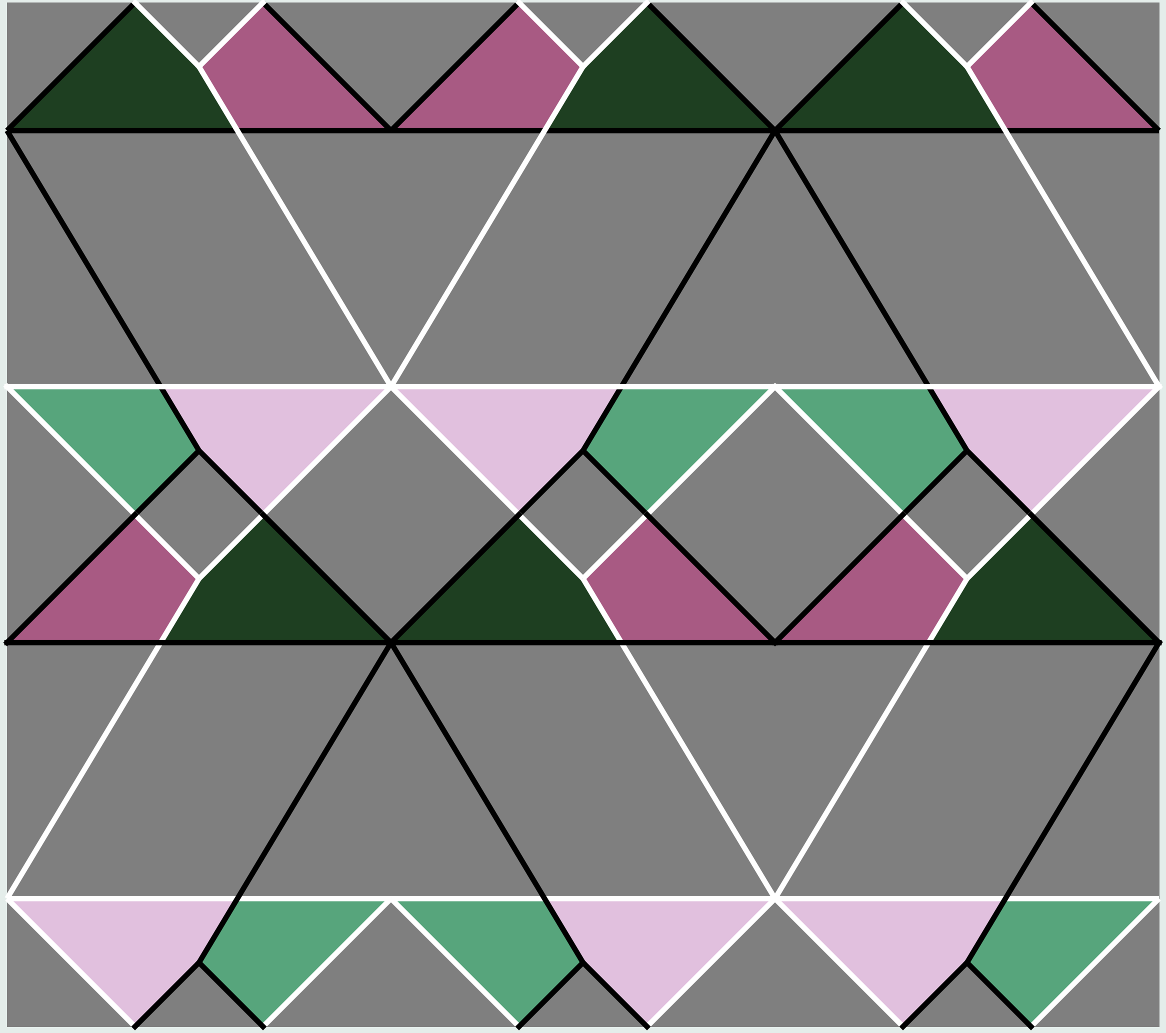
1. If 3-connected – embed and straighten so that the automorphisms are represented by isometries (euclidean, spherical, or hyperbolic).

   Apply geometric methods.

2. Else if two-connected – form the 3–block tree and use the program on each block, and merge with data on the automorphisms of the tree.

3. Else if connected – from the block-cutpoint tree and apply the program to each block, merging with the tree automorphisms.

4. Else apply program to each connected component, and merge with permutations of isomorphic components.



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