Δ-matroids and Matroids

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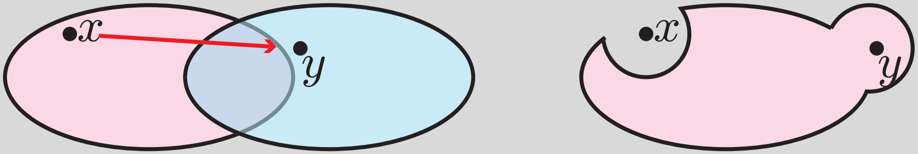
1. Matroids vs. Δ-matroids

M is a Matroid

E a finite set – the *ground set* of M
$\mathcal{B} \subseteq \mathcal{P}(E)$ – the *bases* of M
The *basis exchange axiom:*
$B_1, B_2 \in \mathcal{B}, x \in B_1 \setminus B_2 \Longrightarrow \exists y \in B_2 \setminus B_1$

$$(B_1 \cup \{y\}) \setminus \{x\} = B_1 \triangle \{x, y\} \in \mathcal{B}$$



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The alternate *basis exchange axiom:*
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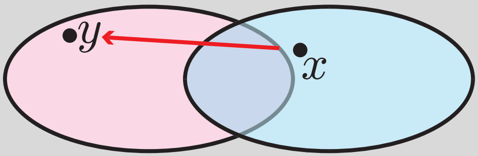

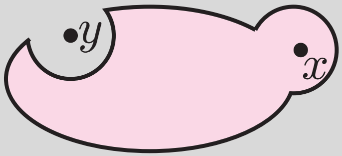

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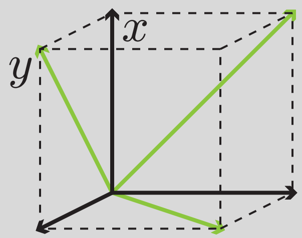

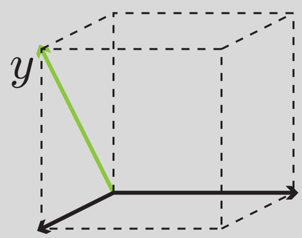

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Bases		–	$B \in \mathcal{B}$.
Independent sets	$\mathcal{I}$	–	$I \subseteq B \in \mathcal{B}$.
Dependent sets	$\mathcal{D}$	–	$D \notin \mathcal{I}$
Cycles (circuits)	$\mathcal{C}$	–	$C \in \mathcal{D}, C \not\subset D \in \mathcal{D}$
Spanning sets	$\mathcal{S}$	–	$S \supset B \in \mathcal{B}$.

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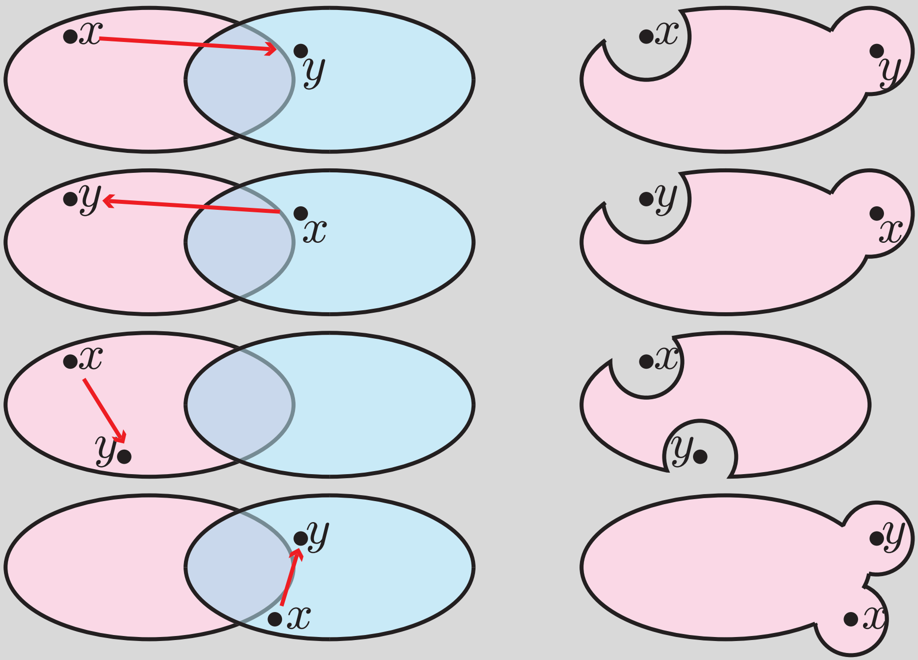
Whitney	1935	[13]	
W. T. Tutte	1971	[11]	(standard text)
D. J. A. Welsh	1976	[12]	(graph theory)
James Oxley	2011	[8]	(geometric/algebraic)
András Recski	1989	[10]	(applied approach)
Leonidas Pitsoulis	2014	[9]	

D is a Δ-matroid

The *symmetric exchange axiom:*

$$F_1, F_2 \in \mathcal{F}, x \in F_1 \triangle F_2 \implies \exists y \in F_1 \triangle F_2$$

$$F_1 \triangle \{x, y\} \in \mathcal{F}$$



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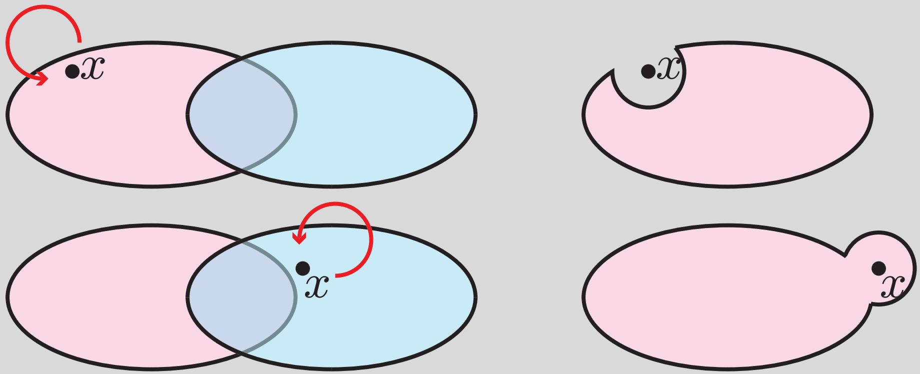
$x = y$:



$$|F_1| - 2 \leq |F_2| \leq |F_1| + 2$$

Feasible sets $\mathcal{F}$ $\quad F \in \mathcal{F}$.

D is a Δ-matroid

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Bouchet	1987	[1]	(Δ-matroids)
Bouchet	1998	[2, 3, 5, 4]	(multimatroids)
Dress & Havel	1986	[7]	(metroids)
Chandrasekaran	1988	[6]	(pseudometroids)

2. Matroids in Δ-matroids

Every matroid M is a Δ-matroid ($\mathcal{F} = \mathcal{B}$)

Every Δ-matroid D with $\mathcal{F} \subseteq \mathcal{P}_n(E)$ is a matroid M, ($\mathcal{B} = \mathcal{F}$)

Given a Δ-matroid D,
M_u, the *upper matroid*, whose bases are the feasible sets with largest cardinality
M_l, the *lower matroid*, whose bases are the feasible sets with least cardinality

Theorem 1 *Let $M = (E, \mathcal{B})$ be a matroid with independent sets $\mathcal{I}$. Then $D = (E, \mathcal{I})$ is a Δ-matroid.*

The upper matroid is $(E, \mathcal{B})$ and the lower matroid $(E, \emptyset)$.

Theorem 2 *Let $M = (E, \mathcal{B})$ be a matroid with spanning sets $\mathcal{S}$. Then $D = (E, \mathcal{S})$ is a Δ-matroid.*

The upper matroid is $(E, \mathcal{P}(E))$ and the lower matroid $(E, \mathcal{B})$.

Theorem 3 *If $D = (E, \mathcal{F})$ is a Δ-matroid, $F \in \mathcal{F}$, then F is spanning in M_l and F is independent in M_u.*

Corollary 1 *If $M_u = (E, \mathcal{B}_u)$ and $M_l = (E, \mathcal{B}_l)$ are matroids, then for M_u and M_l to be upper and lower matroids of a Δ-matroid $D = (E, \mathcal{F})$ it is necessary that*

– every basis of M_u be spanning in M_l and

– every basis of M_l be independent in M_u.

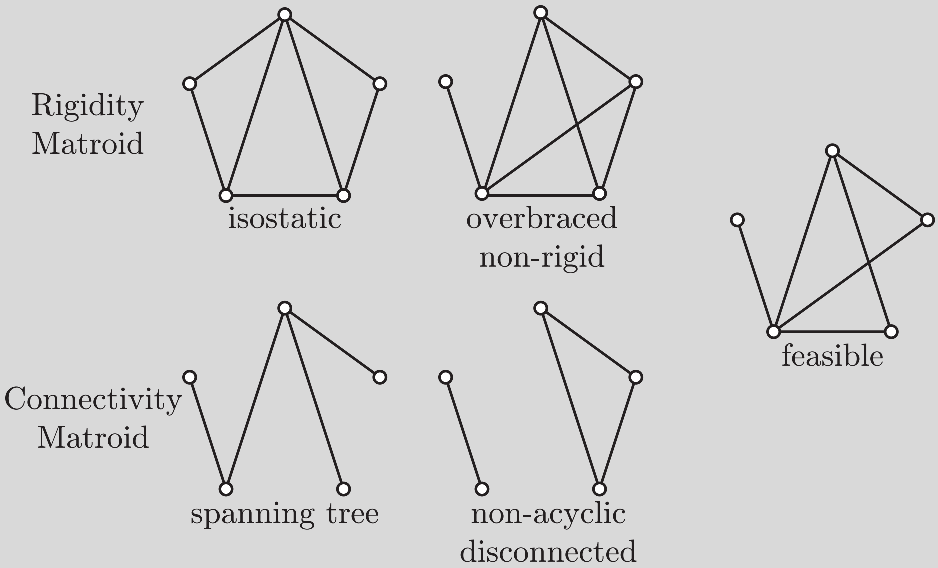
Upper and Lower matroids do not determine the D-matroid:

$$\{\{a,b\},\{a\},\{b\},\emptyset\} \qquad \{\{a,b\},\emptyset\}$$

3. Realization Problem

Given $M_l = (E, \mathcal{B}_l)$ and $M_u = (E, \mathcal{B}_u)$, construct $D = (E, \mathcal{F})$ realizing them.

An example with as many intermediate feasible sets as possible:



Theorem 4 $G = (V, E)$ *a connected simple graph.*
M_c the connectivity matroid (cycle matroid)
M_r the 2-dimensional generic rigidity matroid
$\mathcal{F}$: F connected (spanning in M_c) not-overbraced (independent in M_r)
Then $\mathcal{F}$ satisfies the symmetric exchange property.

Tool: A minimally overbraced graph is 2-connected.

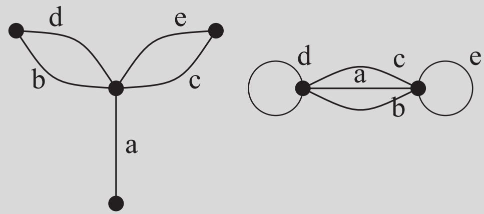
For this construction it is not necessary that a cycle M_u be connected in M_l:

Example

$E = \{1, 2, 3, a, b, c\}$,
$M_u = U_{5,6}(E)$, $M_l = U_{2,3}(\{1, 2, 3\}) \oplus U_{2,3}(\{a, b, c\})$.
$D = (E, \mathcal{B}_u \cup \mathcal{B}_l)$ is a Δ-matroid.
M_u is a cycle.
M_l is disconnected.

A weaker condition: Every cycle in M_u is a union of cycles in M_l.

The weaker condition is necessary:



Two connectivity matroids on the same edge set.
M_u and M_l are matroids.
– Every basis of M_l is independent in M_u
– Every basis of M_u is spanning in M_l
But
Every cycle of M_u is not a union of cycles of M_l.

M_u and M_l are not the upper and lower matroids of any Δ-matroid.

The weaker condition is necessary:

Theorem 5 *Let $D = (E, \mathcal{F})$, with upper matroid M_u and lower matroid M_l.*
Then every cycle in M_u is a union of cycles in M_l.

Theorem 6 *Given $M_u = (E, \mathcal{B}_u)$, $M_l = (E, \mathcal{B}_u)$, with*
Every cycle in M_u is a union of cycles in M_l.
Then every $B \in B_u$ is spanning M_l.
Then every $B \in B_l$ is independent in M_l.

Necessary and Sufficient for Realization

Theorem 7 *Given* $M_u = (E, \mathcal{B}_u)$, $M_l = (E, \mathcal{B}_u)$.

M_u and M_l realize the Δ-matroid $D = (E, \mathcal{F})$
if and only if
Every cycle in M_u is a union of cycles in M_l.

4. Quotients of Matroids

(Oxley [8]) $Q = (E, \mathcal{B}_Q)$ is a *quotient* of $M = (E, \mathcal{B}_M)$ if there is a matroid $N = (E \cup X, \mathcal{B}_N)$, $E \cap X = \emptyset$, with $M = N \setminus X$ and $Q = N/X$.

Theorem 8 *(Oxley) Q is a quotient of M if and only if every circuit of M is a union of circuits of Q.*

Corollary 2 *Given $M_u = (E, \mathcal{B}_u)$, $M_l = (E, \mathcal{B}_u)$.*

*M_u and M_l realize the Δ-matroid $D = (E, \mathcal{F})$
if and only if
M_l is a quotient of M_u.*

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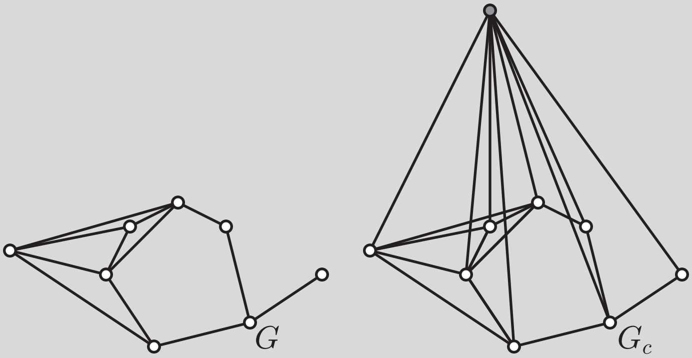
Corollary 3 *The connectivity matroid of a graph is a quotient of the rigidity matroid.*



A graph G and its cone G_c.

Theorem 9 $M_r(G) = M_r(G_c) \setminus X$
$M_c(G) = M_r(G_c)/X$.

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