Equilibrium stressability of multidimensional frameworks

Herman Servatius, Worcester Polytechnic Institute

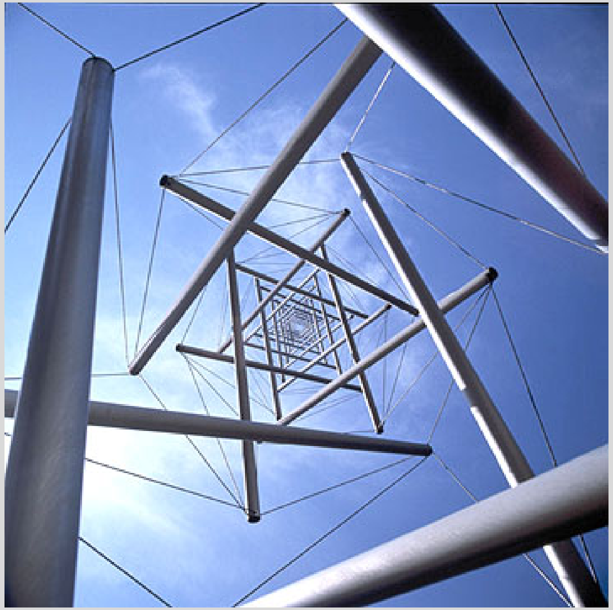
with Oleg Karpenkov, Christian Müller, Gaiane Panina, Brigitte Servatius, and Dirk Siersma

Abstract

Goal: An equilibrium stressability criterium for trivalent multidimensional tensegrities.
The criterium appears in different languages:
(1) stress monodromies,
(2) surgeries
(3) exact discrete 1-forms
(4) in Cayley algebra.

1. Geometric Constraint Systems

Kenneth Snelson's *NeedleTower*



Rigidity

Ingredients: A geometric space
A collection of objects in that space
A structure that associates geometric constrains to particular objects

Example: $\mathbb{R}^3$ $\quad$ $\mathbf{p} : V \to \mathbb{R}^3$. $\quad$ $G(V, E)$

$$(i, j) \in E \qquad (\mathbf{p}_i - \mathbf{p}_j)^2 = \lambda_{ij}^2$$

Solution Set
Rigidity: $\mathbf{p}$ is an 'isolated point'.

Infinitesimal Rigidity

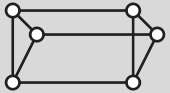
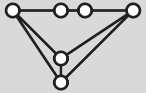
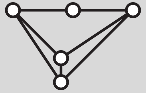
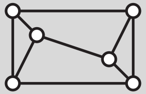
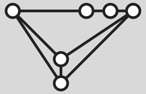
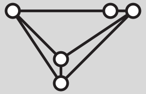
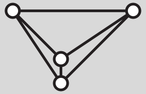
Example: $\mathbb{R}^3$ $\mathbf{p} : V \to \mathbb{R}^3.$ $G(V, E)$

$$(i, j) \in E \qquad (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$$

$$W = \mathbf{f} \cdot \Delta\mathbf{p} = (\omega R)\,\Delta\mathbf{p} = \omega\,(R\Delta\mathbf{p}) = \omega \cdot \Delta\mathbf{e} \qquad (1)$$

Element of the Kernel: Infinitesimal Motion

Element of the Cokernel: Equilibrium Stress

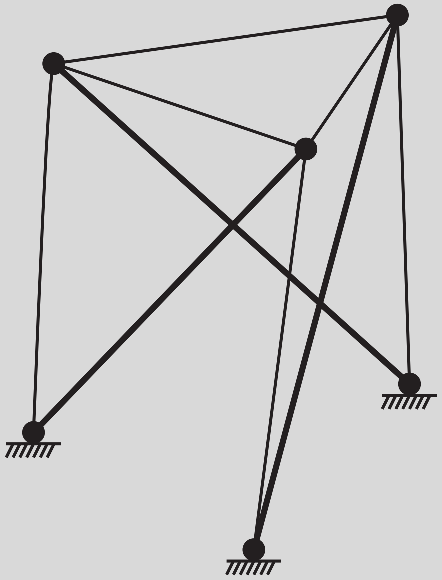
	rigid	globally rigid	infinitesimally rigid	generically rigid
	No	No	No	No
	No	No	No	Yes
	Yes	No	No	No
	Yes	No	No	Yes
	Yes	No	Yes	Yes
	Yes	Yes	No	No
	Yes	Yes	No	Yes
	Yes	Yes	Yes	Yes

2. Tenegrities and Graphs

BUCKMINSTER FULLER coined the term *tensegrity*

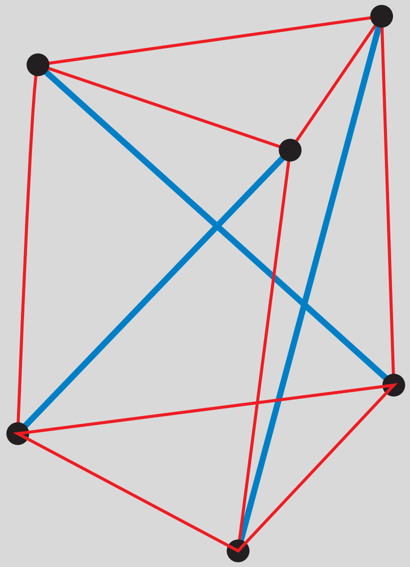
- a combination of 'tension' and 'integrity'
- rigid networks of rods and cables

Kenneth Snelson's $T3$



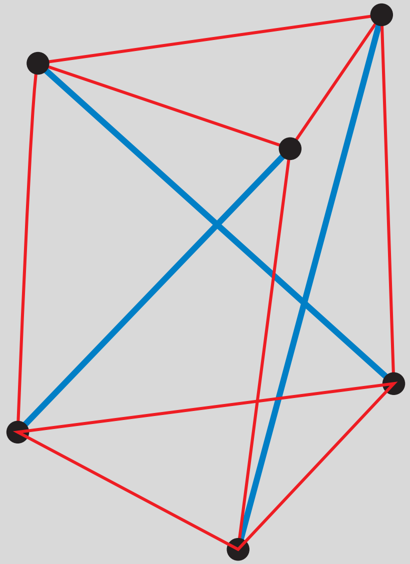
$(V, E) \qquad E = C \cup S \qquad C \cap S = B \qquad \mathbf{p} : V \to \mathbb{R}^D$

- $\mathbf{p}$ gives the initial position for vertices
- Edges in C cannot expand.
- Edges in S cannot contract.



The Three Mysteries of Snelsen's Tensegrities

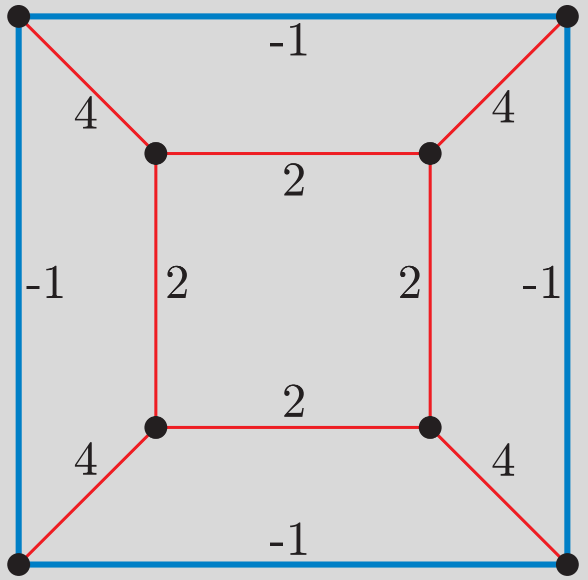
-
-



Geometric . . .
Tenegrities and . . .
Stress
d-framework
Home Page
Title Page
Page 9 of 20
Go Back
Full Screen
Close
Quit

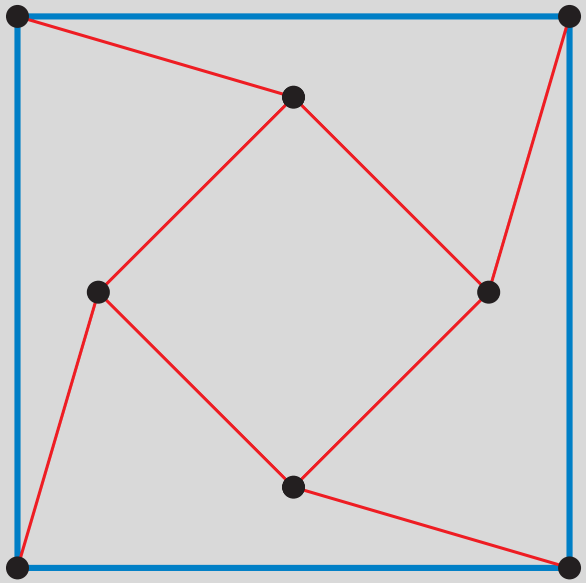
$(V, E) \qquad E = C \cup S \qquad C \cap S = B \qquad \mathbf{p} : V \to \mathbb{R}^D$

- $\mathbf{p}$ gives the initial position for vertices
- Edges in C cannot expand.
- Edges in S cannot contract.



$$(V, E) \qquad E = C \cup S \qquad C \cap S = B \qquad \mathbf{p} : V \to \mathbb{R}^D$$

- $\mathbf{p}$ gives the initial position for vertices
- Edges in C cannot expand.
- Edges in S cannot contract.



3. Stress

Stress $\omega : S \cup C \to \mathbb{R}$
$\omega(s) \geq 0$ and $\omega(c) \leq 0 \qquad c \in C,\ s \in S.$
Equilibrium Stress (Presstress, Self-stress)

$$\forall v \in V : \quad \sum_{(v,w)\in C\cup S} \omega((v,w))(\mathbf{p}(v) - \mathbf{p}(w)) = \mathbf{0}.$$

A non-trivial proper equilibrium stress is necessary for the structural integrity of a strut, cable system.

Theorem

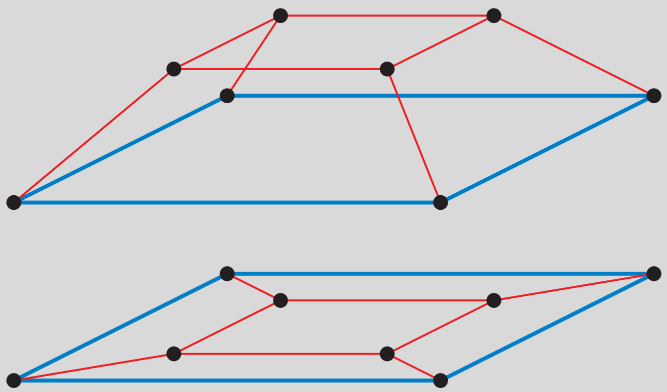
[Roth and Whitely] A framework which is statically rigid as a bar and joint framework and has a proper nowhere zero equilibrium stress, is statically rigid as a cable strut framework.

Theorem

[Connelley and Whiteley] A framework with with a stress passing the *second-order stress test* is second order rigid, hence rigid.

3.1. Liftings of Maxwell-Cremona

A framwork which is plane embebedded and stressed, *lifts*:



With all lifted cells planar in 3-D.

Lee, Whiteley), Lee, Ryshkov, Rybnikov

Connection has an analogue:

- CW-complexes
- dimension D (not necessarily embedded)

4. d-framework

Let $D > d \geq 2$
$\mathcal{F} = (E, F, I, \mathbf{n})$
E, a collection of $(d-1)$-dimensional planes in $\mathbb{R}^D$;
F, a collection of d-dimensional planes in $\mathbb{R}^D$;
$I \subset \{(p, q) \in (E \times F) \mid p \subseteq q\}$; (incidences)
$\mathbf{n} : I \to \mathbb{R}^D$, (normal selection)

$$\mathbf{n}(e, f) \perp e \quad \mathbf{n}(e, f) \in f \quad |\mathbf{n}(e, f)| = 1$$

A d-framework is *generic*
for every $e \in E$, the planes f with $(e, f) \in I$ are distinct.

$T3$: 1-framework in $\mathbb{R}^3$.
The cube graph example: a 1-framework in $\mathbb{R}^2$ lifted into $\mathbb{R}^3$.

A *stress* s on $\mathcal{F} = (E, F, I, \mathbf{n})$

$$s : F \to \mathbb{R}$$

An *equilibrium* stress (prestree, self=stress)

$$\forall e \in E \qquad \sum_{(e,f)\in I} s(f)\mathbf{n}(e, f) = 0.$$

$\mathcal{F}$ is *self-stressable* (a *tensegrity*):
— there exists a non-zero self-stress on it.

$\mathcal{F}$ is *Trivalent*: Each $e \in E$ has 3 incidences.

2-tensegrities (surfaced based) similar to:
minimal (harmonic) surfaces meeting at edges

Think of:

— soap bubbles
— tents
— flat expansive/contractive plates meeting at edges

4.1. Example 1

$d = 2$, $D = 3$.
E: — edges of a K_5 embedded as regular tetrahedron plus centroid in $\mathbb{R}^3$
F: — plane of triangles of K_5.
I: — incidences in K_5.
$\mathbf{n}$: — all point "in".
Note: The 2-framework is generic.
Interior/exterier triangles stressed in ratio $-\sqrt{6}/4$ gives equilibrium stress.
Interior expanding triangles cooling and contracting,
Exterior "skin" triangles expanding

4.2. Example 2

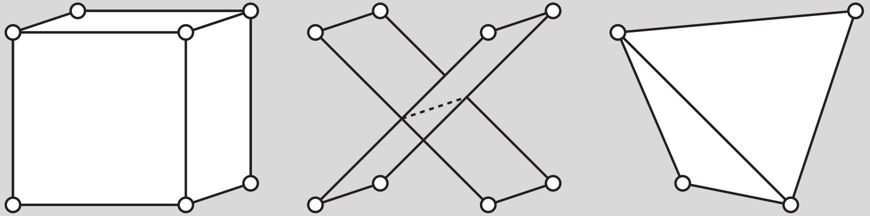
$d = 2$, $D = 3$.
E: — edges of a K_5 embedded as regular tetrahedron plus centroid in $\mathbb{R}^3$
F: — K_4 subgraphs of K_5.
I: — incidences in K_5.
$\mathbf{n}$: — various.

Since any two K_4's intersect in 3 edges, all planes must coincide.

All choices of $\mathbf{n}$ yield only 0 self-stress. Not a tensegrity.

4.3. Example 3

$d = 2$, $D = 3$.
E: — edges of a cube in $\mathbb{R}^3$, and all face diagonals
F: — cube faces and triangles of inscribed tetrahedra
I: — incidences from cube.
$\mathbf{n}$: — pointing "in".



A 3-framework based on the cube with three types of faces. $1 : -\sqrt{2} : \sqrt{3}/4$.

Theorem 1 *Consider a generic face-connected trivalent d-framework. Then the following three statements are equivalent.*

1. *$\mathcal{F}$ has a non-zero self-stress (which is in fact non-zero at any d-plane).*
2. *For every two d-planes f_a, f_z in $\mathcal{F}$ the stress-transition does not depend on the choice of an induced face-path d-framework on $\mathcal{F}$.*
3. *Every induced face-loop d-framework on $\mathcal{F}$ is self-stressable.*

Theorem 2 *A generic trivalent d-framework is self-stressable if and only if the discrete multiplicative 1-form defined by (**??**) is exact.*

Theorem 3 *Let M be the d-skeleton of some (d+1)-dimensional manifold $\overline{M}$.*

1. *If the first homology group of $\overline{M}$ vanishes, that is,*

$$H_1(M, \mathbb{Z}_2) = 0,$$

   then the linear spaces $\mathrm{Lift}(\overline{M}, p)$ and the space of self-stresses $\mathrm{Stress}(M, p)$ are canonically isomorphic.

2. *Liftability of $(\overline{M}, p)$ implies self-stressability of (M, p).* □