Equitable partitions and Kirchhoff graphs

T.M. Reese B. Servatius R.C. Paffenroth J.D. Fehribach

Worcester Polytechnic Institute

Equitable Partitions
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Abstract

We give necessary and sufficient conditions for a matrix to be the quotient of the adjacency matrix $A(G)$ of a graph G with respect to an equitable vertex partition of G. We define equitable edge partitions for multi-digraphs and establish connections to Kirchhoff graphs.

1. Equitable Partitions

Let G be a simple undirected graph with vertices $V(G) = \{v_i\}$. A ***partition*** of $V(G)$ is a set whose elements are disjoint, nonempty subsets of $V(G)$ whose union is $V(G)$.

The elements of a partition π will be called *cells*.
A partition $\pi = (V_1, \ldots, V_k)$ of $V(G)$ is ***equitable*** if
$\forall \quad i, j \in \{1, \ldots, k\}, \qquad v_p \in V_i :$

$$c_{i,j} = \sum_{\substack{q \\ v_q \in V_j}} A(G)_{p,q}$$

— depends only on i and j,

— **not** on the choice of vertex $v_p \in V_i$.

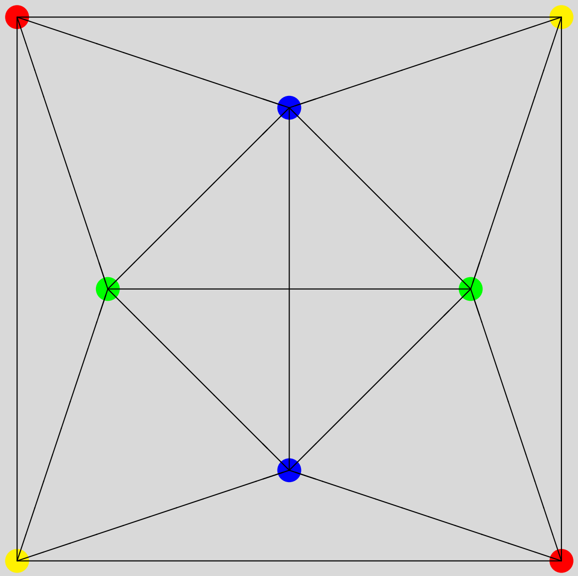
Easy Examples

- The discrete partition is always equitable

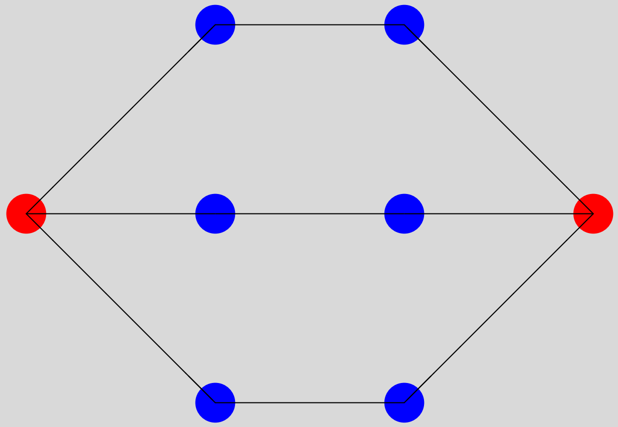
$$c_{i,j} = (A(G))_{i,j}$$

- A single-cell partition is equitable if G is regular.

A Non-trivial Example



A Less Symmetric Example



Easy Facts

Given equitable partition $\pi = (V_1, \ldots, V_k)$

- Two vertices in V_i have equal # of neighbors in cell V_i.
- $\forall\ i \in \{1, \ldots, k\}$, the induced subgraph $G[V_i]$ is regular (degree $c_{i,i}$)
- All vertices in cell V_i have the same degree: $\displaystyle\sum_{j=1}^{k} c_{i,j}$.
- Equitable partition of G is an equitable partition of its complement, $\overline{G}$.

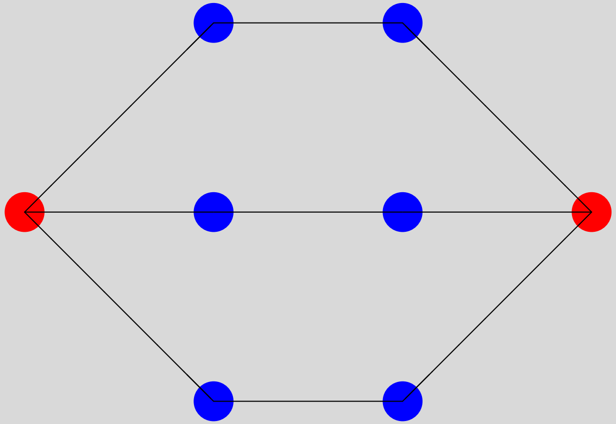
G/π

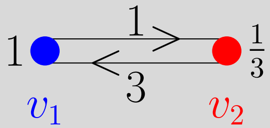
$A(G/\pi)$ adjacency matrix:

- $k \times k$ matrix with (i, j)-entry $c_{i,j}$.
- $A(G/\pi)$ is the ***quotient matrix*** of G with respect to π.

Equitable Partition, Labeled Quotient

$$\begin{array}{cc} Q & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \end{array}$$





Labeled digraph

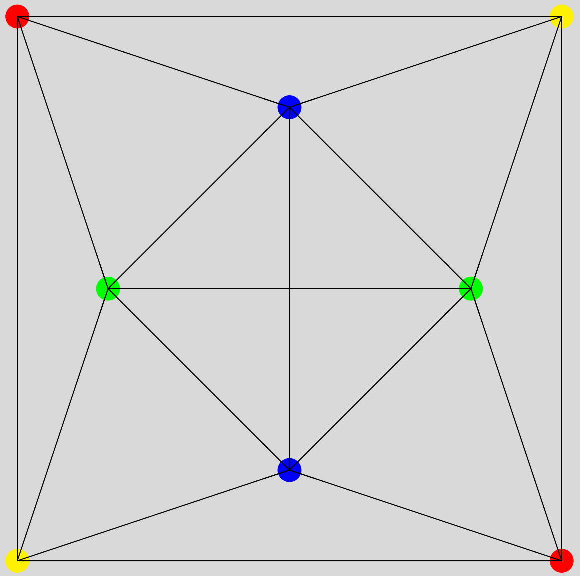
The ***characteristic matrix*** P of partition $\pi = (V_1, \ldots, V_k)$ is a $|V(G)| \times k$ matrix whose j^{th} column is the characteristic vector of set V_j. That is, $P_{i,j} = 1$ if $e_i \in V_j$ and is zero otherwise. Characteristic matrix P provides the relationship between $A(G)$ and $A(G/\pi)$.

Theorem 1 *[?] Let π be a partition of $V(G)$ with characteristic matrix P. Then π is equitable if and only if there exists a $k \times k$ matrix Q such that*

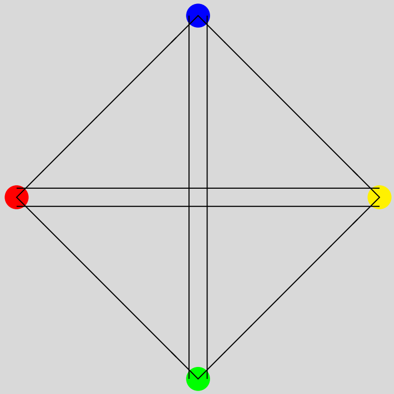
$$A(G)P = PQ$$

in which case $Q = A(G/\pi)$.

Equitable partition – Symmetric Quotient Matrix



$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$



Undirected Quotient

2. How to recognize a quotient matrix?

Theorem 2 *Let A be a $k \times k$ matrix with non-negative integer entries. Then there exists a graph G with equitable partition π such that $A = A(G/\pi)$ if and only if there exist positive integers x_i such that $x_i a_{i,j} = x_j a_{j,i}$ for $1 \leq i, j \leq k$.*

PROOF: $(\Rightarrow)$. Let G be a graph with equitable partition $\pi = (V_1, \ldots, V_k)$. Then clearly $A(G/\pi)$ is a $k \times k$ matrix with non-negative integer entries $c_{i,j}$, and symmetric zeros. By definition of equitability, for each i the induced subgraph $G[V_i]$ must be a $c_{i,i}$ regular graph. Moreover, for any $i \neq j$, the graph formed by taking the vertices of V_i and V_j and all (V_i, V_j) edges must be a $(c_{i,j}, c_{j,i})$-biregular graph. Therefore for all $i \neq j$,

$$c_{i,j}|V_i| = c_{j,i}|V_j|.$$

That is, for all $i \neq j$,

$$c_{i,j}|V_i| - c_{j,i}|V_j| = 0.$$

$\square$

PROOF:

($\Leftarrow$). Conversely, let A be any $k \times k$ matrix with non-negative integer entries $a_{i,j}$. Suppose $\mathbf{x} = x_1, \ldots x_k$ is a vector with positive integer entries so that for all $1 \leq i, j \leq k$ such that $i \neq j$,

$$x_i(a_{i,j}) = x_j(a_{j,i}). \tag{1}$$

. Scale $\mathbf{x}$ by a positive integer so that for each j,

$$\begin{cases} x_j > a_{i,j} & \text{for all } i \in \{1, \ldots, k\} \\ x_j a_{j,j} & \text{is even} \end{cases} \tag{2}$$

Now for each i, let V_i be a set of x_i vertices, and let G be a graph with vertex set $V(G) = V_1 \cup \cdots \cup V_k$. Add edges to V so that for each i, the induced subgraph $G[V_i]$ is $a_{i,i}$-regular, which is possible since $|V_i|a_{i,i}$ is even and $|V_i| > a_{i,i}$ by 2. For each $i \neq j$, add edges so that the edges between V_i and V_j form an $(a_{i,j}, a_{j,i})$-biregular graph. By construction, the resulting graph G has an equitable partition $\pi = (V_1, \ldots, V_k)$ with quotient matrix $A(G/\pi) = A$. $\square$

2.1. A graphical algorithm

Let A be a $k \times k$ matrix with non-negative integer entries. Associate with A a di-graph H with k vertices, $v_1, \ldots, v_k$, and label arc $(v_i, v_j) \in E(H)$ with matrix entry $a_{i,j}$.
For any path $P_0 = v_i, v_j \cdots v_k, v_l$, let $\omega(P_0)$ be the product

$$\omega(P_0) = a_{i,j} \cdot a_{j,k} \cdots a_{k,l}.$$

Now let $C = v_i, v_j, v_k \cdots v_l, v_i$ be any (oriented) cycle of H. We say that C is A-*invariant* if

$$\frac{a_{i,j}}{a_{j,i}} \cdot \frac{a_{j,k}}{a_{k,j}} \cdots \frac{a_{l,i}}{a_{i,l}} = 1.$$

Theorem 3 *There exist positive integers $x_1, \ldots, x_k$ with $x_i a_{i,j} = x_j a_{i,j} \bigvee i, j \in \{1, \ldots k\}$ if and only if every oriented cycle of H is A-invariant.*

Proof: First, suppose that there exists a positive $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_k \end{bmatrix}^t$ such that $R(A)\mathbf{x} = \mathbf{z}$. In particular, in light of (1), for any $i \neq j$,

$$x_i(a_{i,j}) = x_j(a_{j,i}).$$

That is, for any $i \neq j$,

$$\frac{a_{i,j}}{a_{j,i}} = \frac{x_j}{x_i}.$$

Now let $C = v_i, v_{i_1}, v_{i_2} \cdots v_{i_{l-1}}, v_{i_l}, v_i$ be any cycle of H. Then

$$\frac{a_{i,i_1}}{a_{i_1,i}} \cdot \frac{a_{i_1,i_2}}{a_{i_2,i_1}} \cdots \frac{a_{i_{l-1},i_l}}{a_{i_l,i_{l-1}}} \cdot \frac{a_{i_l,i}}{a_{i,i_l}} = \frac{x_{i_1}}{x_i} \cdot \frac{x_{i_2}}{x_{i_1}} \cdot \frac{\cdot}{x_{i_2}} \cdots \frac{x_{i_{l-1}}}{\cdot} \cdot \frac{x_{i_l}}{x_{i_{l-1}}} \cdot \frac{x_i}{x_{i_l}} = 1.$$

Therefore C is A-invariant. As C was an arbitrary cycle in H, all cycles of H are A-invariant.

□

Proof: Conversely, suppose that all cycles of C are A-invariant.
Iteratively assign a positive weight W to each vertex of H.
Initialize by assigning vertex v_1 and any vertex v_j adjacent to v_1 by an arc with nonzero label $a_{i,j}$ the weight $\frac{a_{1,j}}{a_{j,1}}$.
If not all vertices are assigned a weight, choose a vertex with assigned weight with a nonzero arc to an un-weighted vertex. If there is no such vertex, Choose an un-weighted vertex, assign it weight 1 and proceed in this fashion until all vertices are assigned a weight. Then scale the weights, if necessary by a common denominator of the fractional weights assigned. By construction, $W(v)$ is a positive integer for every vertex V. Finally, for all $i \neq j$

$$a_{i,j}W(v_i) = a_{j,i}W(v_j). \tag{3}$$

because of A-invariance. □

It is clear that for a symmetric quotient matrix there always is a solution with all partitions of the same size (uniform).

Example

Let A be the 6×6 matrix

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 & 2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 3 \\ 0 & 0 & 3 & 2 & 0 & 3 \\ 1 & 0 & 0 & 4 & 2 & 0 \end{bmatrix}$$

.

To find such a solution to 1, we assign positive integer weights to the vertices of a digraph of order 6, using the iterative process outlined in the proof of Theorem 3. Vertex 1 is initially assigned weight 1, vertex 2 weight $\frac{2}{3}$, vertex 3 and vertex 6 are assigned weight 1. Vertex 1 is not incident to any other vertex by a non-zero arc, but vertex 2 is incident to vertex 4, which receives weight $2.\frac{2}{3} = \frac{4}{3}$ Now vertex 3 is incident to vertex 5, which receives weight $\frac{2}{3}$ and is in turn incident to vertex 6, receiving weight $\frac{2}{3}\frac{3}{2} = 1$. Scaling the rational weights $(1, \frac{2}{3}, 1, \frac{4}{3}, \frac{2}{3}, 1)$ yields the integer vector $(3, 2, 3, 4, 2, 3)$.

To find such a solution to 1, we assign positive integer weights to the vertices of a digraph of order 6, using the iterative process outlined in the proof of Theorem 3. Vertex 1 is initially assigned weight 1, vertex 2 weight $\frac{2}{3}$, vertex 3 and vertex 6 are assigned weight 1. Vertex 1 is not incident to any other vertex by a non-zero arc, but vertex 2 is incident to vertex 4, which receives weight $2.\frac{2}{3} = \frac{4}{3}$ Now vertex 3 is incident to vertex 5, which receives weight $\frac{2}{3}$ and is in turn incident to vertex 6, receiving weight $\frac{2}{3}\frac{3}{2} = 1$. Scaling the rational weights $(1, \frac{2}{3}, 1, \frac{4}{3}, \frac{2}{3}, 1)$ yields the integer vector $(3, 2, 3, 4, 2, 3)$.

3. What is a Kirchhoff graph?

Let A be a matrix with integer entries, with columns $s_1, s_2, \ldots s_n$. A digraph D, whose arcs are labeled with columns of A is a Kirchhoff graph for A if

- $\lambda(v)$ is in the row space of D for all $v \in D$, where $\lambda(v)$ is a vector in $\mathbb{R}^n$ whose i'th entry is the net number of arcs labeled s_i exiting vertex v.
- The vectors $\{\chi(C), C \in D\}$ span the nullspace of A, where C is a cycle in D and $\chi(C)$ is a vector in $\mathbb{R}^n$ whose i'th entry is the net number of arcs labeled s_i traversed by C.

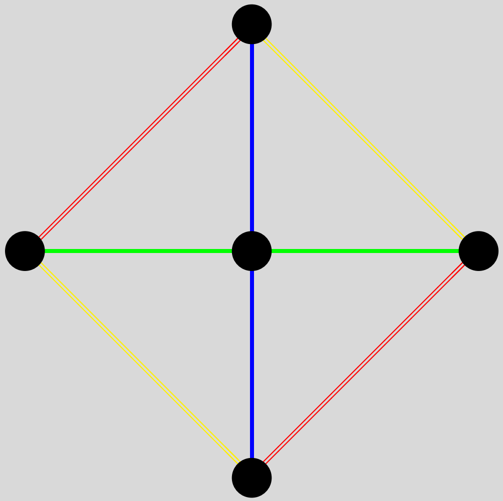
$$\lambda(v) \cdot \chi(C) = 0$$

for each vertex of v and cycle C of D.

3.1. Edge partitions

If we want to partition the edges of a given graph equitably, we can first construct the linegraph $L(G)$ from G, whose vertices are the edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are incident. An edge partition of G is equitable if the corresponding vertex partition of $L(G)$ is equitable.

Harary and Norman [?] defined line-digraphs from di-graphs. If D is a directed graph, its directed line graph, or line-digraph has one vertex for each line of D and vertices (u, v) and (w, x) are adjacent if $v = w$.



equitable edge partition

The adjacency matrix of a line di-graph is not, in general, symmetric. If all the orientations are reversed, one simply gets the negative matrix. There are fewer nonzero entries than in the adjaceny matrix of the unoriented line graph.
Tyler Reese [?] defined a signed adjacency matrix from a digraph. Its connection to the Harary Norman matrix is the following: All entries in the Harary-Norman adjacency matrix are -1 in Tyler's matrix. Nonzero entries of the linegraph adjacency matrix that are 0 in the HararyNorman matrix are the +1's in Tyler's matrix.

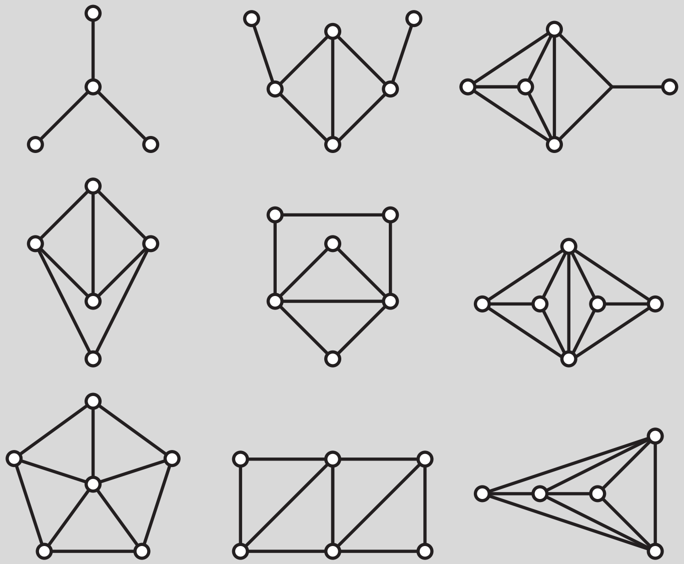
Question

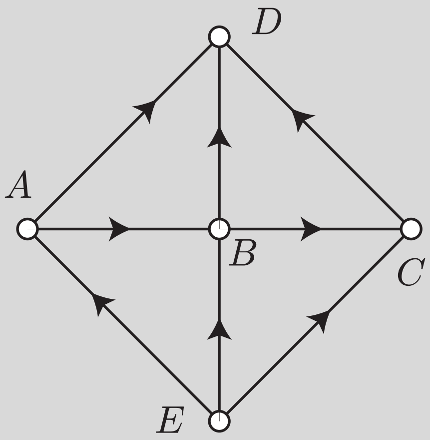
If we start out with a matrix M, the algorithm finds a graph whose partition matrix is M (or shows that no such graph exists). When is this graph a line graph? Is there always a line graph in the infinite family? A forbidden subgraph characterization of line graphs (derived graphs) was obtained by L. Beineke [?].

Line graphs of Simple Graphs

Characterized by L. Beineke, 1978, [?].

Forbidden subgraphs of a simple line graph:

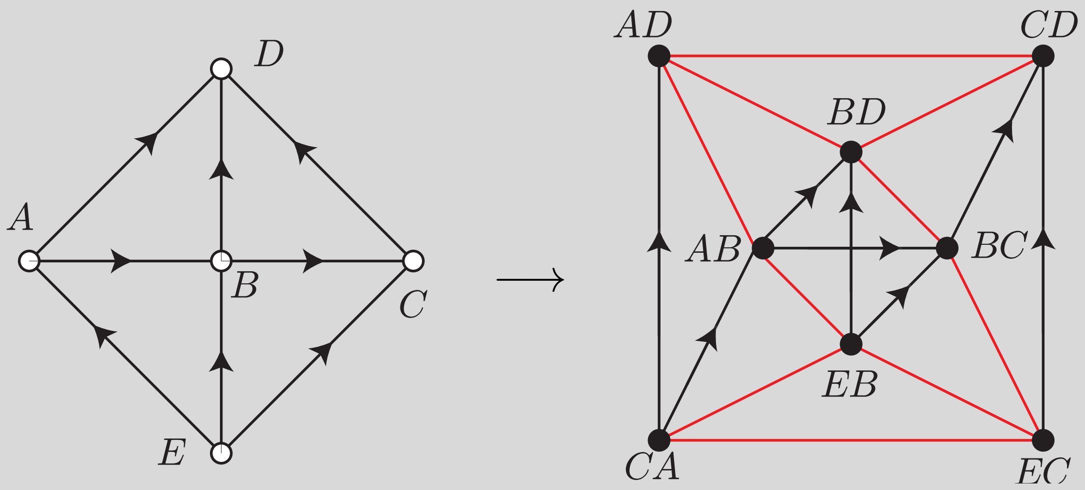




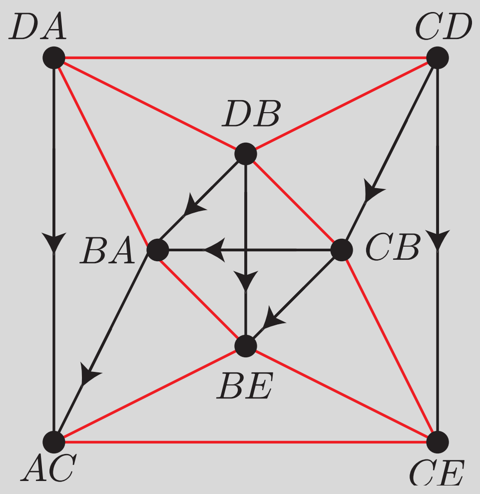
Is a Kirchhoff Graph for

$$\begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Equitable arc partitions from line digraphs



Harrary Norman Line Digraph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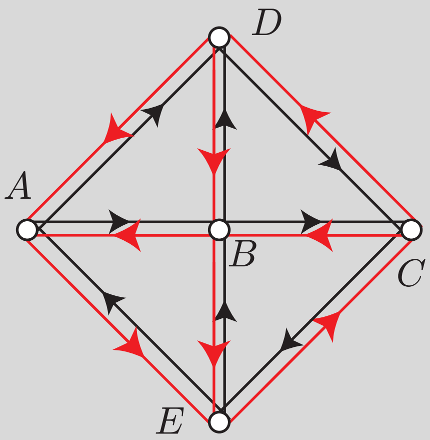
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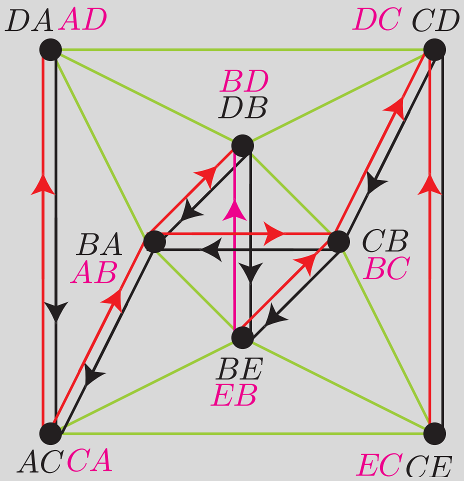
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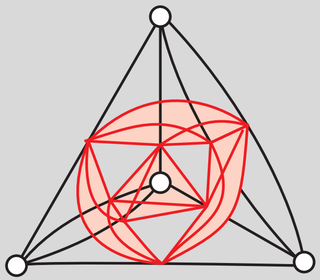
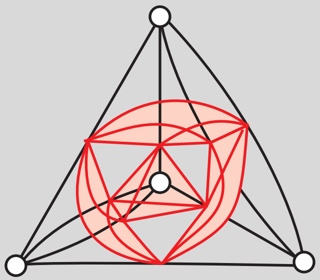
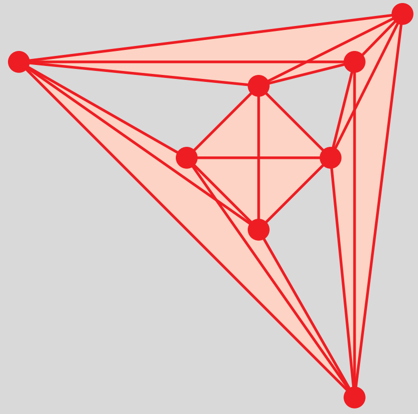
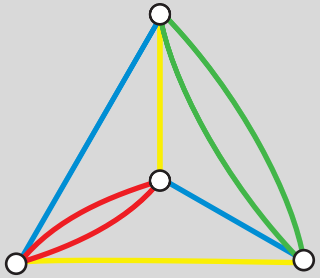
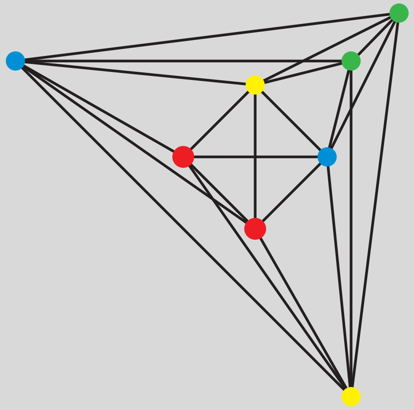
What about line graphs of non-simple graphs

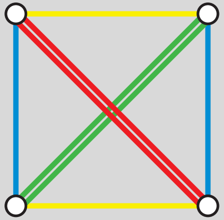












$$Q \quad \begin{bmatrix} 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

Theorem 4 *Let D be a connected digraph with edge partition π. The $A_E(D/\pi)$ is symmetric if and only if π is uniform.*

Corollary 1 *Every uniform equitable edge partition of D is Kirchhoff.*

Corollary 2 *If equitable edge partition π is Kirchhoff and uniform, then $A_E(D/\pi)$ is symmetric.*

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References

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