Graphs, frameworks, molecules, and mechanisms

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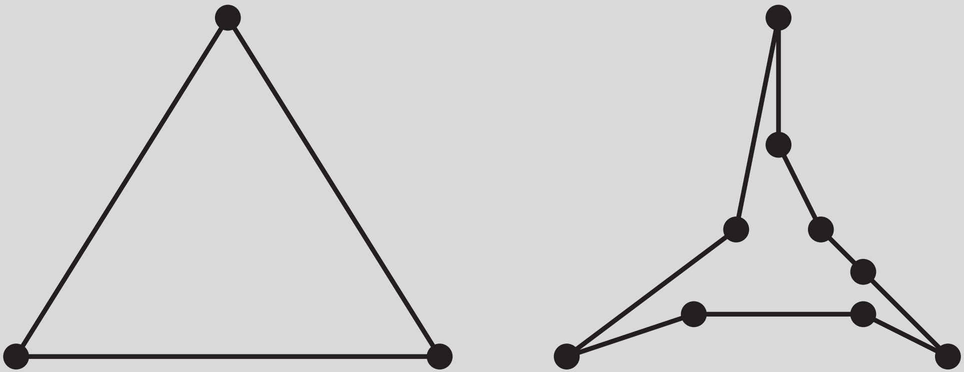
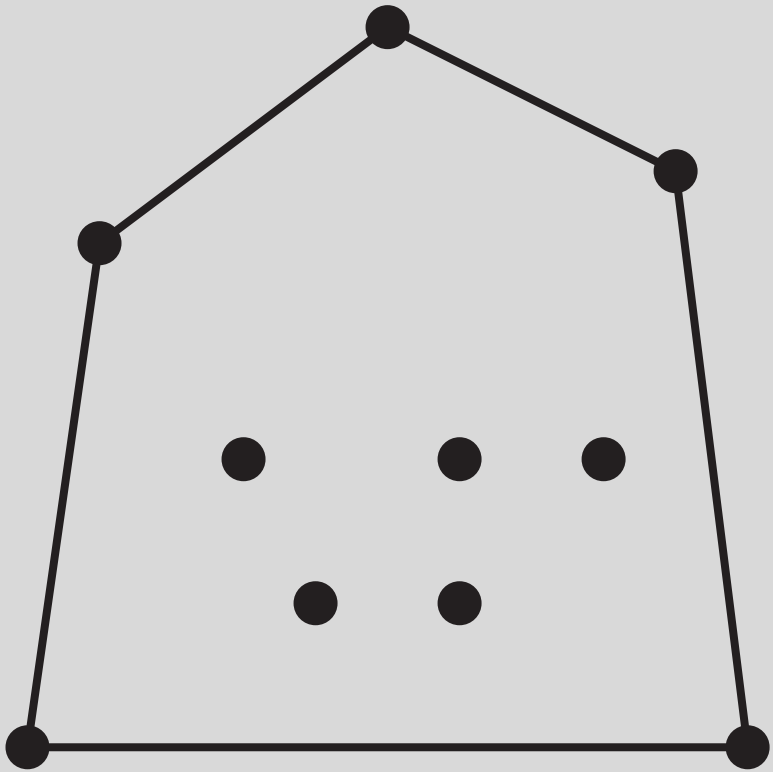
1. Pseudo-triangles



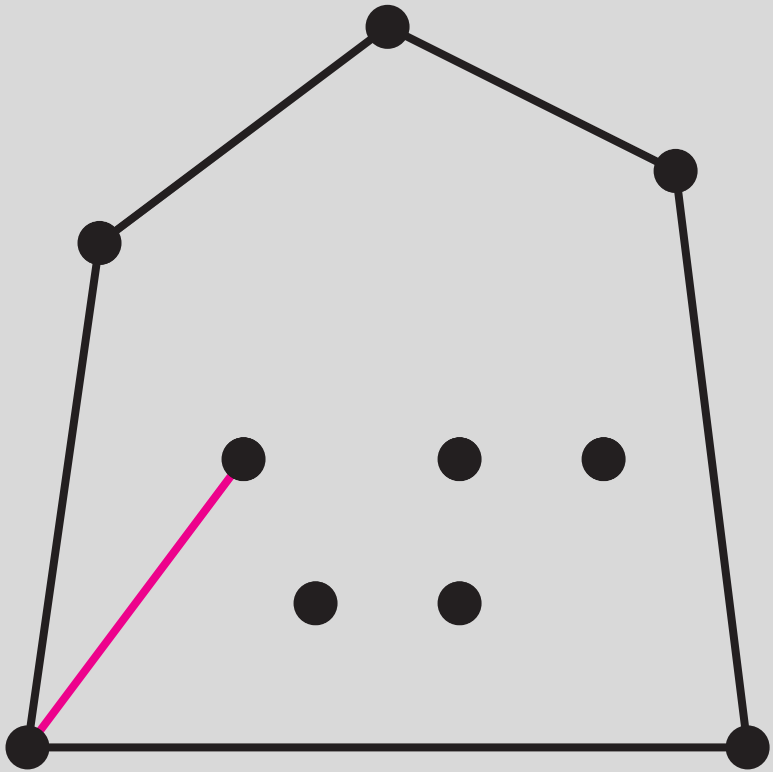
1.1. Pseudo-Triangulating

Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



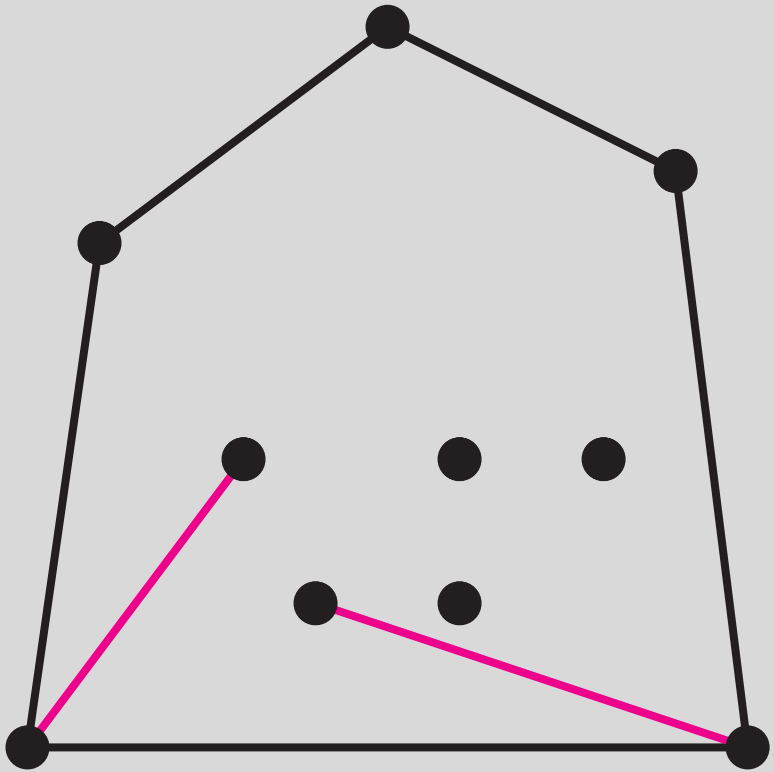
Add edges. . .

Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



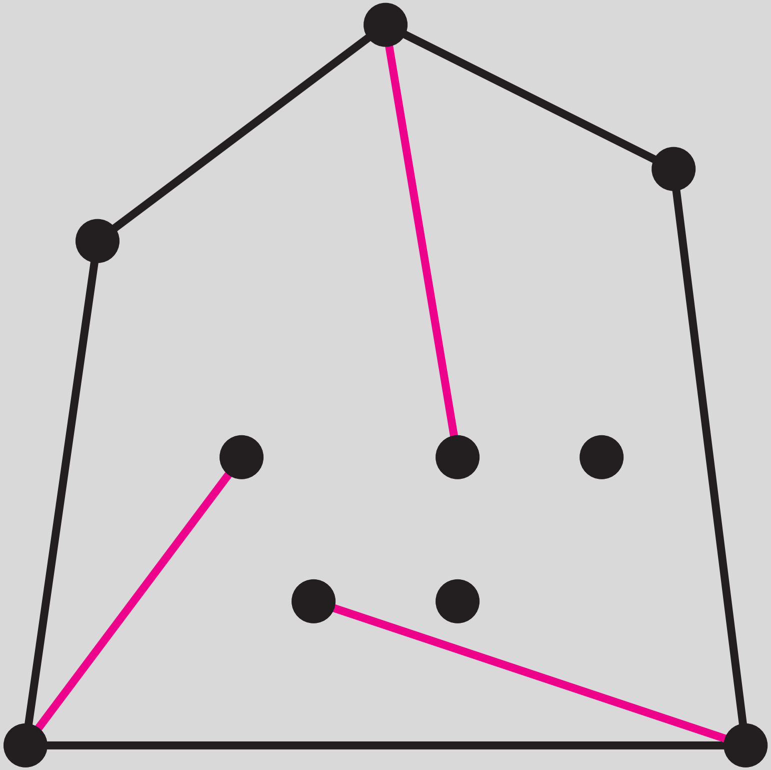
Add one edge

Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



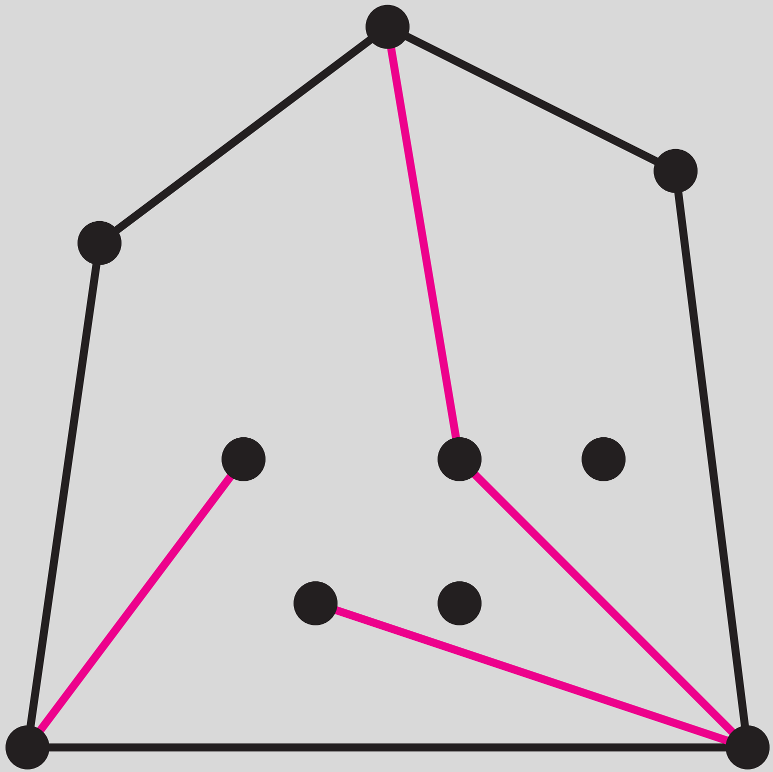
Add two edges

Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add three edges

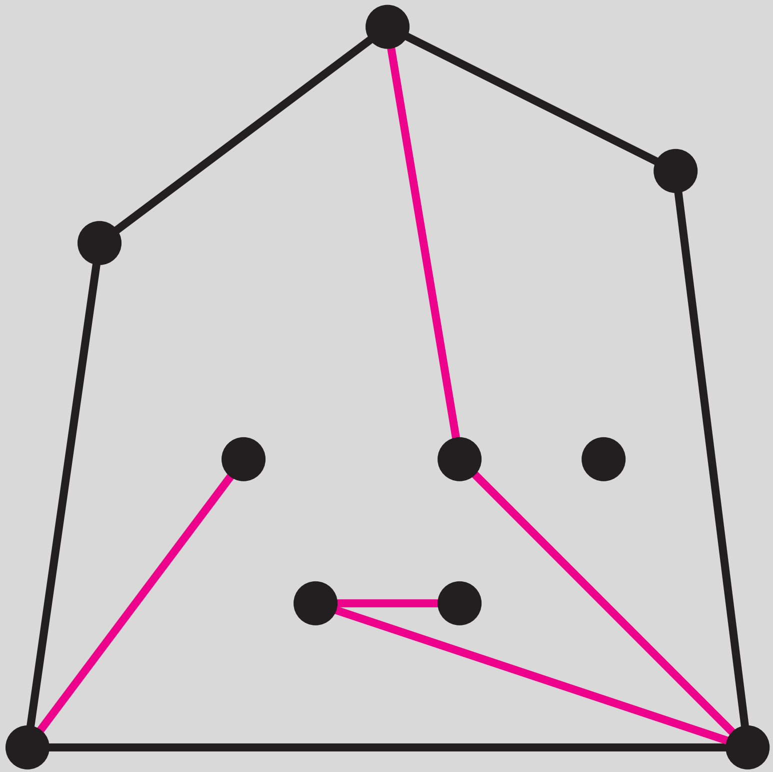
Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add four edges

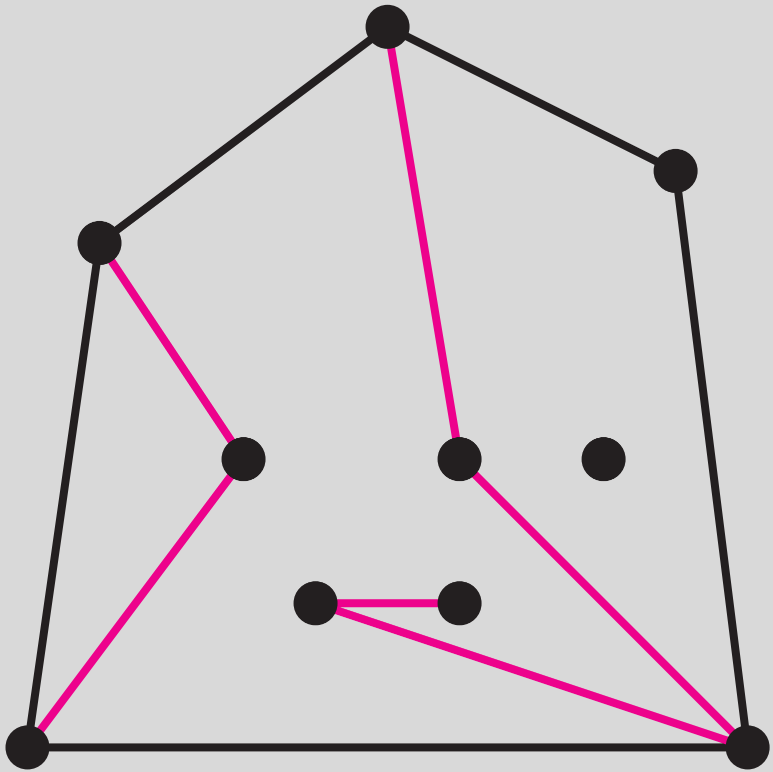
Start with a point set. . . form the convex hull

10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add five edges

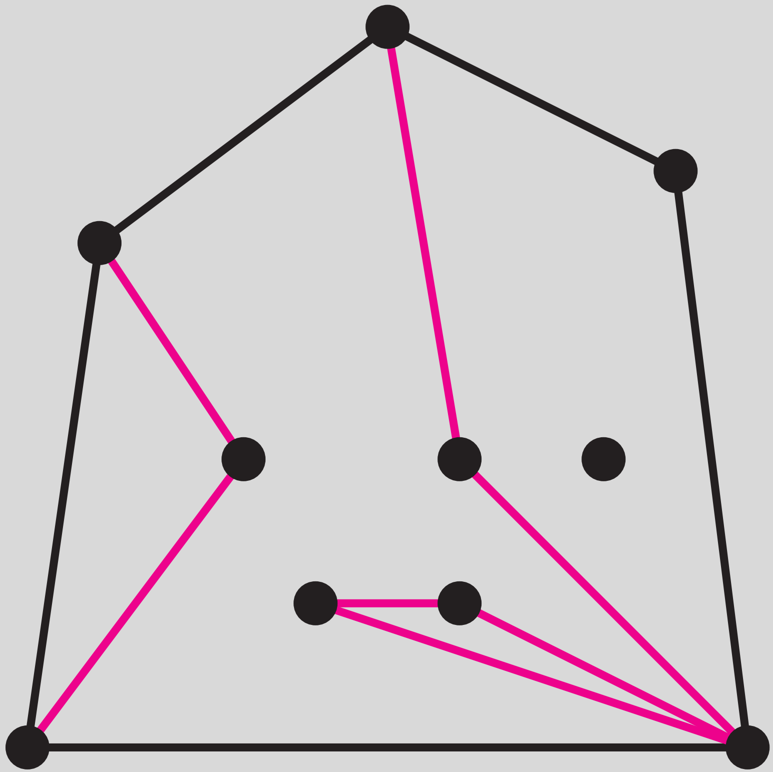
Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add six edges

Start with a point set. . . form the convex hull

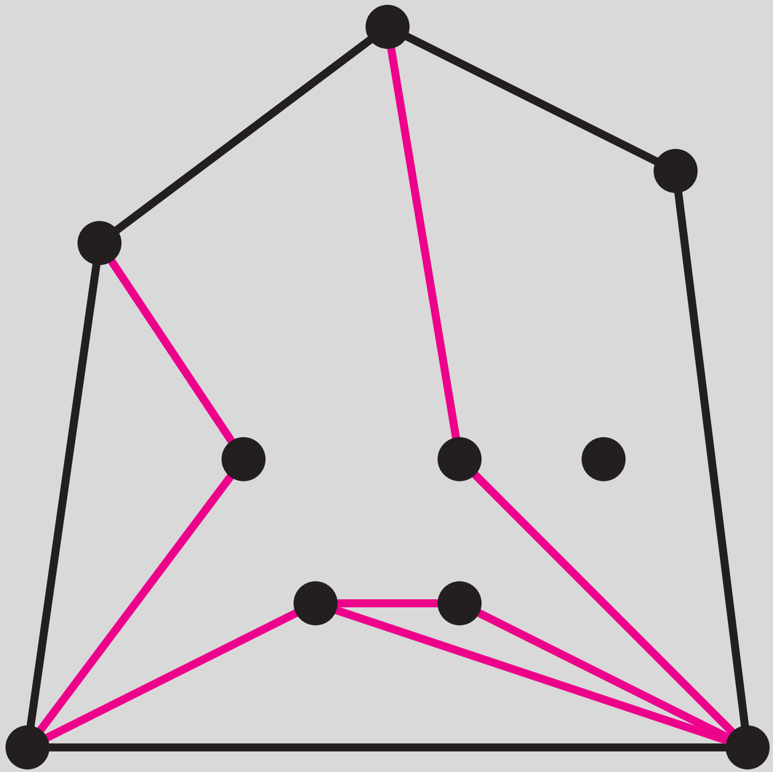
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add seven edges

Start with a point set. . . form the convex hull

10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add eight edges

Pseudo-triangles

Some Examples:

Does planarity . . .

Matroids on Graphs

Matroid connectivity

Configuration Index

Global rigidity

Example

Random graphs

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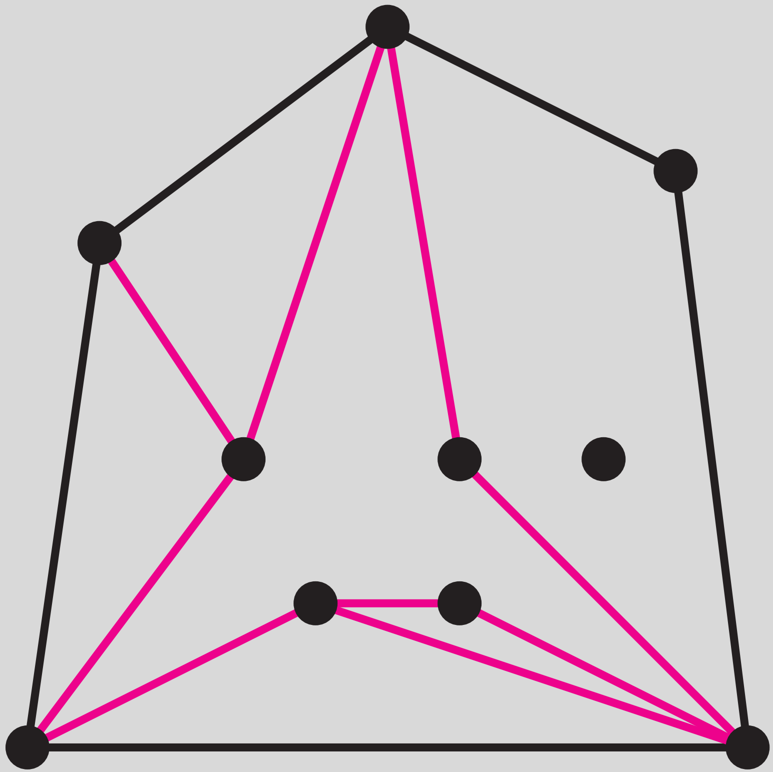
Start with a point set. . . form the convex hull

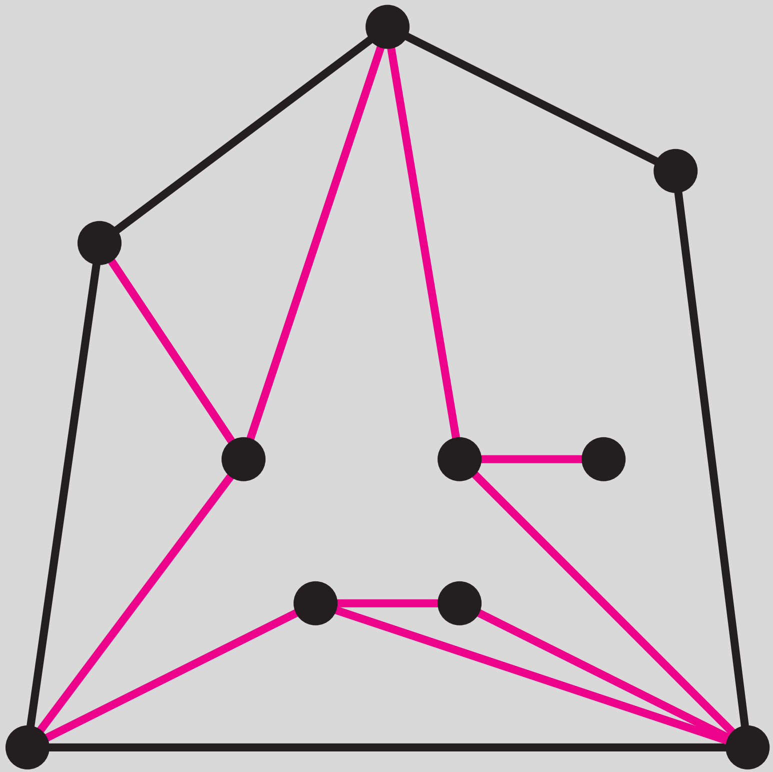
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add nine edges

Start with a point set. . . form the convex hull

10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add ten edges

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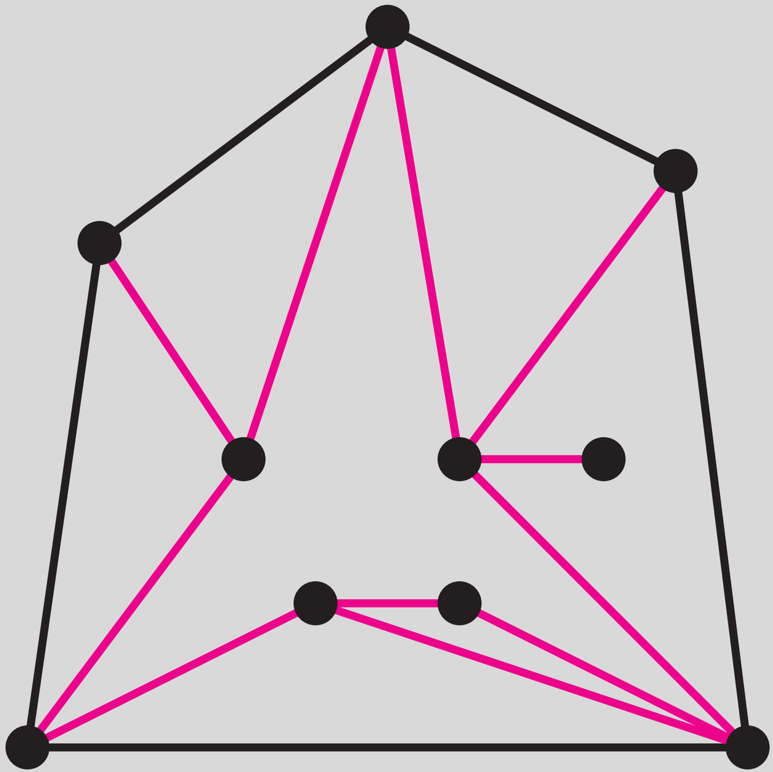
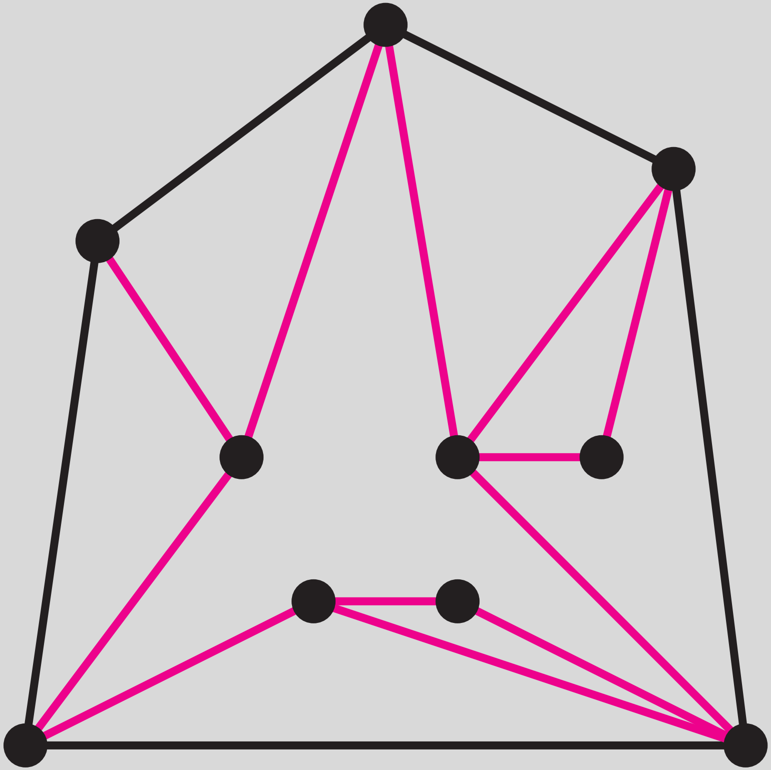
Start with a point set. . . form the convex hull

10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add eleven edges

Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add twelve edges - *Pseudo-Triangulation*

1.2. Properties

Theorem 1 (Streinu - 2000 [15]) *The following are equivalent*

- *G is a pseudo-triangulation with the minimum number of edges.*
- *G is a pointed pseudo-triangulation*
- *G is a pseudo-triangulation with exactly $2n - 3$ edges*
- *G is non-crossing, pointed, and has $2n - 3$ edges*
- *G is non-crossing, pointed, and maximal with this property*

Corollary 1 *If any of the above conditions are satisfied, then G is generically minimally rigid in the plane and any realization of G as a pseudo-triangulation is 1'st order rigid.*

Theorem 2 ([7]) *Every planar graph which is generically minimally rigid has a realization as a pointed pseudo-triangulation.*

Proof 1 uses an inductive construction together with topological information.

Proof 2 uses linear algebra - Tutte's approach to drawing a graph.

1.3. Definition of CPPT

A *combinatorial pointed pseudo-triangulation* (cppt) is an assignment of labels, *big* and *small*, to the angles of a plane graph such that

- every vertex has exactly one big angle,
- every interior face as exactly three small angles
- the outside face has only big angles.

G has

$-n$ vertices,
$-e$ edges and
$-f$ faces.

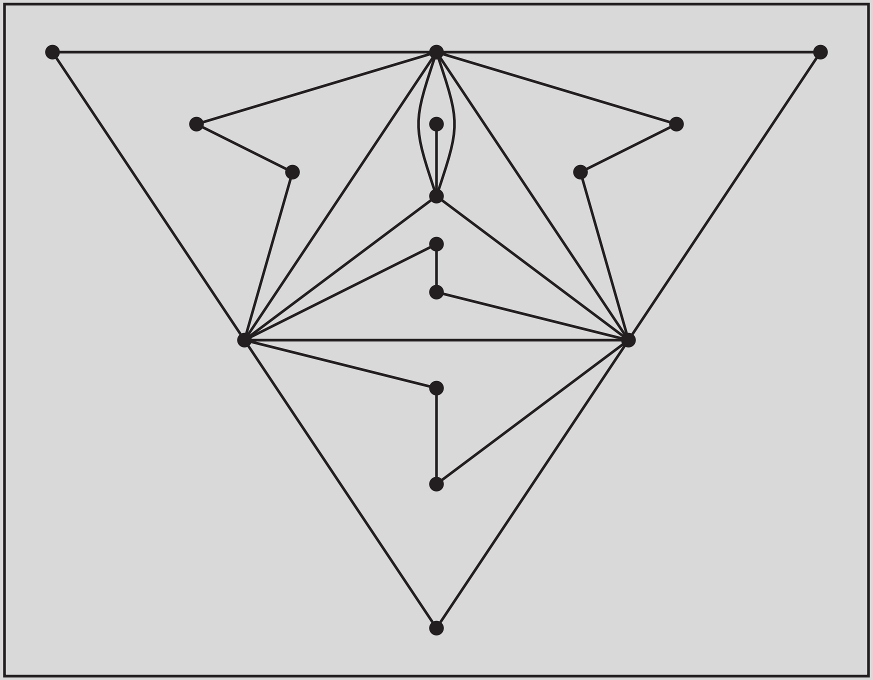
Necessary condition for the existence of a cppt:

$$e = 2n - 3$$

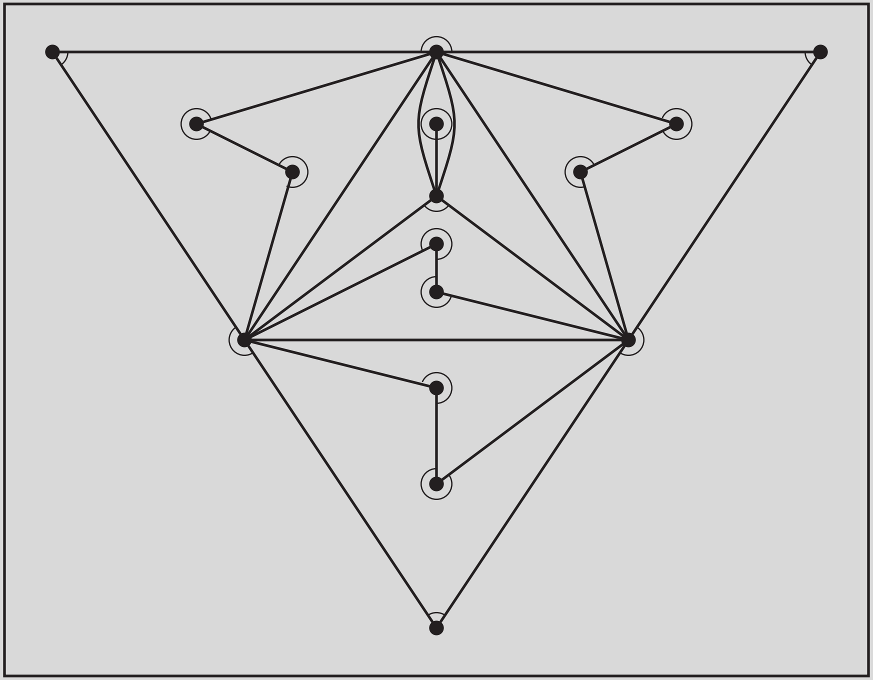
(Since $n - e + f = 2$ and $3f - 3 + n = 2e$.)

1.4. Combinatorial CPPT

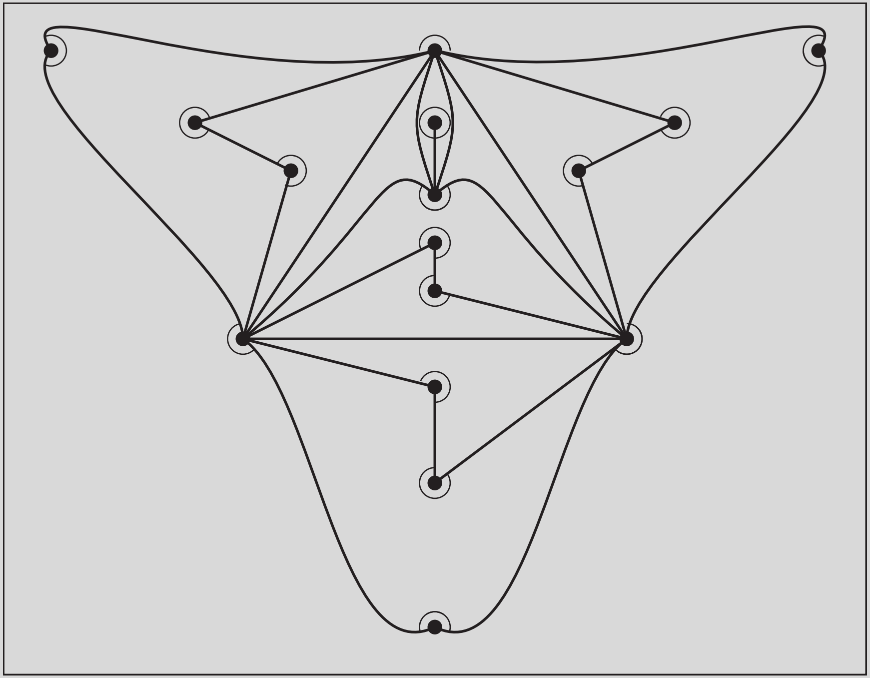
A graph in the plane



A combinatorial pseudo-triangulation



A topological realization



1.5. Generalizations.

A combinatorial pseudo-triangulation has the generalized Laman property if every subset of x non- pointed plus y pointed vertices, with $x + y = 2$, induces a subgraph with at most $3x + 2y - 3$ edges.

Theorem 3 ([12]) *Given a plane graph G, the following conditions are equivalent:*

- *G is generically rigid*
- *G contains a spanning isostatic subgraph,*
- *G can be labelled as a CPT with the generalized Laman property.*
- *G can be stretched as a pseudo-triangulation (with the given embedding and outer face).*

1.6. Reciprocal Figures

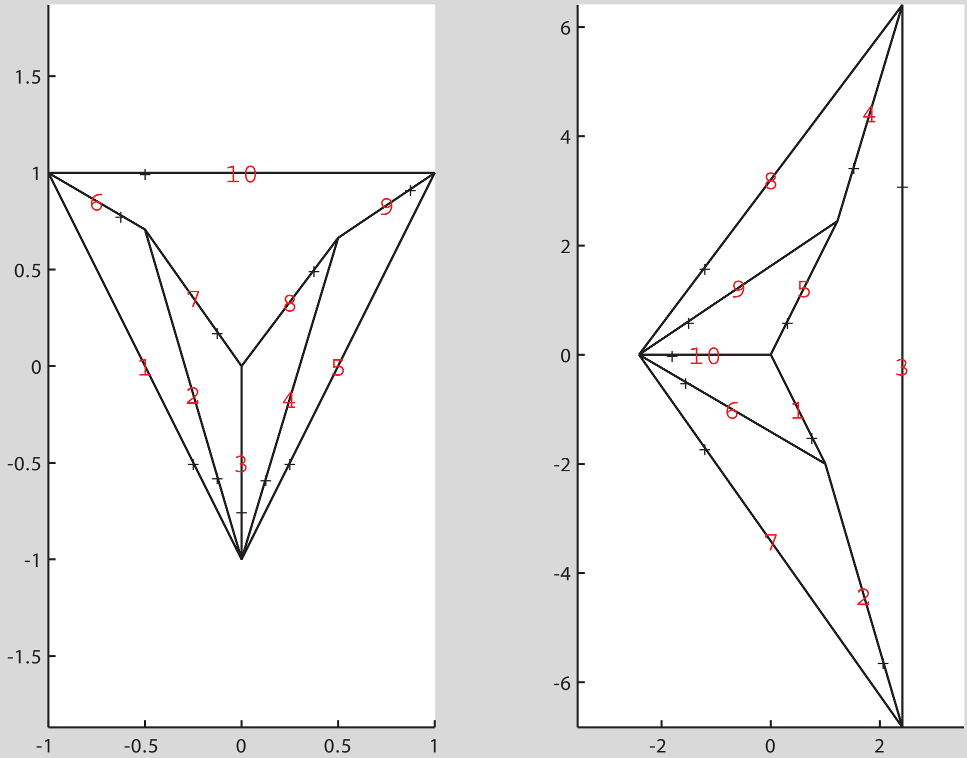
We want to draw the geometric dual using the same edge directions.

Construction

Use a framework with a resolvable stress, non-zero on every edge, for example a cycle in the rigidity matroid.

Such a cycle corresponds to a pseudo-triangulation with one non-pointed vertex.

A Wheel and Its Reciprocal



Theorem 4 *If a generic 2-cycle is realized as a pseudo-triangulation, then the reciprocal diagram is also a pseudo-triangulation.*

2. Some Examples:

Seven Wheel 1:

Seven Wheel 2:

Center segment revolves:

Graph not in a plane embedding.

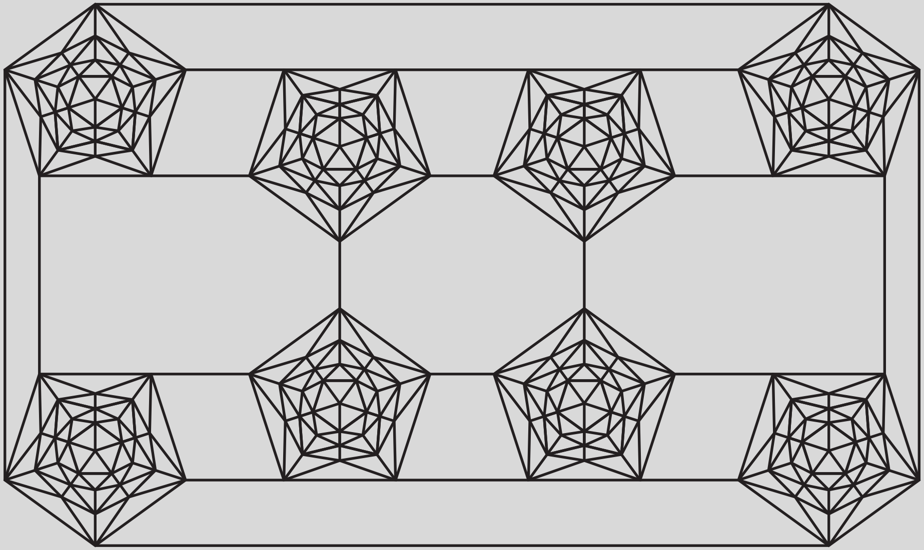
Center segment revolves and rotates:

Graph not in a plane embedding.

A non-planar reciprocal:

3. Does planarity influence rigidity?

In 1982 Lovasz and Yemini showed that 6-connectivity implies planar rigidity, [10]. Can the connectivity requirement be lowered for planar graphs?



A planar 5-connected non-rigid graph.

However,Lovász and Yemini [10] note that their proof technique will show that $G - \{e_1, e_2, e_3\}$ is rigid for all $e_1, e_2, e_3 \in E$, and hence that G is edge 2-rigid. This result was combined with Theorem 9 in [8] to deduce

Theorem 5 *Every 6-connected graph is globally rigid.*

There are easy inductive constructions to generate all 2-connected graphs, namely G is 2-connected if and only if it can be built up from a cycle by sequentially adjoining edges (loops are not allowed) and subdividing edges. A graph is edge-2connected if and only if it can be built up from a vertex by adding edges (loops are allowed) and subdividing edges, see [6].

A graph is 2-rigid if and only if it can be obtained from tetrahedra by a sequence of 1-extension, edge addition and 2-sum [1].

4. Matroids on Graphs

We are studying two matroids on the edge set E of a graph $G(V, E)$, namely the *cycle matroid*, $\mathfrak{C}(G)$, defined by its cycles of G as circuits (or, equivalently, by c-independent sets as the collection $\mathcal{I}$) and the (2-dimensional generic) *rigidity matroid*, $\mathfrak{R}(G)$, defined by r-independent edge sets.

5. Matroid connectivity

Tutte [17] calls a matroid on the ground set E *n-connected*, if for any positive integer $k < n$ there is no partition of E into two sets E_1 and E_2 such that $|E_i| \geq k$ and $\rho(E_1) + \rho(E_2) \leq \rho(E) + k - 1$. With this definition every matroid is 1-connected. A matroid is 2-connected if there is no partition of E into two sets E_1 and E_2 such that $|E_i| \geq 1$ and $\rho(E_1) + \rho(E_2) \leq \rho(E)$, i.e. if it is not the direct sum of its restrictions to the E_i's. Every matroid can be uniquely decomposed into a direct sum such that each of the summands is 2-connected. With Tutte's 2-connectivity of the graph G is equivalent to 2-connectivity of its cycle matroid $\mathfrak{C}(G)$.

It is well known, see for example [13] or [14], that a matroid is 2-connected if and only if for any partition of the ground set into two sets, there is a circuit C intersecting both of them. In fact an even stronger conclusion holds, namely a matroid is 2-connected if and only if any pair of its edges is contained in a circuit.

5.1. The 2-sum

The 2-sum, $M_1 \bigoplus_2 M_2$, of two matroids M_1 and M_2, both containing at least 3 elements and having exactly one element e in common, where e is neither dependent (a loop) or a bridge in either of the M_i, is a matroid on the union of the ground sets of M_1 and M_2 excluding e and the circuits of $M_1 \bigoplus_2 M_2$ consist of circuits of M_i not containing e and of sets of the form $C_1 \bigcup C_2 \setminus e$ where C_i is a circuit of M_i containing e.
A matroid is 3-connected if and only if it cannot be written as a 2-sum.

The 2-sum is also defined for graphs, but here one cannot identify two edges without specifying which pairs of endpoints are to be identified, in other words, without specifying an orientation on the edges to be amalgamated, see Figure 1.

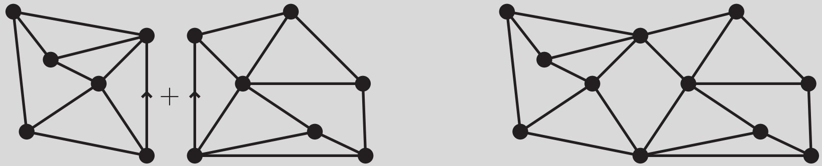


Figure 1 *The 2-sum of two circuits.*

Note that the 2-sum of two cycles is a cycle.

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5.2. The 2-sum and 2-connectivity

Clearly the 2-sum of graphs is associative provided that the edges to be amalgamated are distinct, and so it is convenient to represent the result of a succession of 2-sums as a tree in which the nodes encode the graphs to be joined, and the edges encode the (oriented) edges to be amalgamated, see Figure 2.

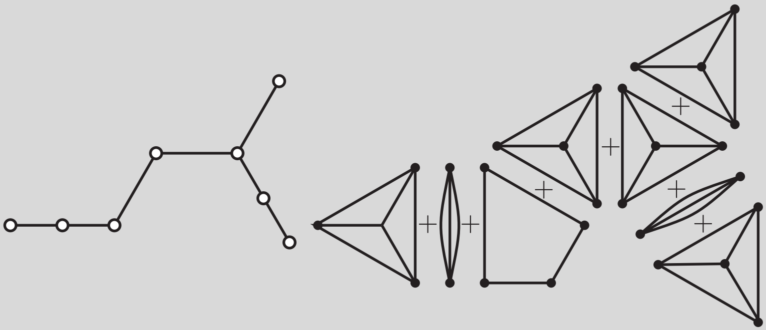


Figure 2 *A 2-sum tree.*

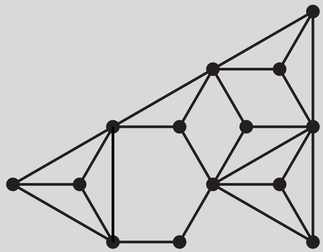
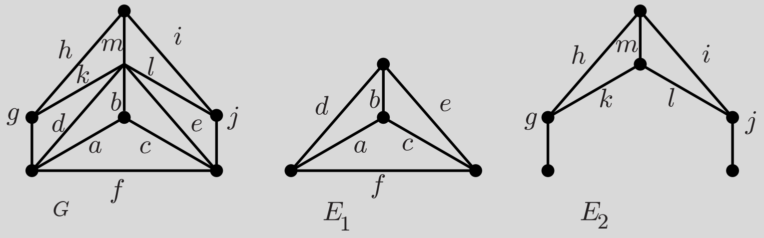


Figure 3 *The three block tree in Figure 2 encodes this graph.*

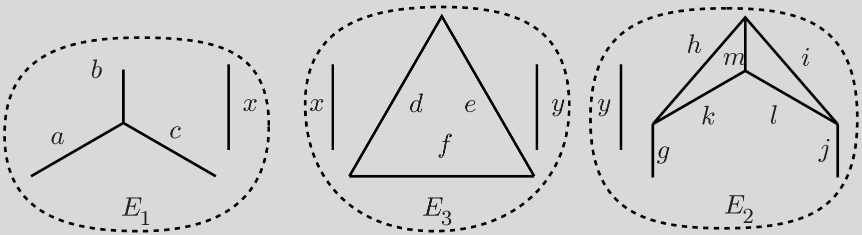
Tutte proved the following deep theorem characterizing finite 2-connected graphs, see [16, 5].

Theorem 6 *A 2-connected graph G is uniquely encoded by its 3-block tree.*

This result has been generalized for matroids. Every 2-connected matroid has a unique encoding as a 3-block tree in which the 3-blocks are 3-connected matroids, bonds (matroids in which every 2-element subset is a circuit) and polygons (matroids consisting of a single circuit) [4] Theorem 18.



The rigidity matroid is not closed under 2-sum decomposition.



Theorem 7 *Let G be a rigid graph with connected rigidity matroid $\mathfrak{R}(G)$. Then the 3-blocks of G are multilinks or globally rigid graphs on at least four vertices.*

Proof: If G is 3-connected it is globally rigid. If G is not 3-connected, we compute its 3-block tree T. Consider a leaf node G_L of T. G_L cannot be a multilink because G is simple, and it cannot be a cycle, because G is redundantly rigid. Therefore G_L is a 3-connected graph, which is redundantly rigid, hence globally rigid. Now the theorem follows by induction on the number of nodes of T. $\square$

6. Configuration Index

6.1. Definition and Examples

The *configuration index* $\iota(G, \mathbf{p})$ of a graph $G(V, E)$ whose vertices are embedded in the plane by $\mathbf{p} : V \to \mathbb{R}^2$ is the cardinality of the set of congruence classes of embeddings of G with the same edge lengths as in $(G, \mathbf{p})$. We call $\mathbf{p}$ *generic* if the coordinates of $\mathbf{p}(V)$ as point in $\mathbb{R}^{2|V|}$ are algebraically independent over $\mathbb{Q}$. If $\mathbf{p}$ is generic, $\iota(G, \mathbf{p}) = 1$ exactly when G is globally rigid.

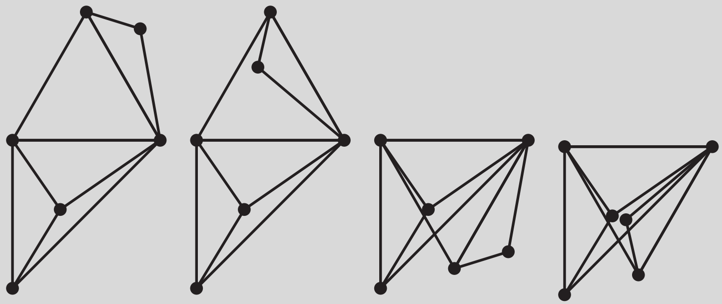


Figure 4

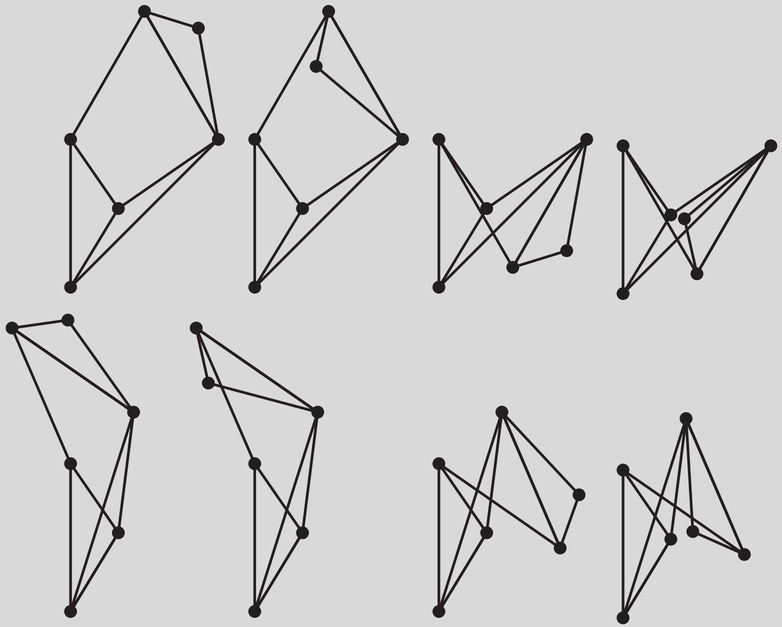


Figure 5

6.2. The configuration index of a graph with 2-connected rigidity matroid

Let G be rigid and let $\mathfrak{R}(G)$ be 2-connected. From Theorem 7 we know that its 3-blocks are globally rigid or multilinks, which makes it easy to compute their configuration index.

Theorem 8 *Let $G(V, E)$ be rigid, $|V| \geq 4$, and let $\mathfrak{R}(G)$ be 2-connected. If k is the number of globally rigid 3-blocks of $\mathfrak{R}(G)$ (which are not multi-links), then $\iota(G, \mathbf{p}) = 2^{k-1}$ for any generic embedding $\mathbf{p}$.*

Proof: Given an embedding of G, we can reflect its 3-blocks about axes determined by the endpoints of edges along which the 2-sum is taken, so 2^{k-1} is a lower bound for the configuration index of G. However, the 3-blocks are not necessarily subgraphs of G and the subgraphs of G induced by the vertex sets of the 3-blocks need not even be rigid or connected, they might in fact consist of isolated vertices. Let $\mathbf{p}$ be a generic embedding of the vertices. All edge-lengths are in the algebraic closure of $\mathbb{Q}(\mathbf{p}(V))$ and the edge lengths of a base of $\mathfrak{R}(G)$ are also algebraically independent. Now if we prune a leaf F of the 3-block tree along e, then, since both 2-summands F and $G \setminus F$ are rigid after deletion of e, the length of e can be computed from the edge length information in either summand alone. $F - e$ might not be globally rigid, but since F is generically embedded, e will have different length in non-congruent embeddings. Any equality of the length of e in a re-embedding of F with the length of e in a re-embedding of $G \setminus F$ can be described as a non-trivial polynomial equation in the vertex coordinates, contradicting genericity. The theorem now follows by induction on k. $\square$

7. Global rigidity

$G(V, E)$ is called *redundantly rigid* if $G(V, E - e)$ is rigid for all $e \in E$, i.e. the removal of a single edge e from the rigid graph G does not destroy rigidity. Redundant rigidity is a key to characterize global rigidity.

Theorem 9 *[8] Let G be a graph. Then G is globally rigid if and only if G is a complete graph on at most three vertices, or G is both 3-connected and edge 2-rigid (redundantly) rigid.*

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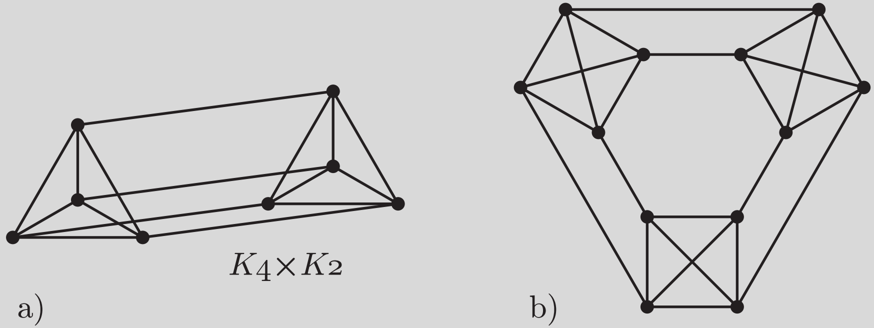
7.1. Vertex transitive graphs

Theorem 10 *A four-regular vertex transitive graph is generically rigid in the plane if and only if it contains no subgraph isomorphic to K_4, or is K_5 or one of the graphs in the following figure.*

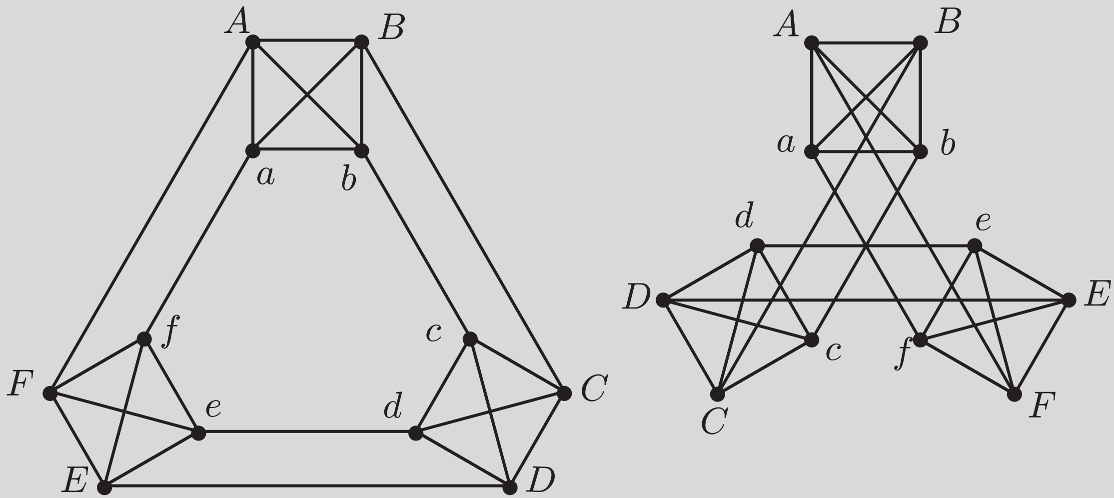


Vertex transitive rigid graphs containing K_4.

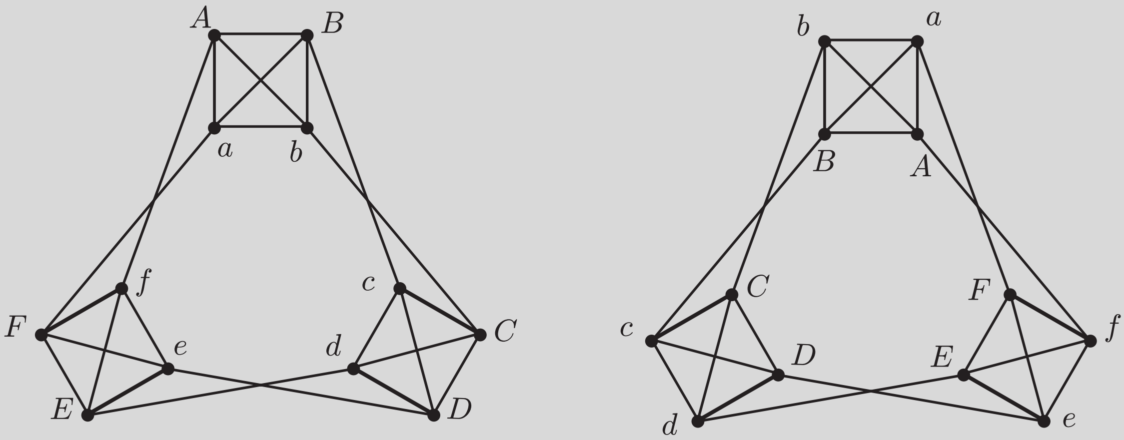
Theorem 11 *Let G be a vertex transitive non-rigid graph. Then G is k-regular with $k \leq 6$, and contracting the non-trivial rigid components of G produces a vertex transitive graph of regularity at most 5.*

PROOF: By the lemma, there is only one kind of non-trivial rigid component, say on s vertices. Such a component is attached to its complement in G by s independent edges. Contracting the non-trivial rigid components will, by our transitivity assumptions, produce an s-regular graph on v vertices, where v is the number of non-trivial rigid components in G. The contracted graph has $sv/2$ edges. The rank of G is $v(2s-3)+sv/2$, which must be smaller than $2sv-3$. This yields the inequality $6 < v(6-s)$, thus each rigid component has at most 5 vertices and the regularity of G is at most 6. □

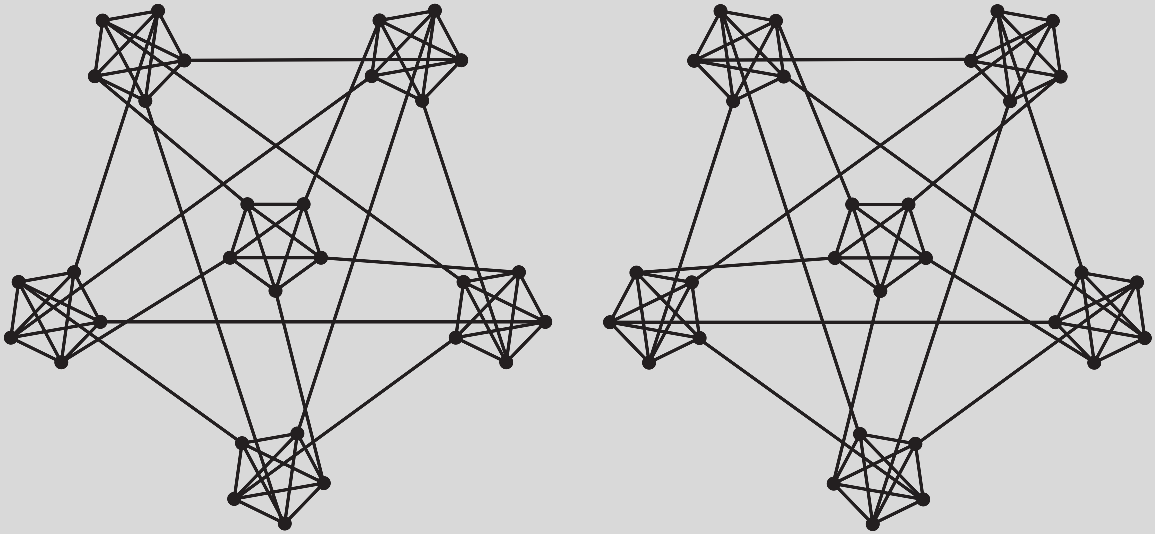
Theorem 12 *Let G be a connected k-regular vertex transitive graph on n vertices. Then G is not rigid if and only if either:*
(a) $k = 2$ and $n \geq 4$.
(b) $k = 3$ and $n \geq 8$.
(c) $k = 4$ and G has a factor consisting of s disjoint copies of K_4 where $s \geq 4$
(d) $k = 5$ and G has a factor consisting of t disjoint copies of K_5 where $t \geq 8$.



Two embeddings which are rigid, but neither infinitesimally rigid nor globally rigid.



Two embeddings which are rigid and infinitesimally rigid but not globally rigid.

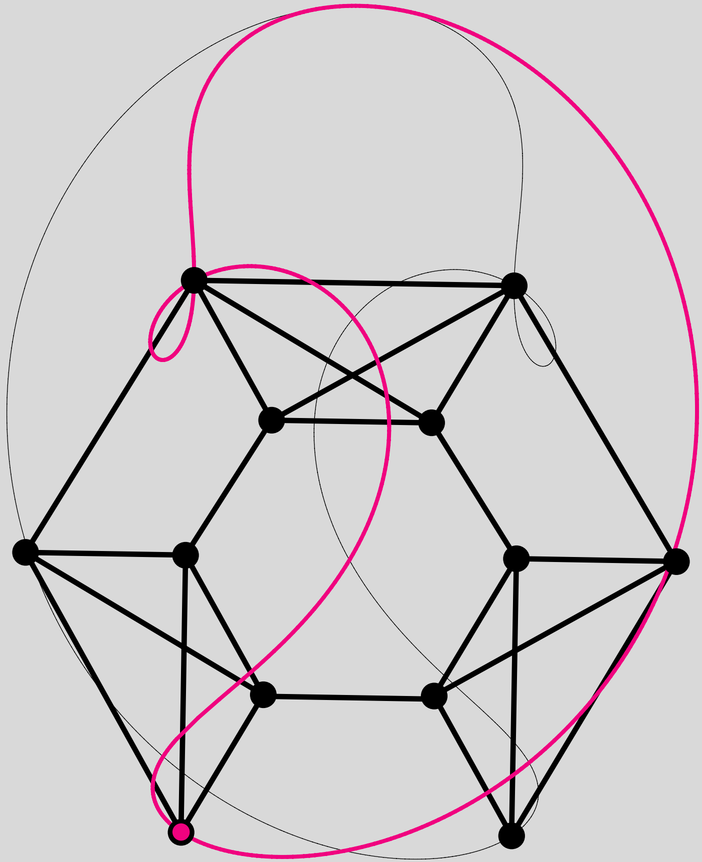


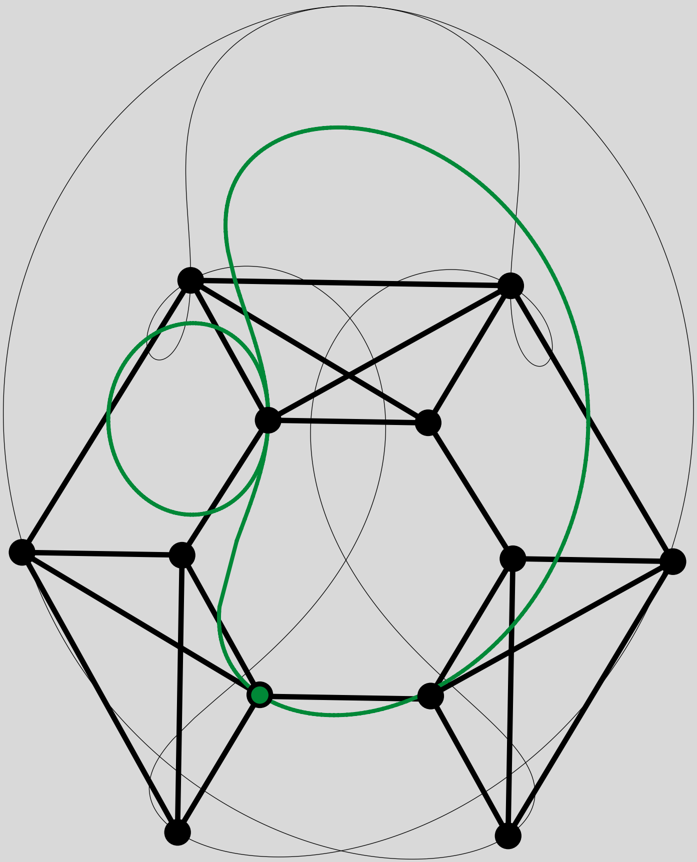
Two non-congruent embeddings with same edge lengths.

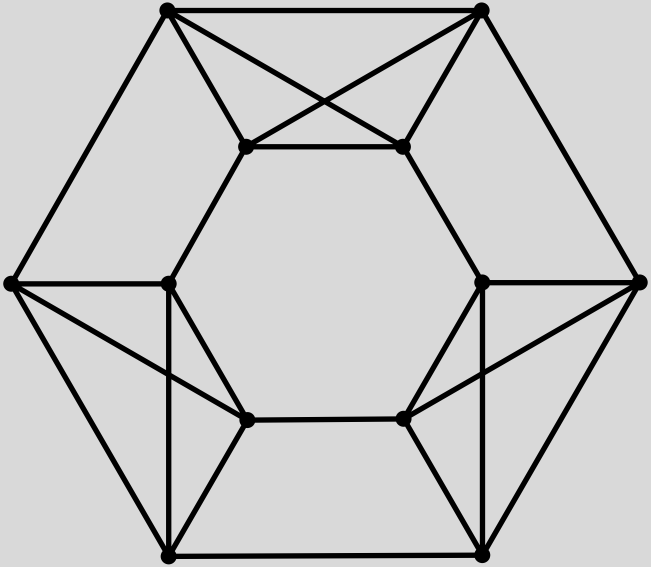
We observe that for a rigid G which is not redundantly rigid, $\mathcal{M}(G)$ is not connected. It is in fact the direct sum over the maximal redundantly rigid subgraphs (or singleton edges). The arguments in the preceding proofs are unaltered if we replace rigid components by redundantly rigid subgraphs and we obtain A vertex transitive rigid graph is also globally rigid unless it has a factor consisting of 3 copies of K_4 or 6 copies of K_5.

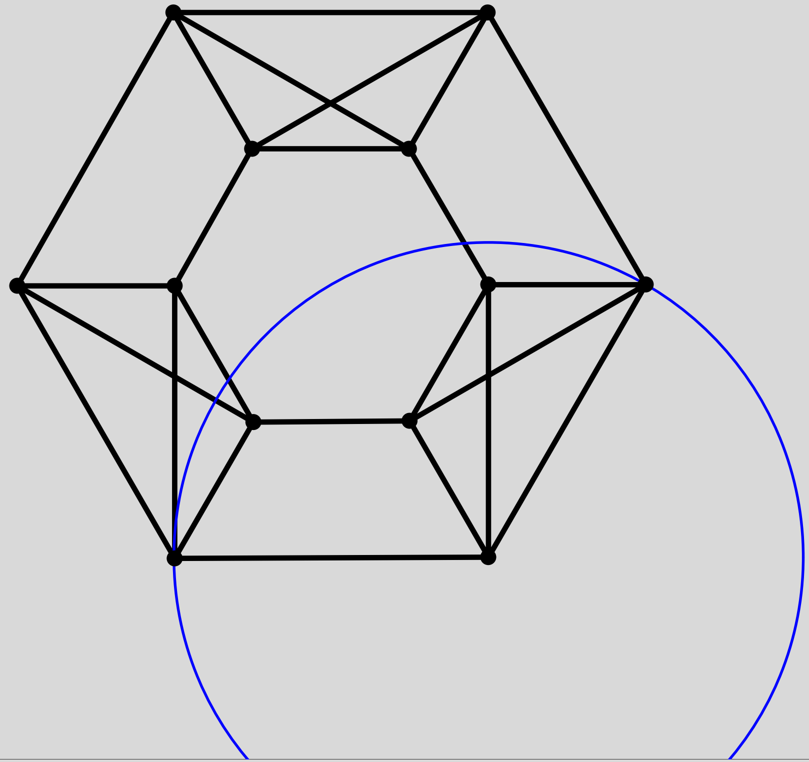
Proof: For rigid but not globally rigid graphs, equality holds in the last inequality of the proof of Theorem 11, and the two solutions yield the two exceptions stated. □

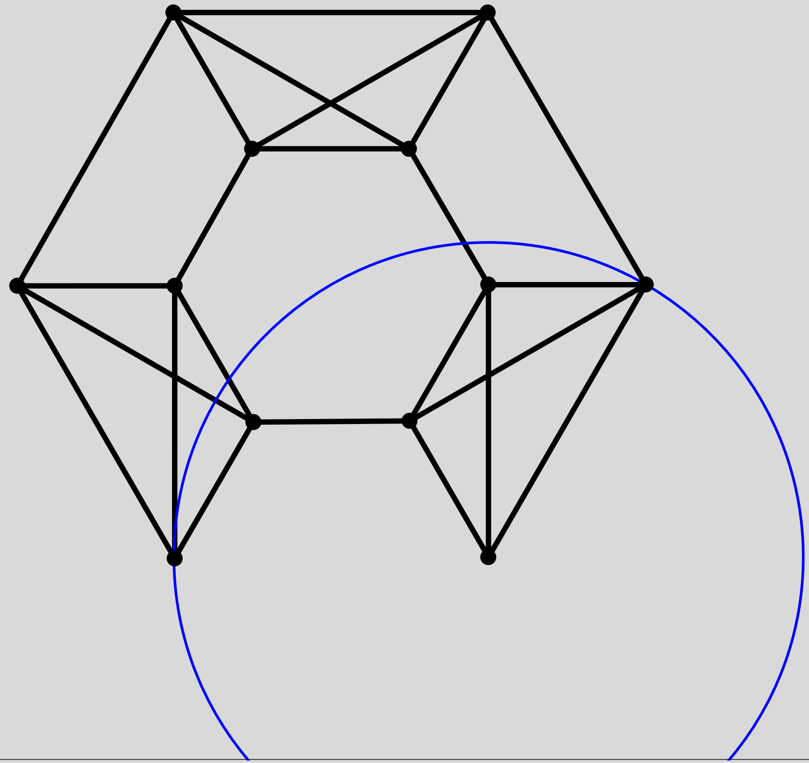
8. Example

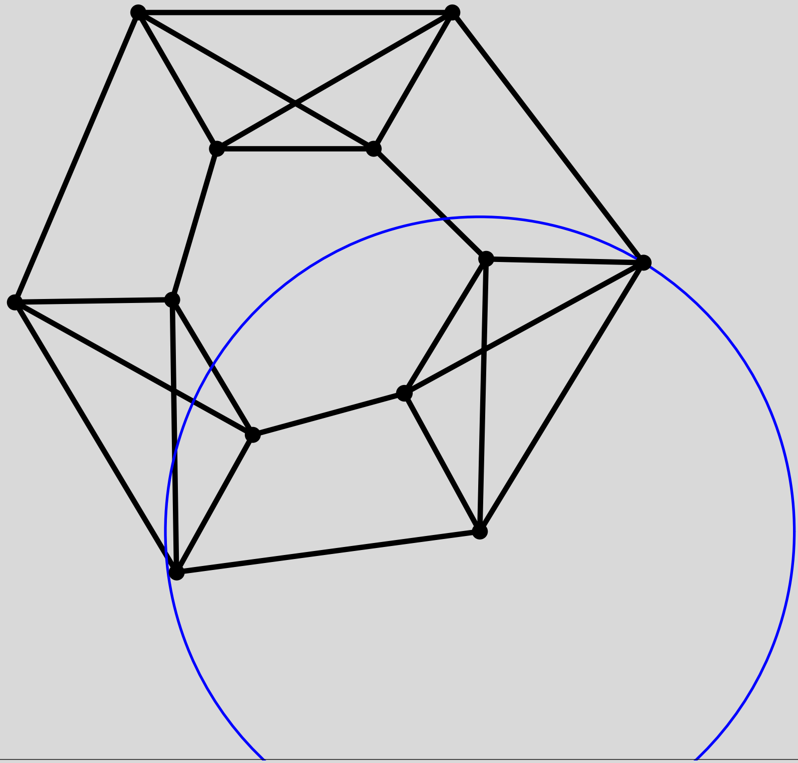


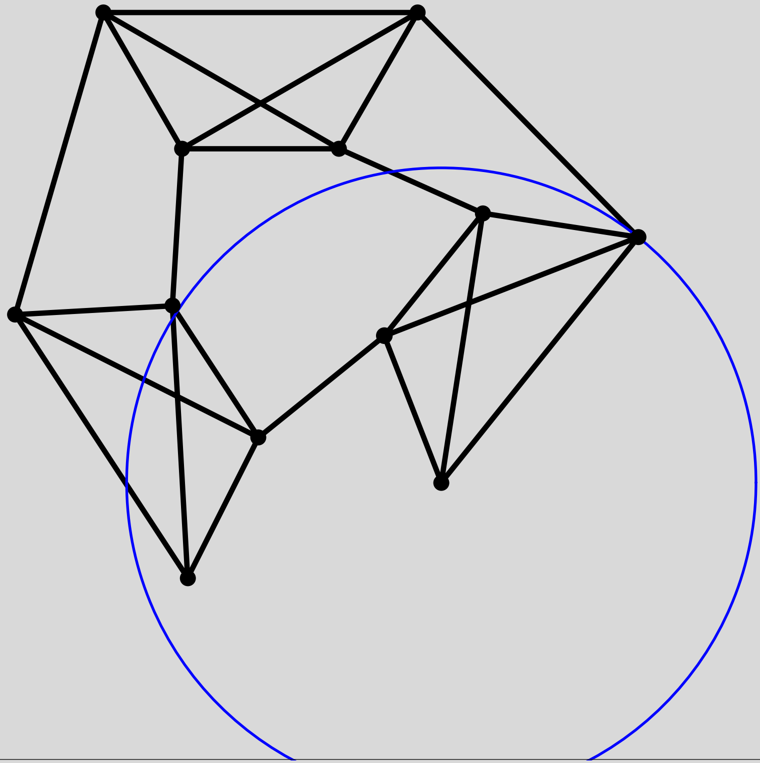


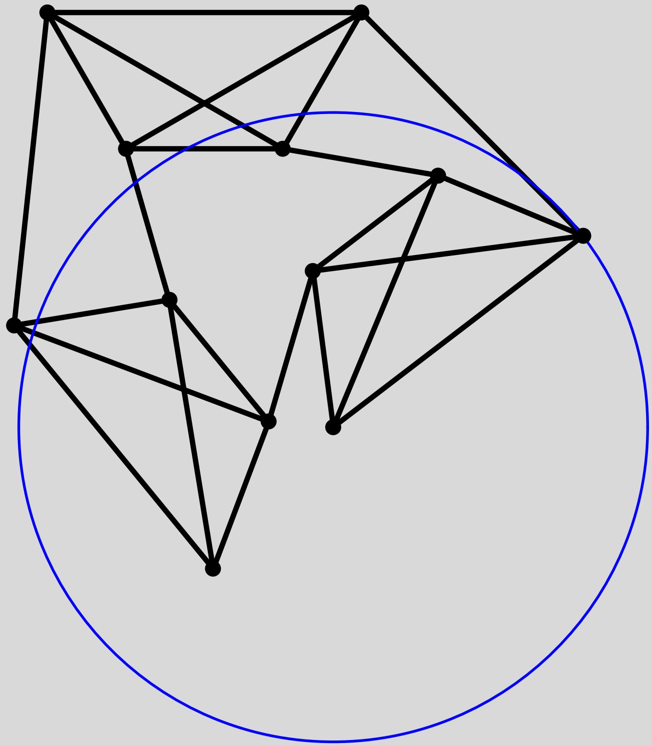


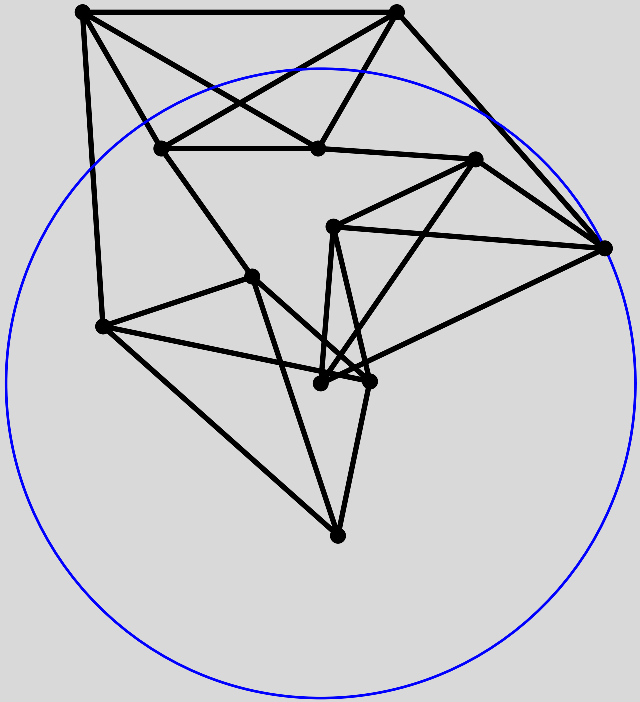


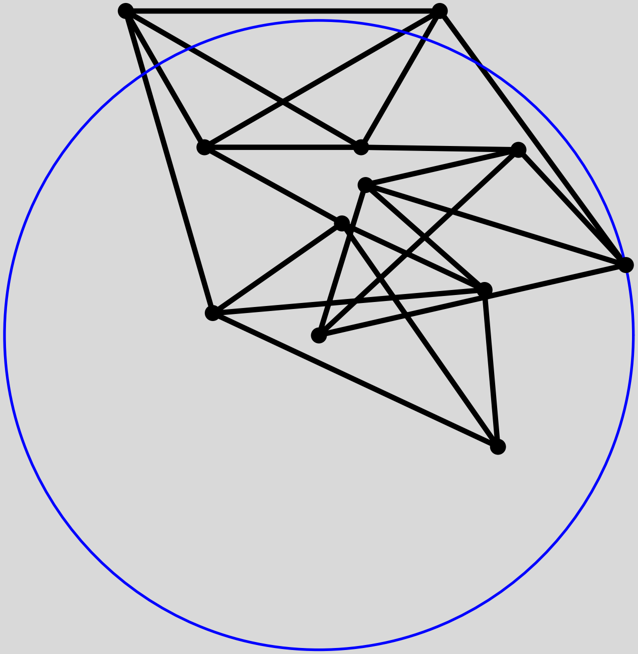


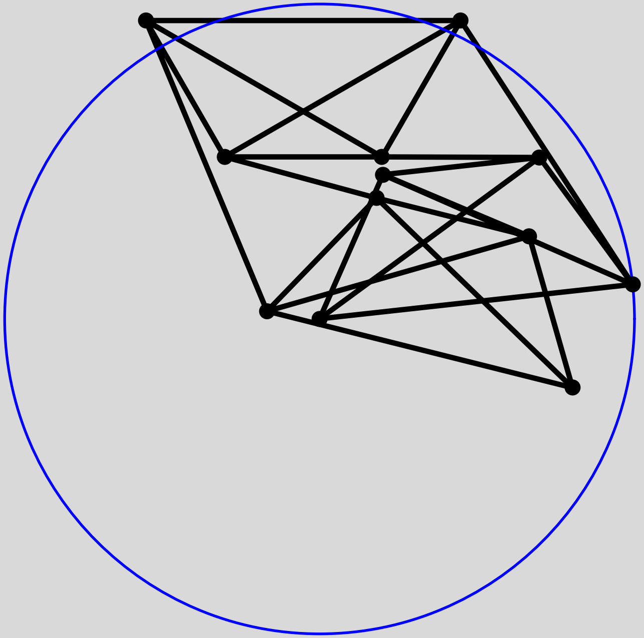


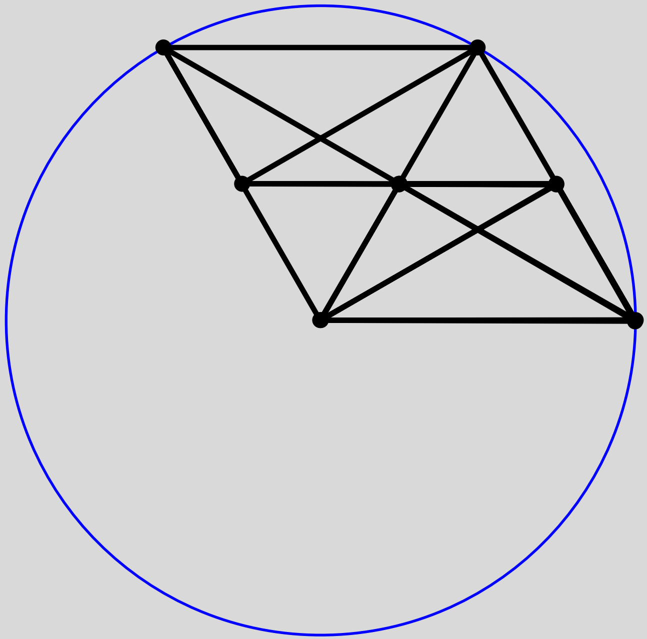












9. Random graphs

Let $G_{n,d}$ denote the probability space of all d-regular graphs on n vertices with the uniform probability distribution. A sequence of graph properties A_n holds asymptotically almost surely, or a.a.s. for short, in $G_{n,d}$ if $\lim_{n\to\infty} \Pr_{G_{n,d}}(A_n) = 1$. Graphs in $G_{n,d}$ are known to be a.a.s. highly connected. It was shown by Bollobás [2] and Wormald [19] that if $G \in G_{n,d}$ for any fixed $d \geq 3$, then G is a.a.s. d-connected. This result was extended to all $3 \leq d \leq n-4$ by Cooper et al. [3] and Krivelevich et al. [9]. Stronger results hold if we discount ‘trivial’ cutsets. In [18], Wormald shows that if $G \in G_{n,d}$ for any fixed $d \geq 3$, then G is a.a.s. cyclically $(3d-6)$-edge-connected.

Theorem 13 *If $G \in G_{n,4}$ then G is a.a.s. globally rigid.*

In fact this result holds for all $d \geq 4$.

Theorem 14 *If $G \in G_{n,d}$ and $d \geq 4$ then G is a.a.s. globally rigid.*

PROOF: If $d \geq 6$ then G is a.a.s. 6-connected by [3, 9] and the result follows from Theorem 5. If $d = 5$ then the result follows from Theorem 13 by contiguity, see [18]. □

Let $G(n,p)$ denote the probability space of all graphs on n vertices in which each edge is chosen independently with probability p. In the following we will assume that all logarithms are natural. We will need the following results on $G(n,p)$.

Lemma 1 *Let $G \in G(n,p)$, where $p = (\log n + k \log\log n + w(n))/n$, $k \geq 2$ is an integer and $\lim_{n\to\infty} w(n) = \infty$. For each fixed integer t, let S_t be the set of vertices of G of degree at most t. Then, a.a.s.*
(a) S_{k-1} is empty,
(b) no two vertices of S_t are joined by a path of length at most two in G,
(c) $G - S_{t-1}$ is non-empty and t-connected.

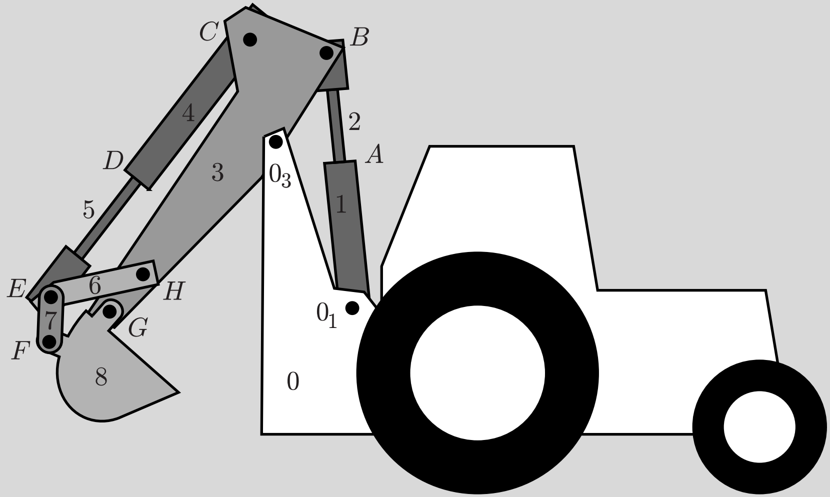
Proof: Facts (a) and (b) are well known, see for example [2]. Fact (c) follows from (a), (b) and [11, Theorem 4] □

Theorem 15 *Let $G \in G(n,p)$, where $p = (\log n + k \log\log n + w(n))/n$, and $\lim_{n\to\infty} w(n) = \infty$.*
(a) If $k = 2$ then G is a.a.s. rigid.
(b) If $k = 3$ then G is a.a.s. globally rigid.

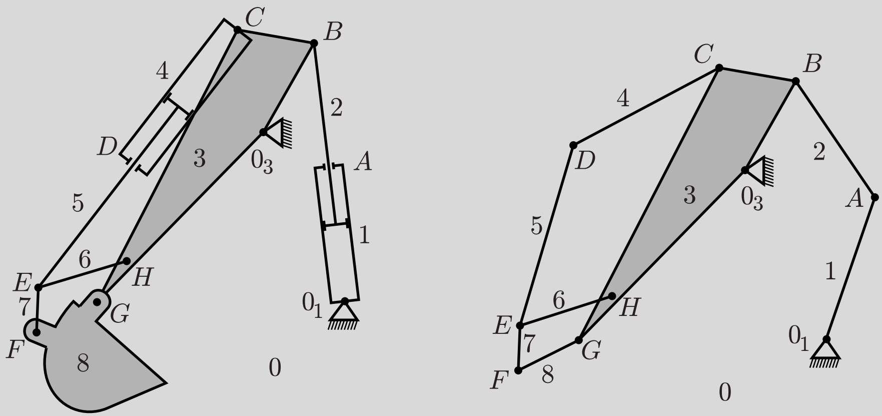
PROOF: (a) We adopt the notation of Lemma 1. It follows from Lemma 1 that a.a.s. $S_1 = \emptyset$ and $G - S_5$ is a.a.s. 6-connected. Hence $G - S_5$ is a.a.s. (globally) rigid by Theorem 5. Since adding a new vertex joined by at least two new edges to a rigid graph preserves rigidity, it follows that G is a.a.s. rigid.
(b) This follows in similar way to (a), using the facts that $S_2 = \emptyset$ and that adding a new vertex joined by at least three new edges to a globally rigid graph preserves global rigidity. □

The bounds on p given in Theorem 15 are best possible since if $G \in G(n,p)$ and $p = (\log n + k \log \log n + c)/n$ for any constant c, then G does not a.a.s. have minimum degree at least k, see [2].

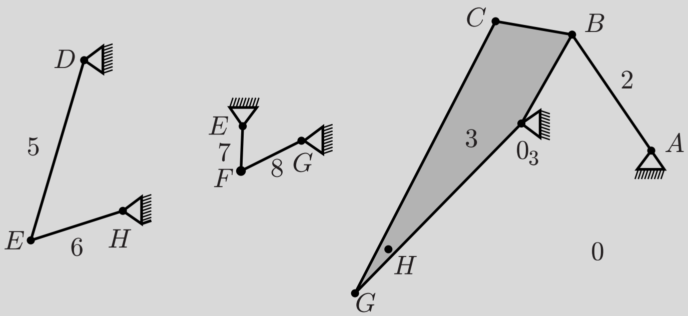
10. Assur Decomposition



The excavator with its kinematic system.



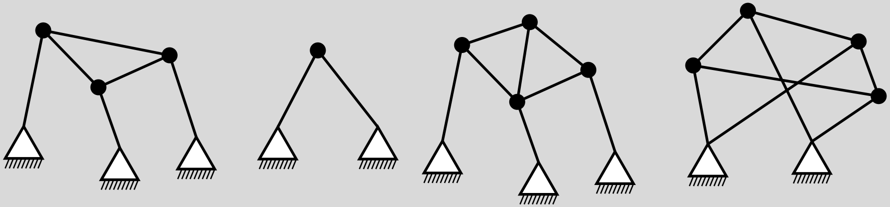
The unified structural scheme of the kinematic system of the excavator.



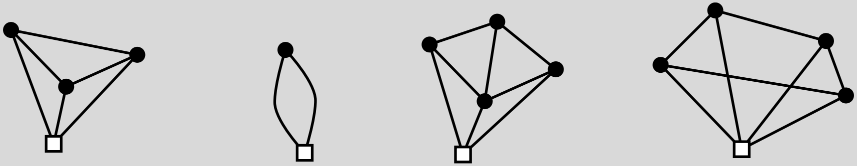
The unique decomposition into Assur groups of the structural scheme of the kinematic system of the excavator.

10.1. Basic Characterization of Assur Graphs

Theorem 16 *Assume $G = (V, P; E)$ is a pinned isostatic graph. Then the following are equivalent:*
(i) $G = (V, P; E)$ is minimal as a pinned isostatic graph: that is for all proper subsets of vertices $V' \cup P'$, $V' \cup P'$ induces a pinned subgraph $G' = (V' \cup P', E')$, $|E'| \leq 2|V' \cup P'| - 1$.
(ii) If the set P is contracted to a single vertex $p*$, inducing the unpinned graph $G*$ with edge set E, then $G*$ is a rigidity circuit.
(iii) Either the graph has a single free vertex of degree 2 or each time we delete a vertex, the resulting pinned graph has a motion of all free vertices (in generic position).



Assur Graphs



Corresponding circuit for Assur Graphs

10.2. Singular realizations of Assur Graphs

Theorem 17 *If we have a planar Assur Graph (minimal isostatic pinned framework: - property (i)) then*

1. *we have realizations* $\mathbf{p}$*, such that there is a single self stress, and this self-stress is non-zero on all edges.*

2. *at that configuration* $\mathbf{p}$ *there is a unique (up to scalar) non-trivial first-order motion and this is non-zero on all inner vertices.*

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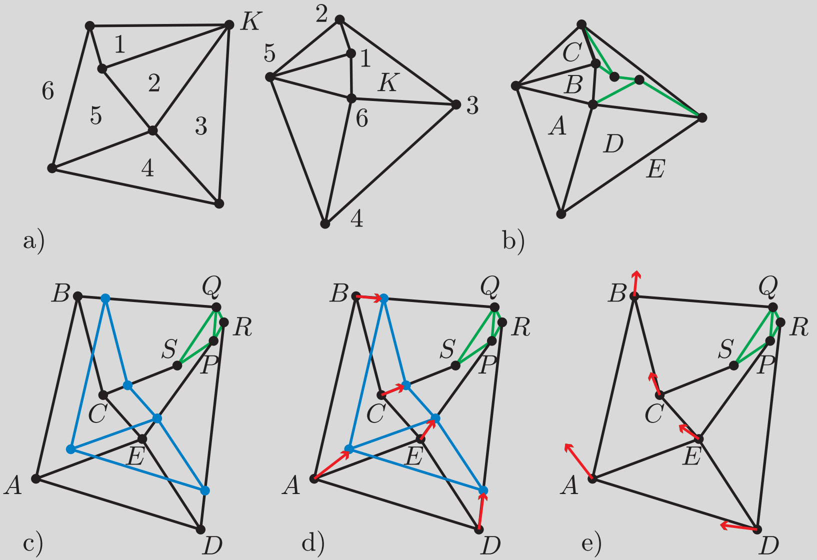

The sequence of steps for producing the configuration for a planar Assur Graph which has both a non-zero self-stress and a non-zero motion: (a) take a generic realization of the underlying circuit and form its reciprocal; (b) Split the reciprocal face K in order to separate the ground vertex into distince points, still with a self-stress; (c) use the additional self-stress to form a parallel drawing; (d) use the parallel drawing to create difference vectors, and (e) turn these difference vectors to create the first-order motion which is non-zero on all inner vertices.

Theorem 18 *If we have an arbitrary Assur Graph then*

1. *we have realizations* p, *such that there is a single self stress, and this self-stress is non-zero on all edges.*
2. *at that configuration* p *there is a unique (up to scalar) non-trivial first-order motion and this is non-zero on all inner vertices.*

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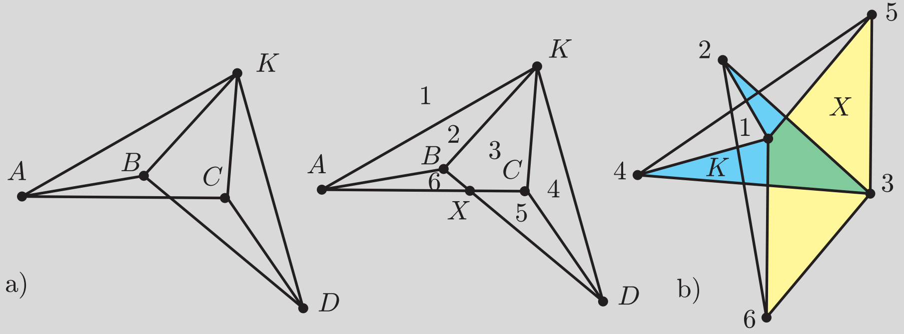

Given a non-planar graph (or drawing) (A))we can insert crossing points to create a planar graph (Bow's Notation). Working on this planar graph we have a reciprocal (B) which also is a non-planar drawing of the planar reciprocal (C).

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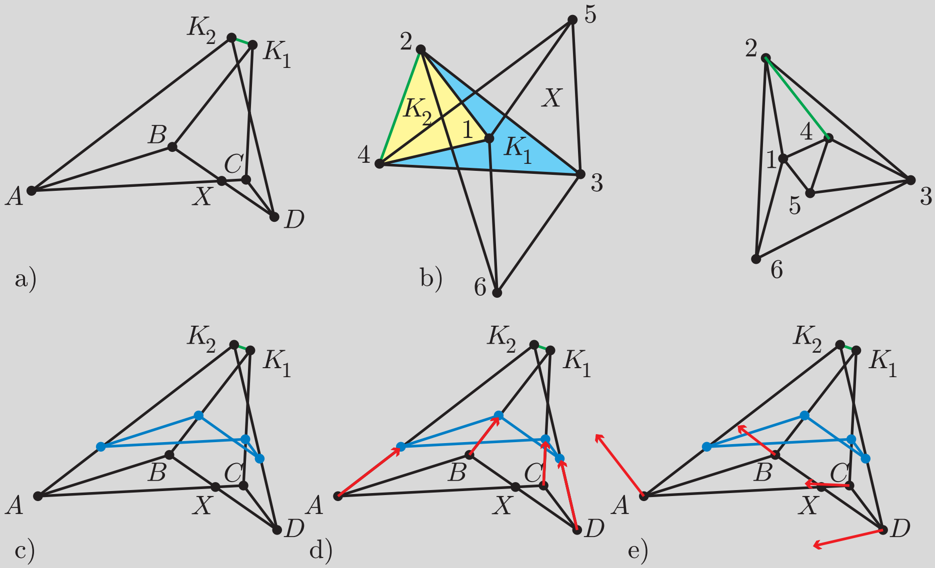
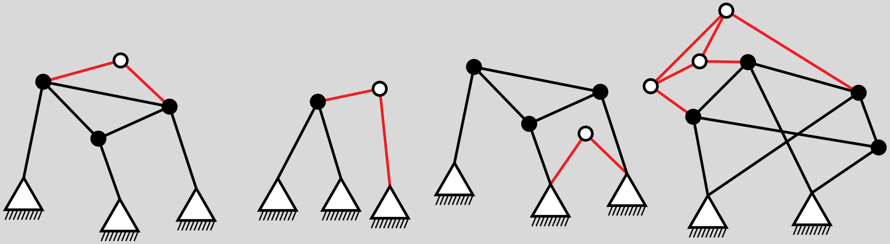
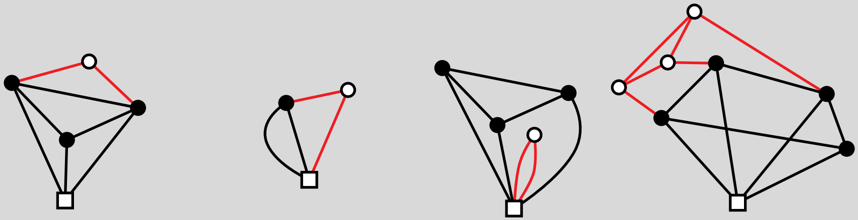


Given the reciprocal pair, we can again split the face K (b) and split the ground vertex in the original. This configuration of the Bowed graph has a non-zero stress, as does the non-planar original. The non-trivial parallel drawing of the Bowed graph is a non-trivial parallel drawing of the non-planar original (c) and induces the required first-order motion on the non-planar original (d,e).

10.3. Decomposition of one-degree of freedom linkages



Decomposible - not Assur Graphs



Decomposition - with identified subcircuit(s).

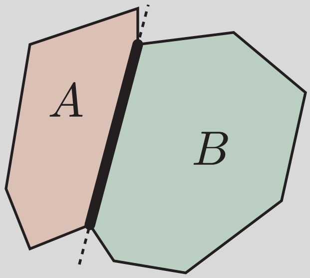
11. Molecular conjectures

3-space: Generic Body and Hinge Frameworks

Solved

Alternate generalization to 3-space:

Two bodies, joined on linear hinge
6 degrees for each body.
Each hinge removes 5 degrees of freedom



Graph $G = (C, H)$
C: vertices for abstract bodies,
H: for pairs which share a hinge.
Necessary count for independence becomes: $5|H| \leq 6|C| - 6$

Theorem 19 (Tay and Whiteley (84)) *Also sufficient for generic independence with hinges, with*

$$5|H| \leq 6|C| - 6$$

Algorithms:

$$6|C| - 6 = 6(|C| - 1)$$

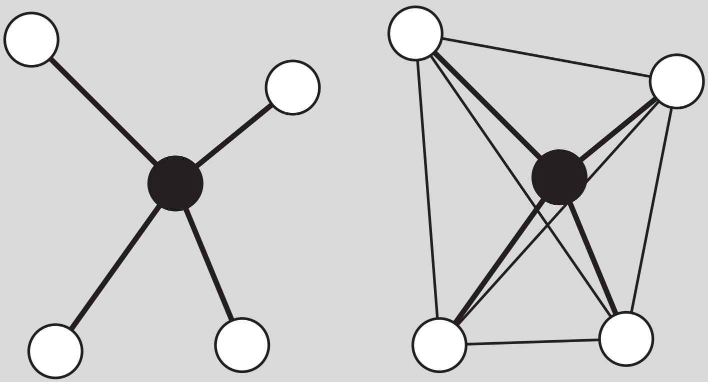
or

6 spanning trees if replace 'hinge edge' by five edges for multi-graph.

11.1. Modeling molecules

(special graphs) - can we predict rigidity?

Single atom and associated bonds

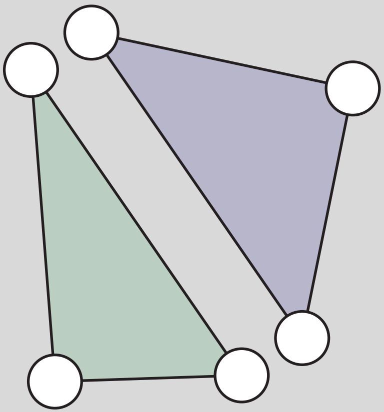


$$|V| = 5 \qquad |E| = 10$$

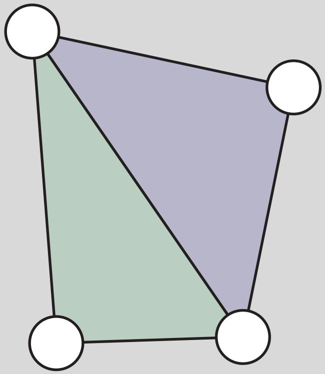
$$|E| = 3|V| - 5 \qquad \text{overbraced}$$

Adjacent atom clusters

Flexible



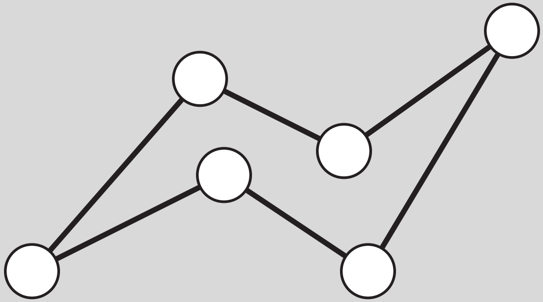
$$|C| = 2, |H| = 1,$$
$$|V| = 4, |E| = 5$$



$$5|H| = 6|C| - 7,$$
$$|E| = 3|V| - 7$$

Rings of atoms:

Ring of 6 atoms and bonds



Body and hinge: $|B| = 6$, $|H| = 6$, $5|H| = 6|B| - 6$
Just the right number to be rigid - generically.

Graph G of atoms and covalent bonds
Body and hinge model
Atoms are bodies
bonds are hinges
count as body and hinge structure
Problem: Special geometry with hinges concurrent
Special geometry may lower rank!

Graph G of atoms and covalent bonds
Form G^2
atoms are vertices
bonds are edges
second neighbor bond bending pairs are edges count as

$$3|V| - 6$$

priority system on bond edges.
Problem: for general graphs G the rank may be lower. (May work for G^2?)

Lots of experimental evidence;
Proofs of correctness for special classes of graphs
Plausibility arguments related to other conjectures on 3-space rigidity
Sketched proof of equivalence of the two conjectures.
Conjectures embedded in implemented algorithms: FIRST on the web (Arizona State University)
Seek additional graph models for applications biochemical constraints:
Apply to other problems in biochemistry, chemistry

11.2. Molecular Conjecture in the Plane

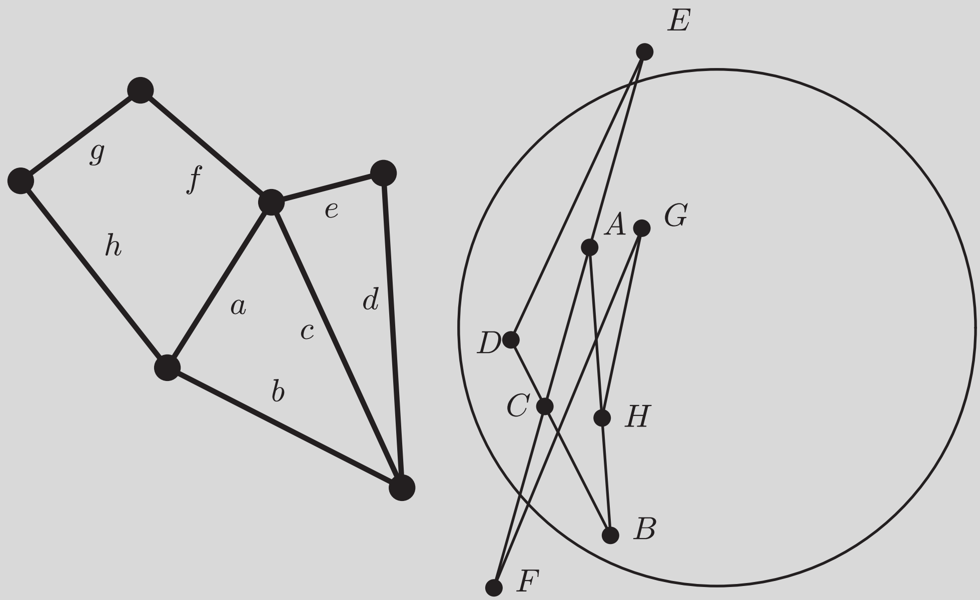
Given: Simple graph $G = (V, E)$.

- Regard as a body and pin graph of a structure in the plane:
  Vertices are bodies.
  Edges denote pins.
- Note: Each pin connects just two bodies. Otherwise we would need a hypergraph.
- Realizations:
  - Amorphous bodies. Embedding specifies the location of the pins.
  - Line bodies. Embedding may specify either lines or pins.
  - Question: Does the line realization always exist?

11.3. Realization in the Plane.

Theorem 20 *If $G = (V, E)$ is simple, then a pin collinear structure exists.*

Take any generic embedding of the structure graph $G = (V, E)$ in $\mathbb{R}^2$. Form the polar of that embedding.

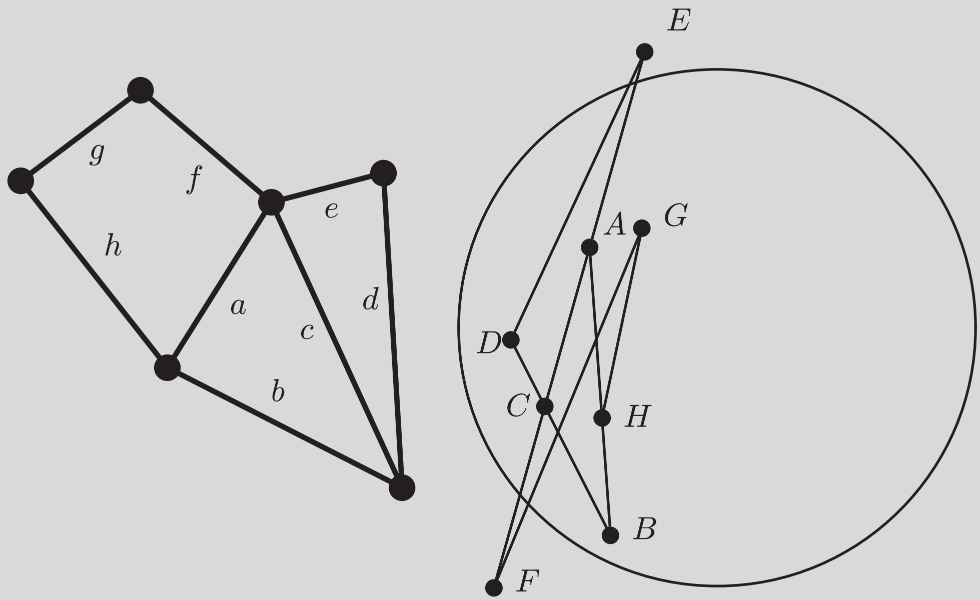


Question:

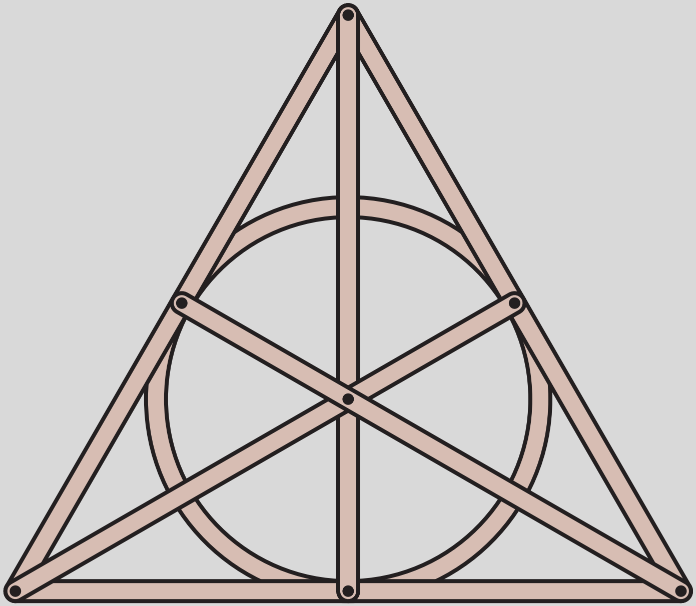
Is the polar generic as a line pin structure?

Question:

Is it generic as a body pin structure?



Note: A general pin body structure may have no pin collinear realization:



The incidence structure is a hypergraph.

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Pin-collinear Body-and-Pin Frameworks and the Molecular Conjecture

Bill Jackson, Tibor Jordàn

Abstract

T-S. Tay and W. Whiteley independently characterized the multigraphs which can be realized as an infinitesimally rigid d-dimensional body-and-hinge framework. In 1984 they jointly conjectured that each graph in this family can be realized as an infinitesimally rigid framework with the additional property that the hinges incident to each body lie in a common hyperplane. This conjecture has become known as the Molecular Conjecture because of its implication for the rigidity of molecules in 3-dimensional space. Whiteley gave a partial solution for the 2-dimensional form of the conjecture in 1989 by showing that it holds for multigraphs $G = (V, E)$ in the family which have the minimum number of edges, i.e. satisfy $2|E| = 3|V| - 3$. In this paper, we give a complete solution for the 2-dimensional version of the Molecular Conjecture. Our proof relies on a new formula for the maximum rank of a pin-collinear body-and-pin realization of a multigraph as a 2-dimensional bar-and-joint framework.

Theorem 21 *A multigraph G can be realized as an infinitesimally rigid body and hinge framework in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)G$ has $\binom{d+1}{2}$ edge-disjoint spanning trees. (Tay and Whiteley, 1984)*

Recent Advances in the Generic Rigidity of Structures, Tiong-Seng Tay and Walter Whiteley Structural Topology # 9, 1984
Many body and hinge structures are built under additional constraints. For example in architecture flat panels may be used in which all hinges are coplanar. In molecular chemistry, we can model molecules by rigid atoms hinged along the bond lines so that all hinges to an atom are concurrent. This is the natural projective dual for the architectural condition.
Conjecture: A multigraph is generically rigid for hinged structures in n-space iff it is generically rigid for hinged structures in n-space with all hinges of body v_i in a hyperplane H_i of the space.

Remarks: For the plane (n=2 with all pins along a line), this conjecture was made in 1979 but remains unsolved. With the recent breakthrough for real structures in 3-space the problem becomes more important.

Only in 3-space does projective duality convert a hinge structure to a new hinge structure.

Jackson-Jordan prove first:

Theorem 22 *Let $G(V, P)$ be a graph with no isolated vertices. Then the maximum rank of a pin-collinear body and pin realization of G as a bar and joint framework is $2(|V| + |P|) - 3 - \mathit{def}(G)$.*

A pin-collinear body and pin realization of $G(V, P)$ is the square of a subdivision of G.
The deficiency of $G(V, P)$ is $3V - 3 - r_2(G)$, where r_2 is the rank in the associated 2-polymatroid of G.

2-polymatroid associated to $G(V,P)$ with a body and pin realization G^* embedded in $\mathbb{R}^2$: An infinitesimal motion of G is a map $S : V \to \mathbb{R}^3$ satisfying the constraints that for all $p = uv \in P$ we have $S(u) - S(v) = \langle (x(e), -y(e), 1) \rangle$.
The set of infinitesimal motions is the nullspace of a $2|P| \times 3|V|$ matrix.

Jackson and Jordan show further, that a pin-line generic pin-collinear body-and-pin realization of G has maximal rank over all pin-collinear body and pin realizations of G.
The main theorem is then:

Theorem 23 *Let $G(V,P)$ be a graph with no isolated vertices. Then the maximum rank of a pin-collinear body and pin realization of G as a bar and joint framework is $2(|V|+|P|)-3-\mathit{def}(G)$.*

Finally Jackson and Jordan show that the body-and-pin and rod-and-pin 2-polymatroids of a graph are identical. As a solution to the molecular conjecture they formulate

Theorem 24 *Let $G(V,E)$ be a multigraph. Then the following statements are equivalent:*
(a) G has a realization as an infinitesimally rigid body and hinge framework in $\mathbb{R}^2$.
(b) G has a realization as an infinitesimally rigid body-and-hinge framework (G,q) in $\mathbb{R}^2$ with each of the sets of points $\{q(e) : e \in E_G(v)\}, v \in V$, collinear.
(c)$2G$ contains 3 edge disjoint spanning trees.

In the body-and hinge frameworks so far investigated, each hinge is shared by exactly two bodies. Can one generalize from body-pin graph to body and pin incidence structure?
J-J conjecture yes and point out that Whiteley proved this for independent structures and made a similar conjecture in 1989.

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