Combinatorial Maps, delta-matroids and Rigidity

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1. Combinatorial Maps

Tutte [8] defined maps axiomatically. If we have three fixed point free permutations τ_0, τ_2, and $\mathcal{V}$ on a set Φ of *flags* such that

A1 $\tau_0^2 = \tau_2^2 = \mathrm{Id}$

A2 $\tau_0\tau_2 = \tau_2\tau_0$

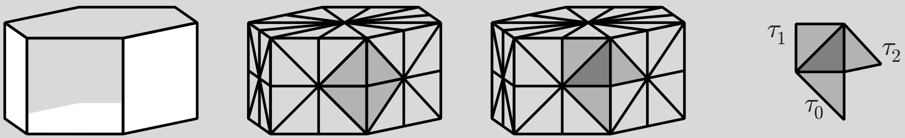
A3 $\mathcal{V}\tau_2 = \tau_2\mathcal{V}^{-1}$

A4 $\{\mathcal{V}^i\phi\} \cap \{\mathcal{V}^i\tau_2\phi\} = \emptyset$

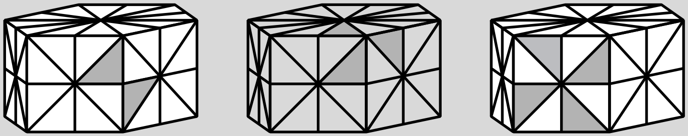
A5 τ_0, τ_2 and $\tau_0\tau_2$ are fixed point free

A6 $\langle\tau_0, \tau_2, \mathcal{V}\rangle$ acts transitively on Φ

then we can define a graph G whose vertices are the orbits of Φ under $\langle\tau_2, \mathcal{V}\rangle$ and whose edges are the orbits of Φ under $\langle\tau_0, \tau_2\rangle$. The orbits of $\langle\tau_0, \tau_2\rangle$ each have four elements and intersect either one or two orbits of $\langle\tau_2, \mathcal{V}\rangle$, defining the endpoints of the edge. $M(G, \tau_0, \tau_2, \mathcal{V})$ is a combinatorial map.



Decomposition of the hexagonal prism into flags.



Orbits under $\tau_2\tau_0$, $\tau_2\tau_1$ and $\tau_0\tau_1$.

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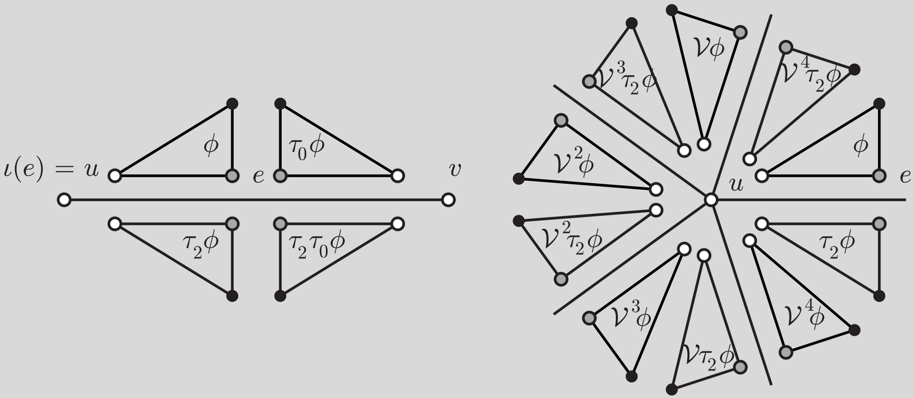

Flags at an edge, and at a vertex.

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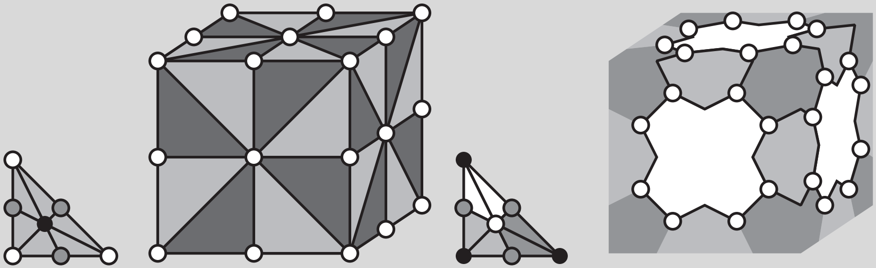
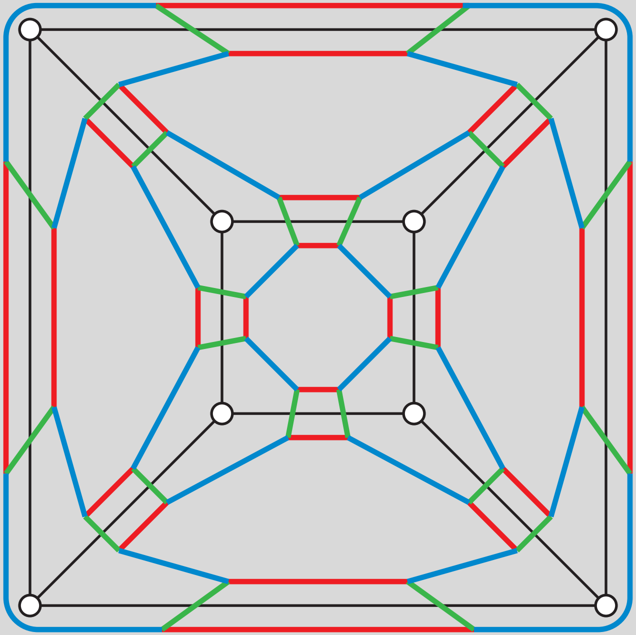
2. The flag graph



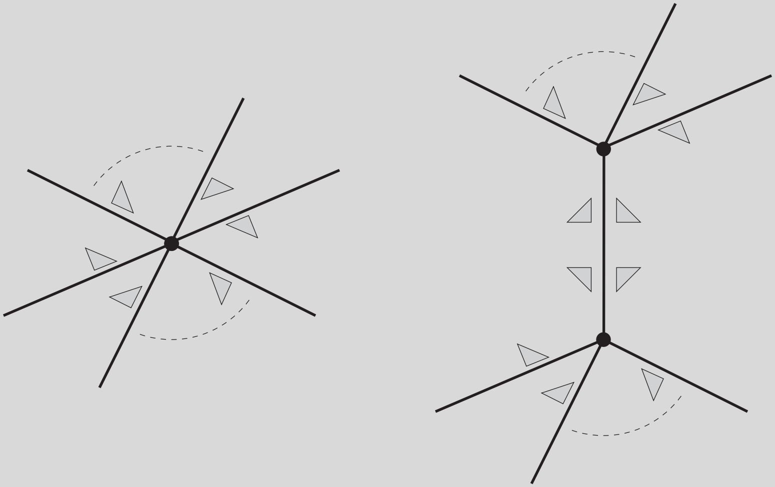
$BS(\mathcal{M})$ and $Co(\mathcal{M})$.

A map is orientable if and only if its flag graph is bipartite.



τ_0 red
τ_2 green
τ_1 blue

3. Vertex splitting and edge contraction



Vertex split/edge contraction.

Number of faces stays the same. Contraction of a loop is undefined.

4. Reduction to unitary maps

A map is called *unitary* if it has exactly one vertex and one face. By a sequence of edge contractions one can reduce the number of vertices of a map M, or the number of faces of M^*.

The flags $\{x, \tau_0 x, \tau_2 x, \mathcal{E}x\}$ of a unitary map are distributed among the two disjoint cycles of $\mathcal{V}$ by

$$(x, A, \mathcal{E}x, B)(\tau_2 B, \tau_0 x, \tau_2 A, \tau_2 x)$$

or

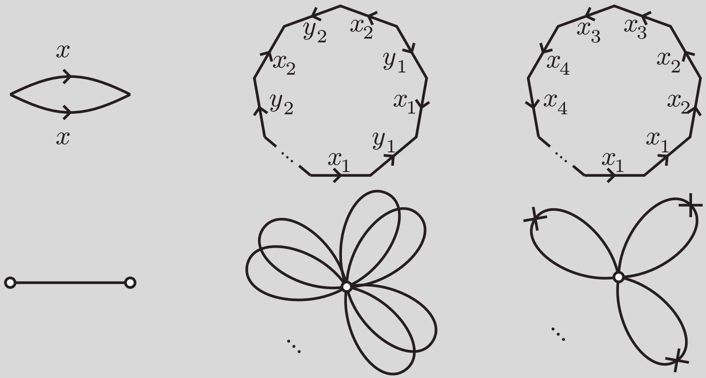
$$(x, A, \tau_0 x, B)(\tau_2 B, \mathcal{E}x, \tau_2 A, \tau_2 x)$$

Crosscap

A crosscap is assembled if A or B is empty. Crosscaps may be assembled by a sequence of vertex splits and edge contractions.

5. Classification of Surfaces

5.1. Normal forms



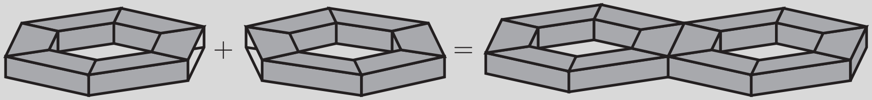
Map projection and polygon models of canonical normal forms.

Theorem 1 *Every closed surface has the topological type of either*

1. *The sphere.* $(\chi(S) = 2)$.
2. *A connected sum of n tori.* $(\chi(T^n) = 2(n-1))$.
3. *A connected sum of n projective planes.* $(\chi(P^n) = 2-n)$.

6. Joining two maps

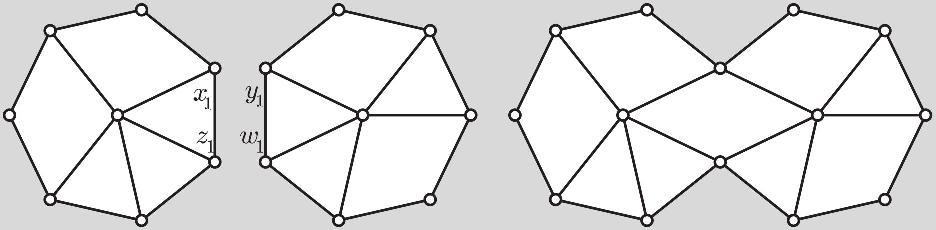
6.1. Connected sum



The connected sum of two tori.

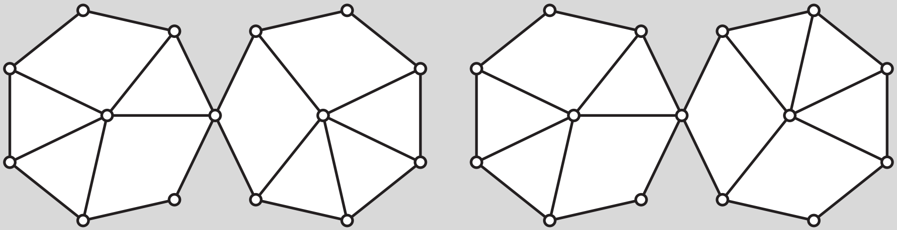
$$\chi(S\sharp S') = \chi(S) + \chi(S') - 2.$$

6.2. Edge join

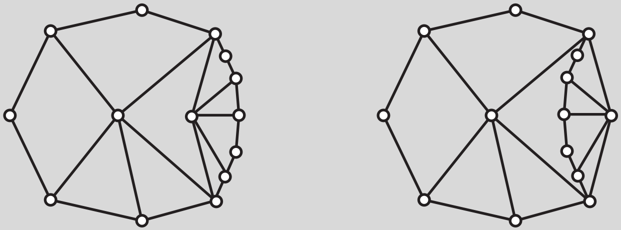


$\mathcal{M}_1 +_{x_1,y_1} \mathcal{M}_2$, joining two maps along an edge.

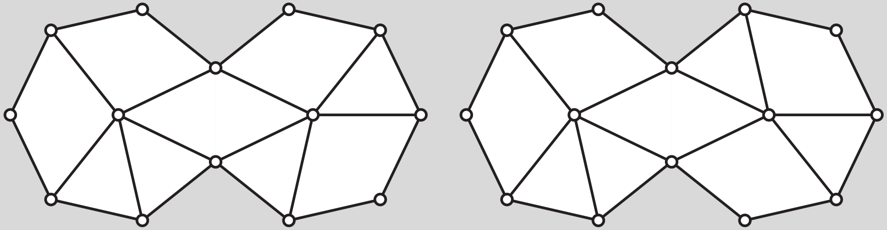
6.3. Vertex join



Two ways to join two maps at a vertex. Observe the sequence of vertex valences around the exterior face.



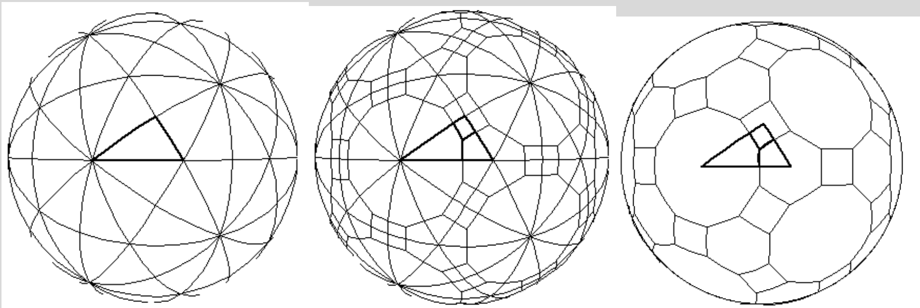
A Whitney twist: $\mathcal{M}_1 +_{x_1, \tau_2 y_1} \mathcal{M}_2$



A Whitney flip: $\mathcal{M}_1 +_{x_1,\tau_0 y_1} \mathcal{M}_2$.

7. Regular maps on the sphere

If M is a regular map, then the flag graph is a Cayley graph of $Aut(M)$ where the generating set S has three elements, namely the three group elements which map a given flag ϕ to $\tau_0\phi$, $\tau_1\phi$, and $\tau_2\phi$.



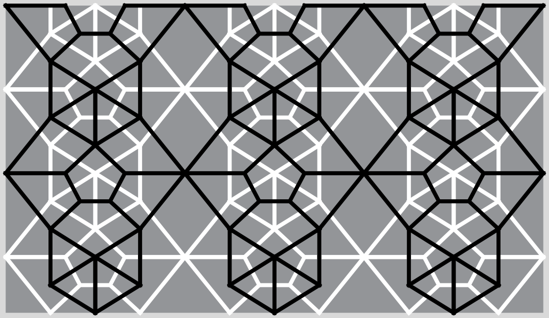
Constructing a Cayley map for the icosahedron.

7.1. Straightening Lemma

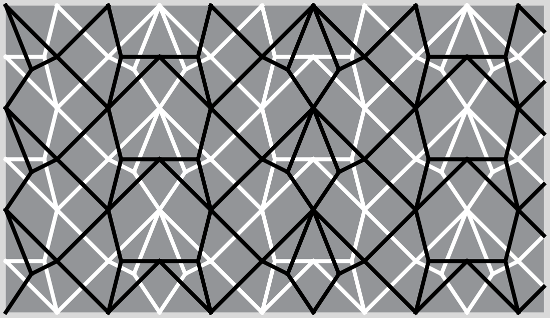
A complex $\mathcal{M}$ is *harmonious* if $Aut\mathcal{M} = \mathrm{Isom}(\mathcal{M})$. In [6] it was shown that every planar combinatorial map can be straightened to a harmonious map.

Theorem 2 *Let $\mathcal{M}$ be a complex of polygons piecewise linearly embedded on the sphere S^2, the Euclidean plane $\mathbb{R}^2$, or the hyperbolic plane $\mathbb{H}^2$, and $\mathcal{M}$ is the associated map. If $Aut\,\mathcal{M}$ has finitely many face orbits, then there is a complex Ideal($\mathcal{M}$) of $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ which is combinatorially isomorphic to $\mathcal{M}$ and such that every automorphism of $\mathcal{M}$ is expressed as an isometry of Ideal($\mathcal{M}$), so Isom(Ideal($\mathcal{M}$)) $= Aut(\mathcal{M})$.*

Theorem 3 *Every self-dual map $\mathcal{M}$ with finitely many face orbits is the map of a harmoniously self-dual complex.*



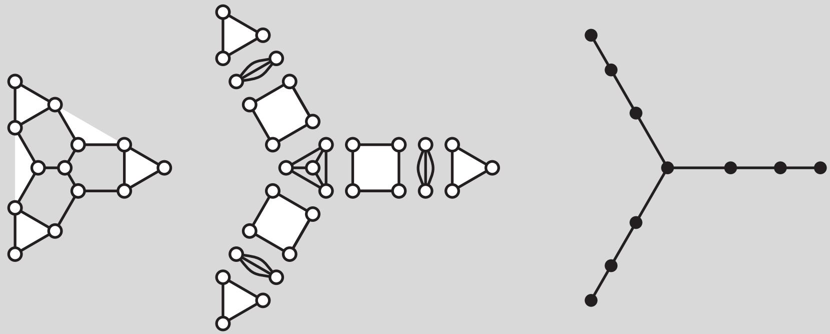
$(pm, pm)a.$



$(pm, pm)b.$

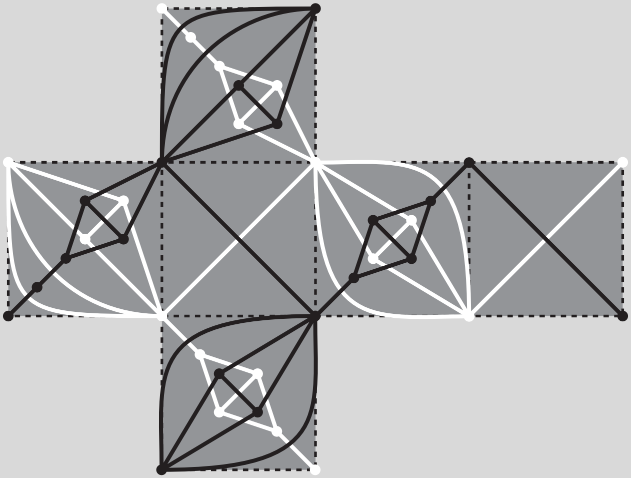
8. The 3-block tree

For planar graphs which are 3-connected, Whitney's theorem [9] says that all its planar or spherical maps are combinatorially isomorphic and therefore, by the straightening lemma, can be studied with geometric methods alone.
For graphs of lower connectivity, faces are not uniquely defined.



The 3-block tree.

For a planar graph G, we may now analyze the automorphisms of its 3-blocks via geometric methods, and then piece these symmetries together combinatorially via its structure trees. This program has been followed for planar Cayley graphs [3] and self-dual graphs and matroids, see [6] and [7], where the construction of all self-dual graphs are described, even those which have no associated self-dual map to be straightened.



A self-dual graph drawn on an unfolded cube with no corresponding self-dual map.

A refinement of the 24 symmetry pairings of maps on the sphere was obtained by Graver and Hartung in [5] by considering grid patches.

9. Matroids

Whitney [10] defined a matroid on a finite set E to be a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a collection of subsets of E such that

I1 $\emptyset \in \mathcal{I}$;

I2 If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$

I3 If I_1 and I_2 are members of $\mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element e in $I_2 - I_1$ such that $I_1 + e$ is a member of $\mathcal{I}$.

Because of condition [I2], all of the maximal independent sets have the same cardinality. These maximal independent sets are called the *bases* of the matroid. The bases may be described directly: Let E be a finite set, a nonempty collection $\mathcal{B}$ of subsets of E is called a *basis system* for E if

B1 $\mathcal{B} \neq \emptyset$

B2 For all $B_1, B_2 \in \mathcal{B}$, $|B_1| = |B_2|$

B3 For all $B_1, B_2 \in \mathcal{B}$ and $e_1 \in B_1 - B_2$, there exists $e_2 \in B_2 - B_1$ such that $B_1 - e_1 + e_2 \in \mathcal{B}$.

Condition [B3] is sometimes called the *exchange axiom*. It also has a slightly different but equivalent formulation:

B3' For all $B_1, B_2 \in \mathcal{B}$ and $e_2 \in B_2 - B_1$, there exists $e_1 \in B_1 - B_2$ such that $B_1 - e_1 + e_2 \in \mathcal{B}$.

Complements of bases also satisfy [B3], these complements are bases of the dual matroid. Every matroid M has a dual M^*.

Let M be a matroid on E with independent sets $\mathcal{I}$ and define $r_{(\mathcal{I})}$, a function from the power set of E into the nonnegative integers by $r_{(\mathcal{I})}(S) = \max\{|I| : I \in \mathcal{I}, I \subseteq S\}$. The function $r = r_{\mathcal{I}}$ is called the *rank function of M*.

In general, let E be a finite set and r a function from the power set of E into the nonnegative integers so that

R1 $r(\emptyset) = 0$;

R2 $r(S) \leq |S|$;

R3 if $S \subseteq T$ then $r(S) \leq r(T)$;

R4 $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$;

then r is called a *rank function on E*. If r is a rank function on E we define $\mathcal{I}(r) = \{I \subseteq E \mid r(I) = |I|\}$

Condition [R4] is called the *submodular inequality*.

Given a finite set E, we call a collection $\mathcal{C}$ a *cycle system* for E, if the following three conditions are satisfied:

Z1 If $C \in \mathcal{C}$ then $C \neq \emptyset$

Z2 If C_1 and C_2 are members of $\mathcal{C}$ then $C_1 \not\subseteq C_2$

Z3 If C_1 and C_2 are members of $\mathcal{C}$ and if e' is an element of $C_1 \cap C_2$, then for each $e \in (C_2 - C_1)$ there is an element $C \in \mathcal{C}$, such that $e \in C \subseteq (C_1 \cap C_2 - e')$.

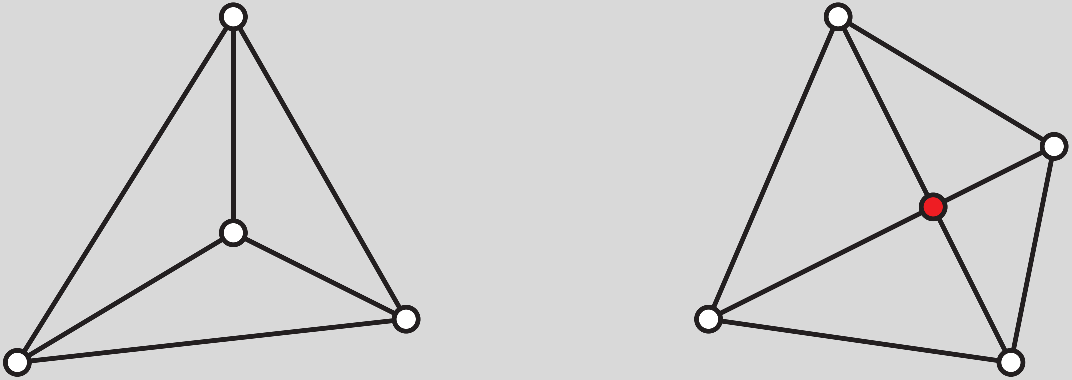
A matroid is *graphic* if it is isomorphic to the cycle matroid on a graph G. Non-isomorphic graphs may have the same cycle matroid, but 3-connected graphs are uniquely determined by their matroids.

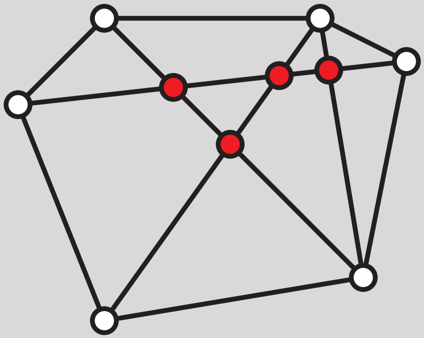
M is co-graphic if M^* is graphic.

M is graphic as well as co-graphic iff G is planar. Map duality (geometric duality) agrees with matroid duality. If $G(V, E)$ is planar and connected, its cycle matroid has rank $|V| - 1$, its co-cycle matroid has rank $|F| - 1$, so $|V| - 1 + |F| - 1 = |E|$. The facial cycles generate the cycle space.

Miracle in the Plane

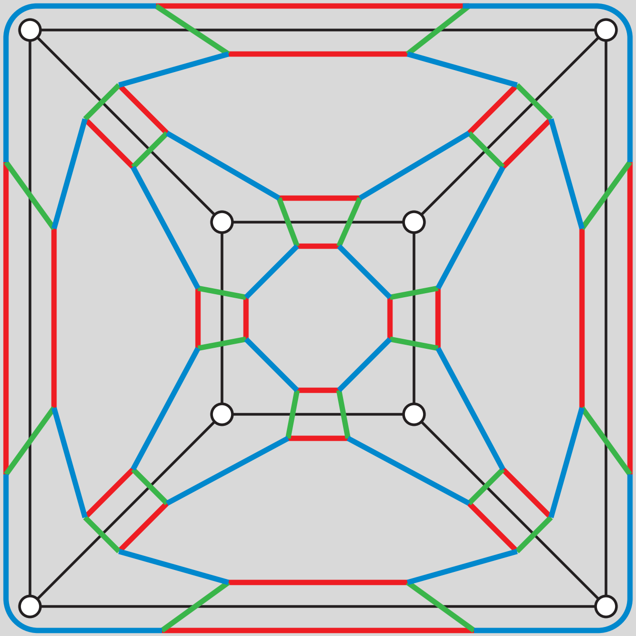
Planar rigidity cycles dualize into rigidity cycles.



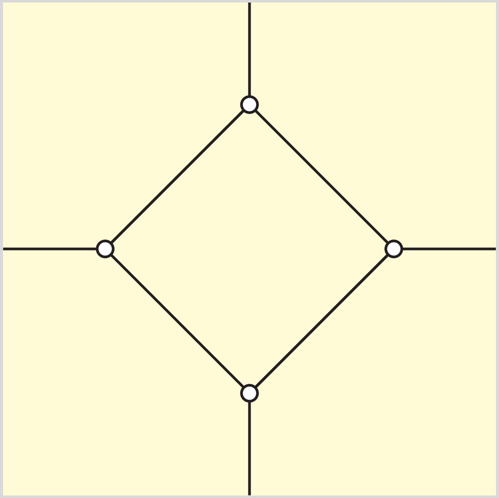


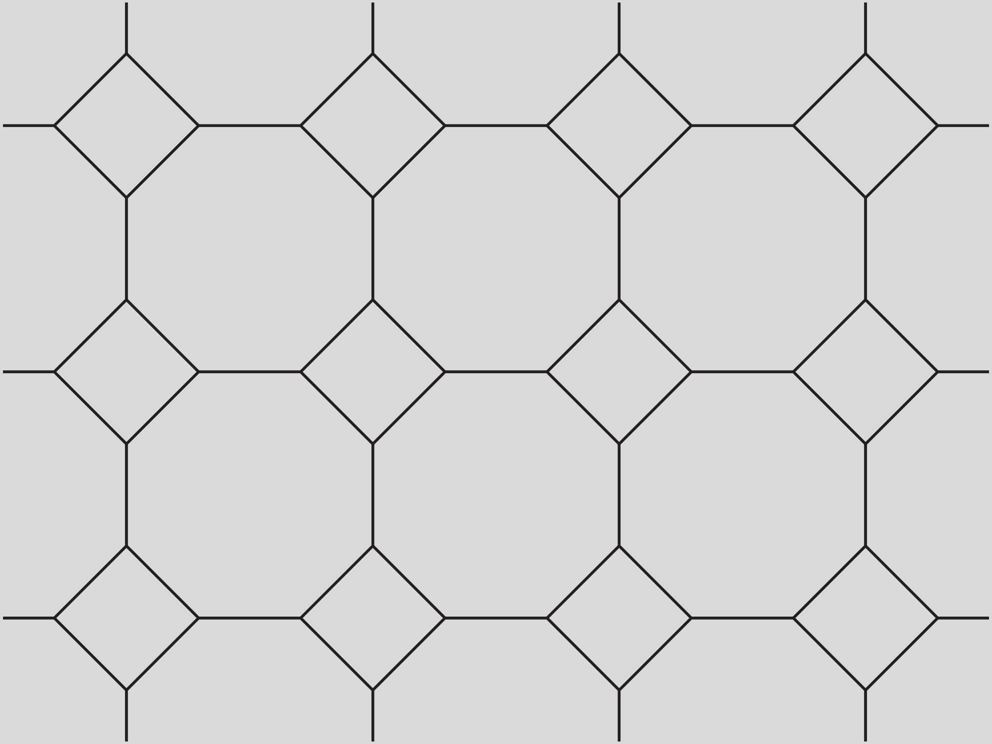
10. Δ-matroids

Introduced by Bouchet as set systems satisfying the symmetric exchange axiom
For $F', F'' \in \mathcal{F}$, $x \in F' \Delta F''$, there exists $y \in F'' \Delta F'$ such that $F' \Delta \{x, y\} \in \mathcal{F}$.
Feasible sets need not be equicardinal.
In [2] Bouchet associates a Δ-matroid to a map on a topological surface S by defining edge sets F feasible if $S - cl(F \cup \overline{F}^*)$ is connected. This easily translates to connectivity properties of the flag graph.

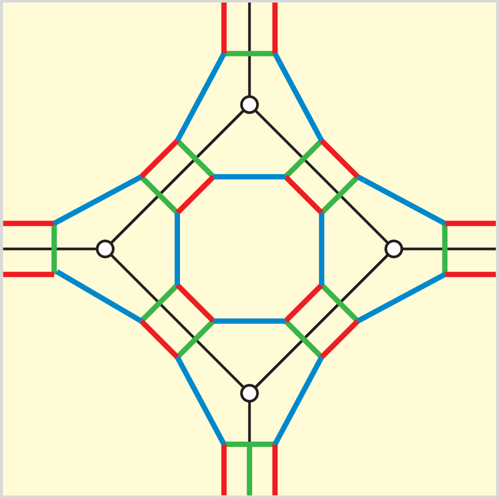


For example consider K_4 embedded on a torus.





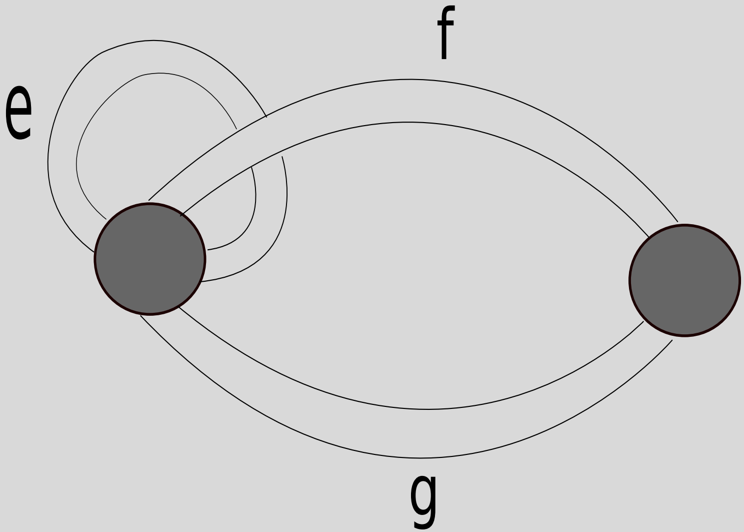
From each red-green square both edges of one color must be deleted without destroying connectivity of the flag graph.



Theorem 4 *Let $D(M)$ be the Δ-matroid of a map $M(G, S)$ and let $M^*(G^*, S)$ be the dual map. Then*

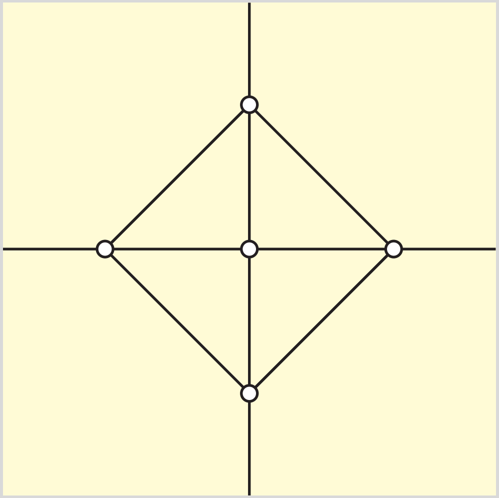
- $D(M^*) = D(M)^*$;
- *the lower matroid of D is the cycle matroid of G;*
- *the upper matroid of D is the co-cycle matroid of G^*;*
- $w(D) = 2 - \chi(S)$.

Note that there are connections to 2-matroids and ribbon graphs. A good reference for ribbon graphs is [4], see also [1].



A ribbon graph.

K_5 on the torus



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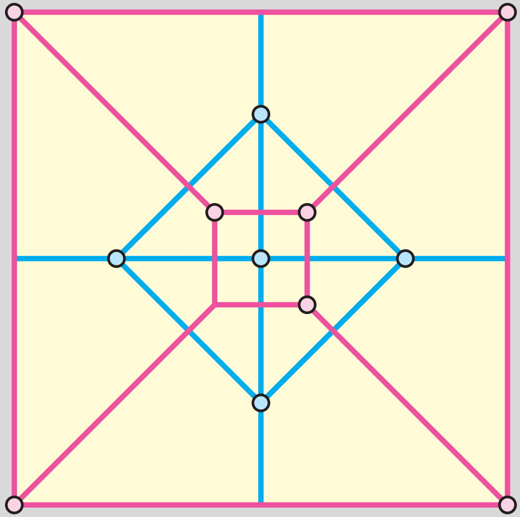

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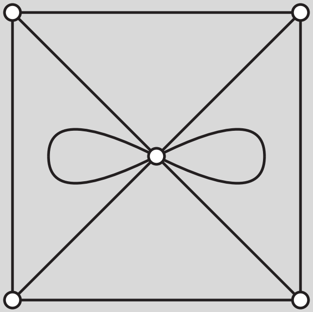
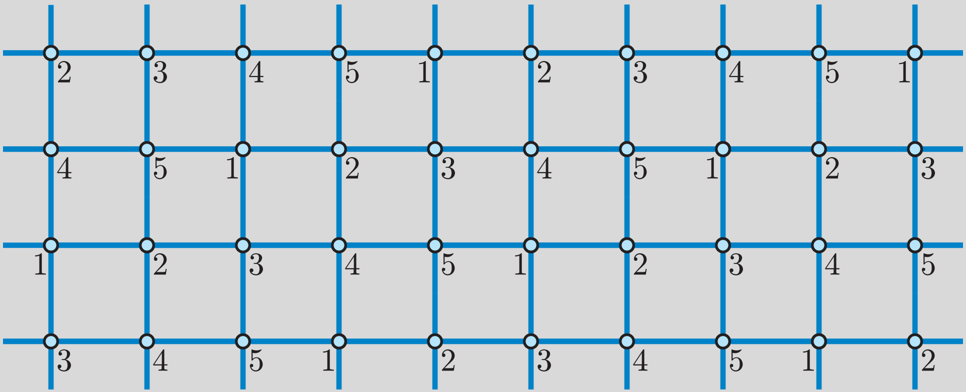
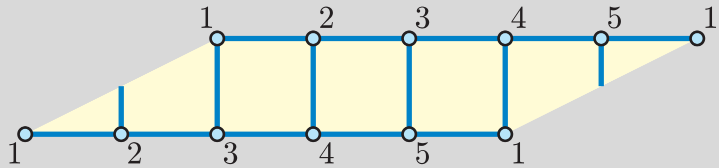


Another way to embed K_5 on the torus

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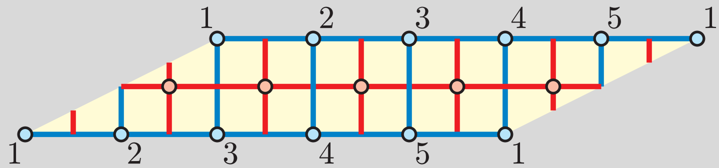
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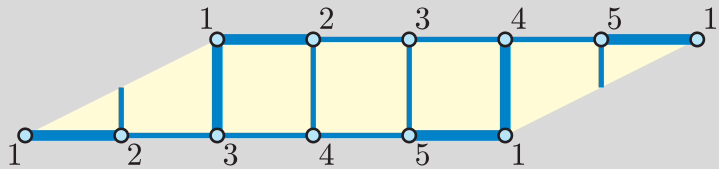
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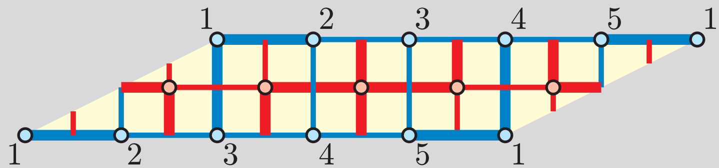
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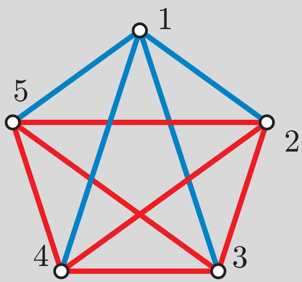
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