Pseudotriangulations and Rigidity

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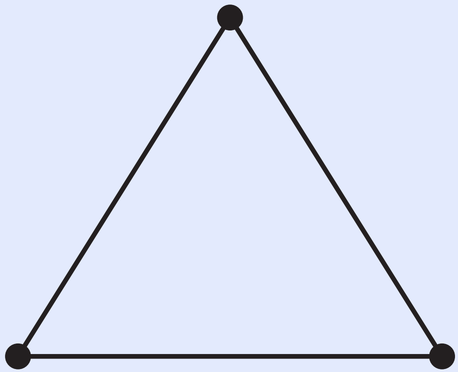

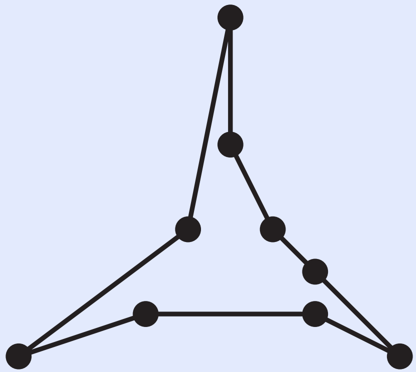

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- Herman Servatius
- Ileana Streinu
- Walter Whiteley

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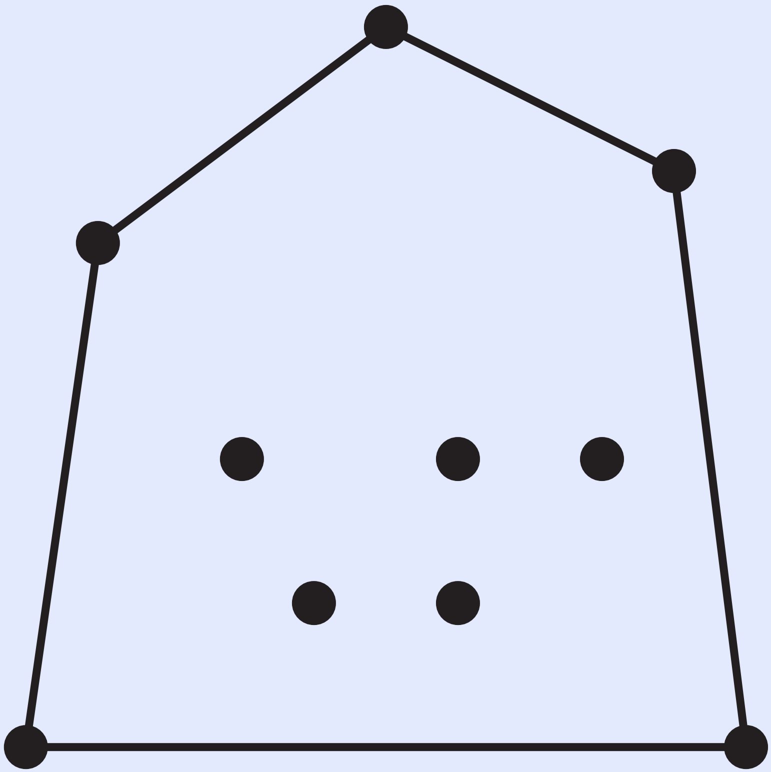
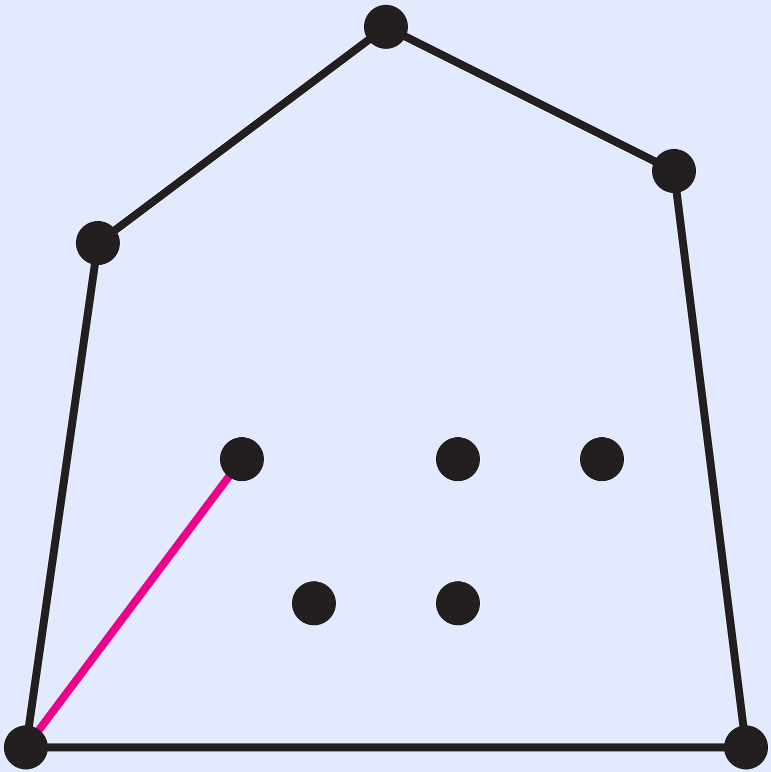
1. Pseudo-Triangulating

Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add edges. . .

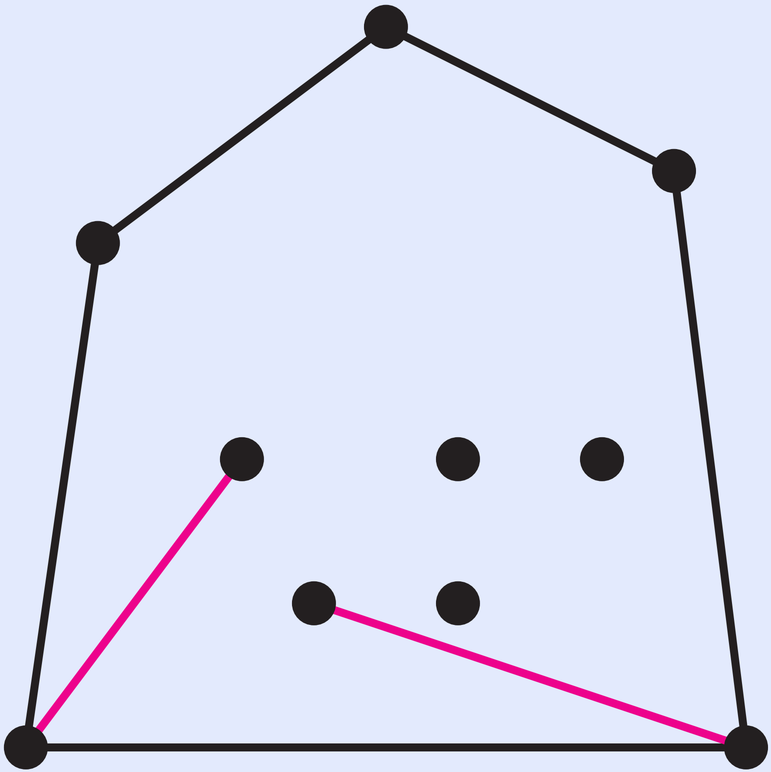
Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add one edge

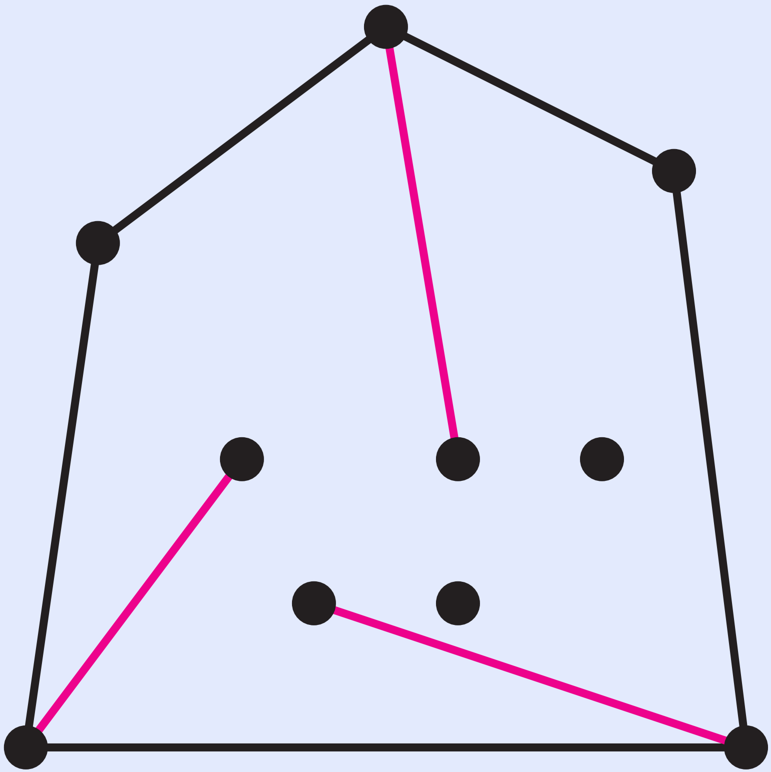
Start with a point set. . . form the convex hull

10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add two edges

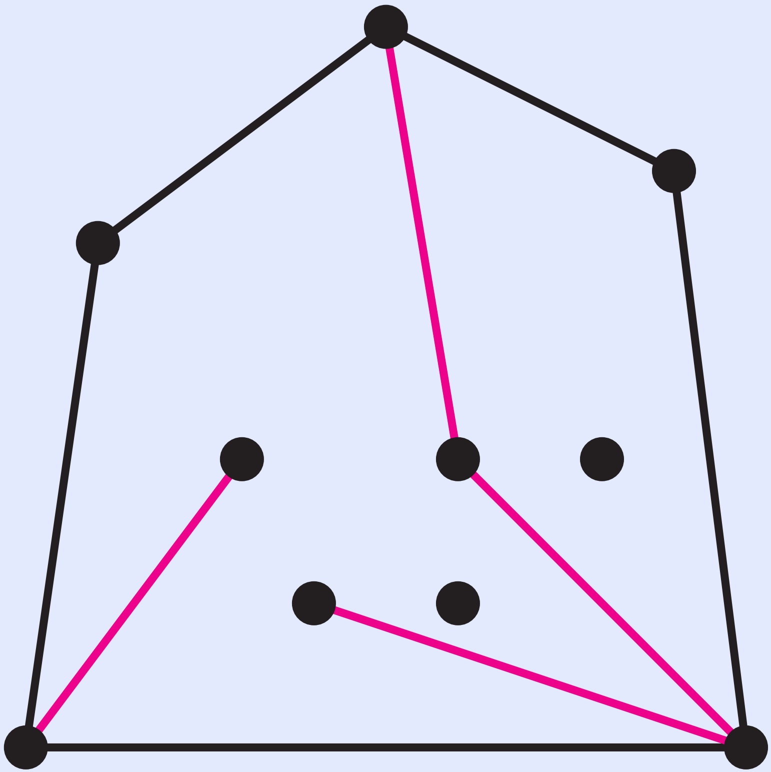
Start with a point set...form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add three edges

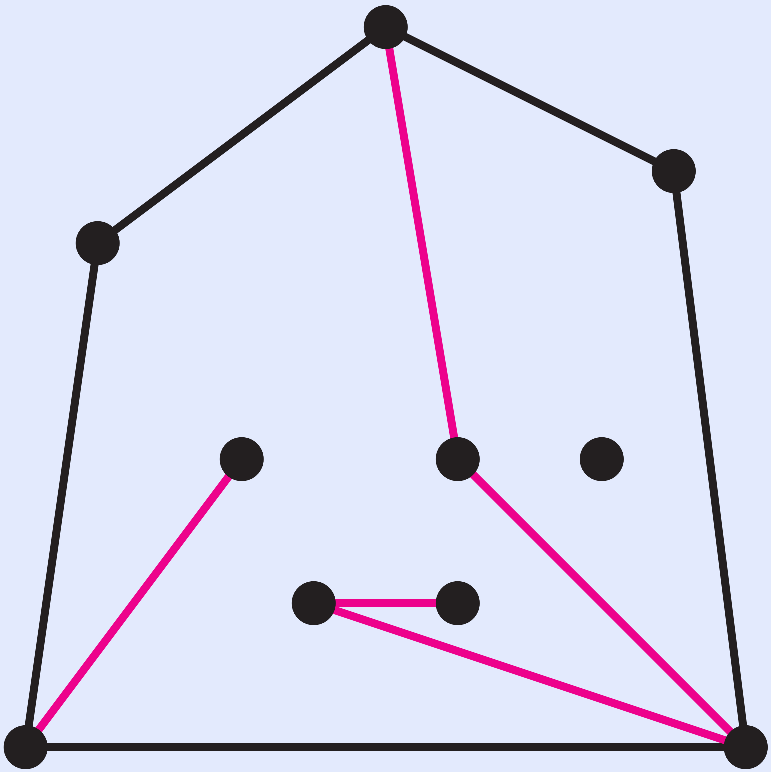
Pseudo-Triangulating

Start with a point set...form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add four edges

Start with a point set…form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom

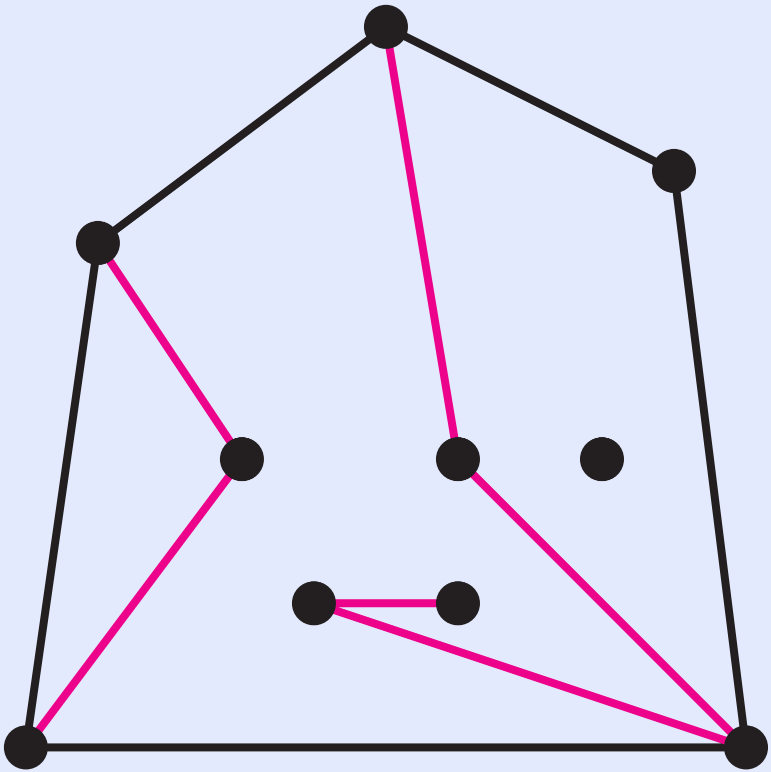


Add five edges

Pseudo-Triangulating

Start with a point set. . . form the convex hull

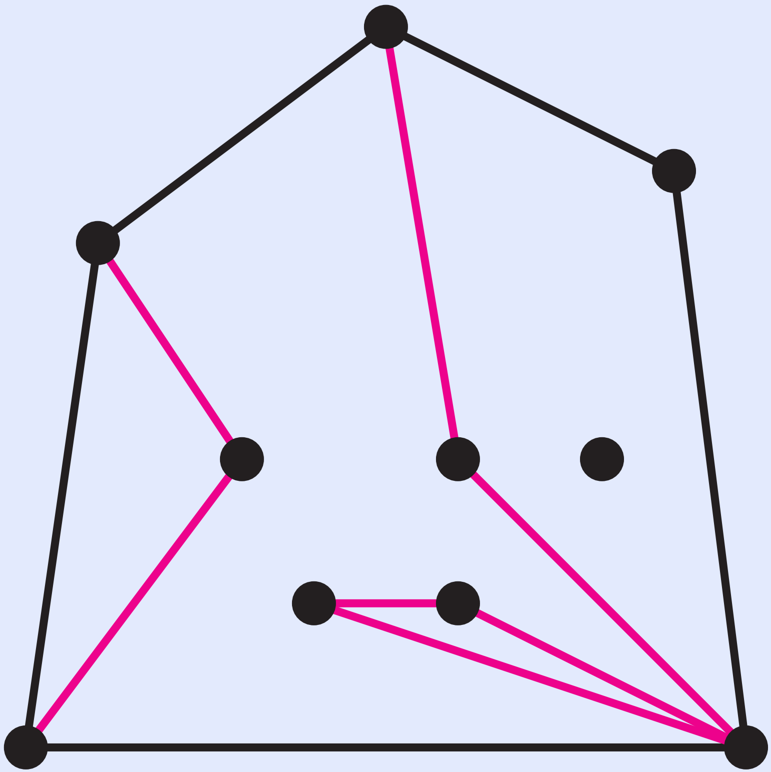
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add six edges

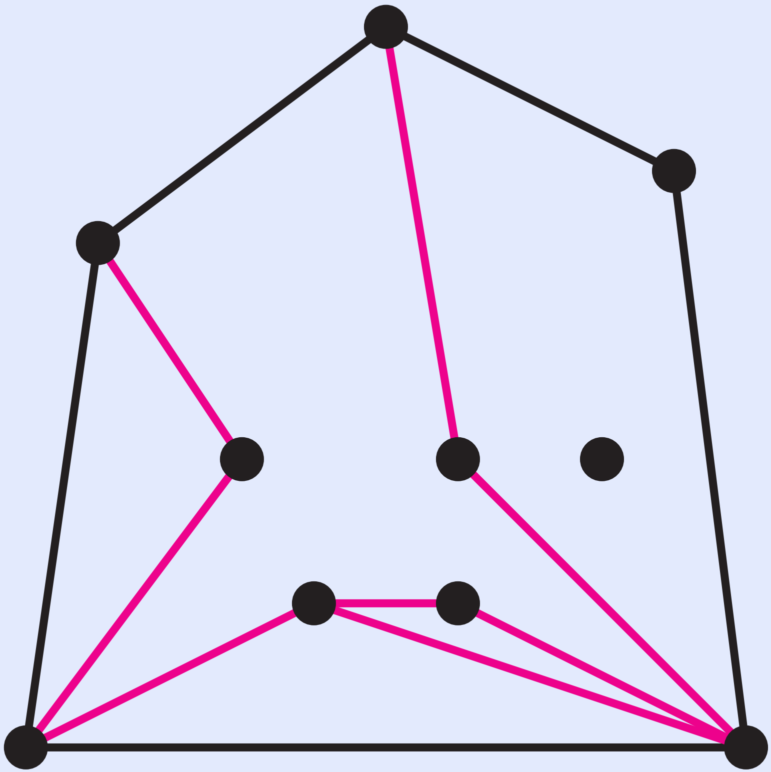
Start with a point set. . . form the convex hull

10 vertices: $2 \cdot 10 - 3$ degrees of freedom



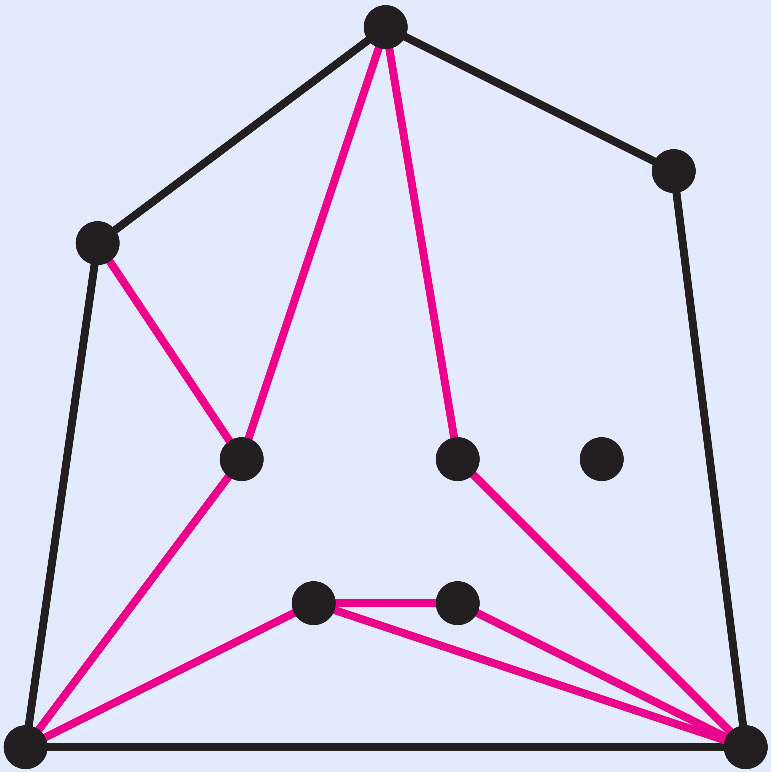
Add seven edges

Start with a point set. . . form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



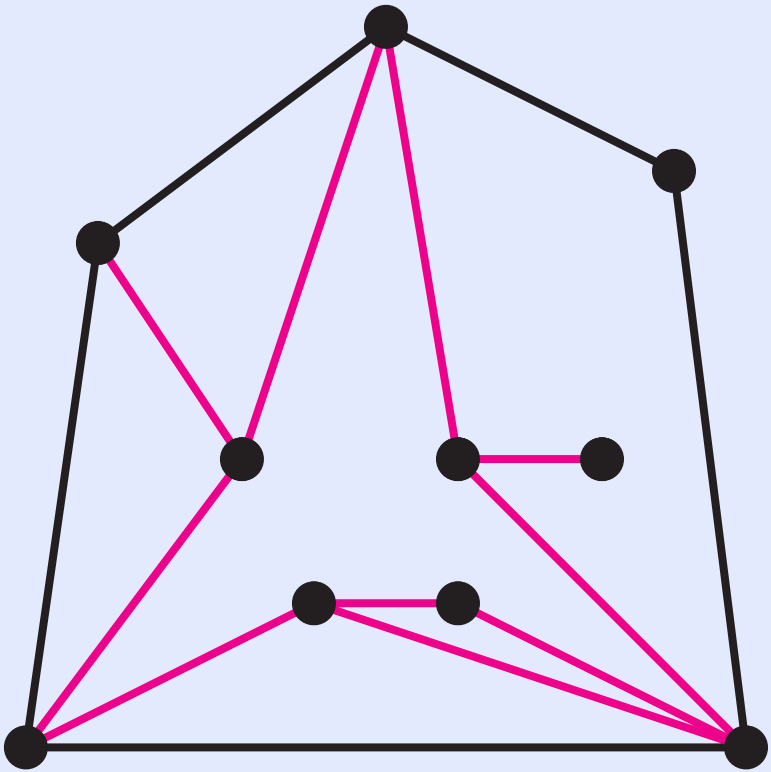
Add eight edges

Start with a point set…form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add nine edges

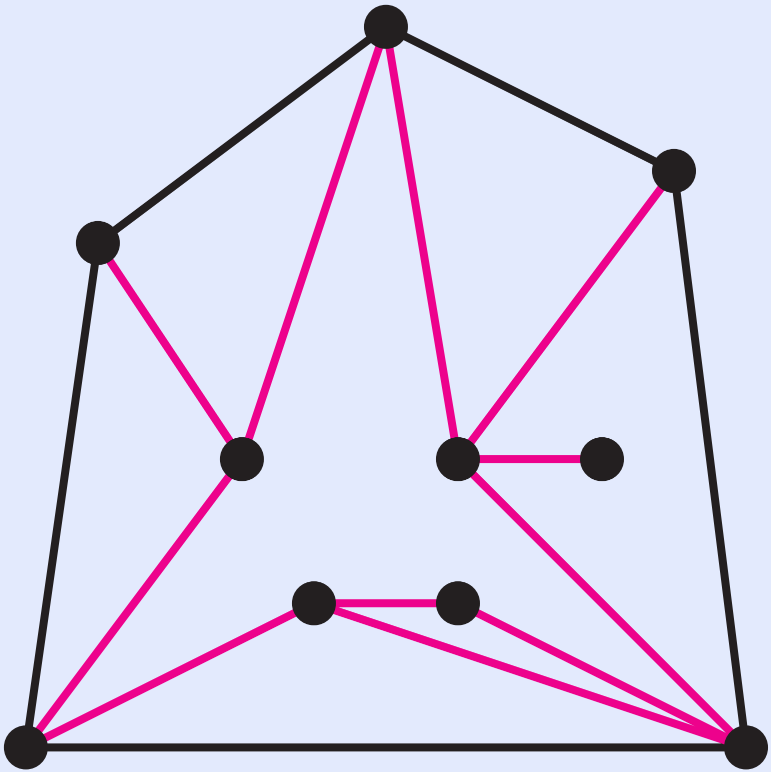
Start with a point set...form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add ten edges

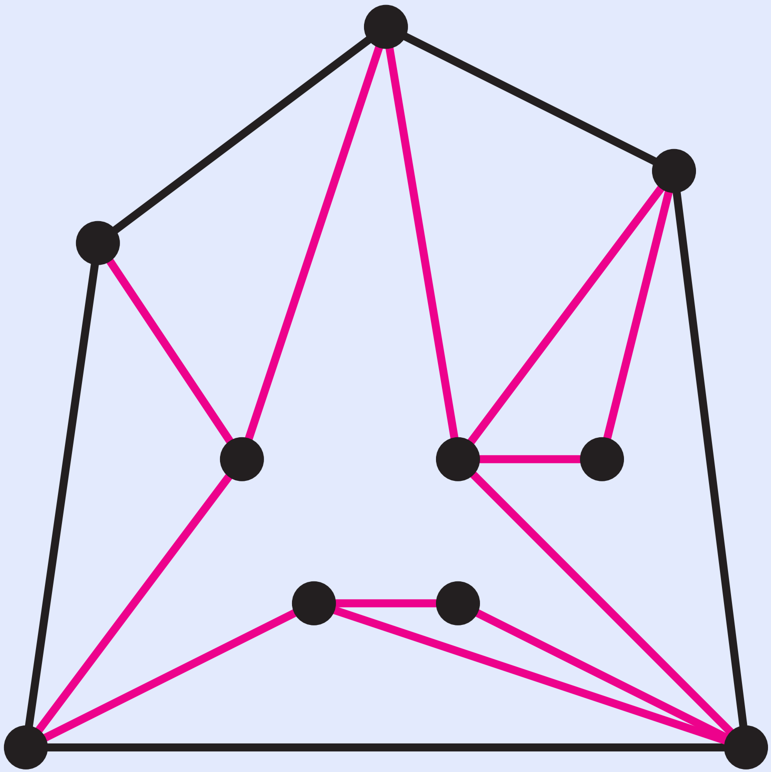
Start with a point set. . . form the convex hull

10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add eleven edges

Start with a point set... form the convex hull
10 vertices: $2 \cdot 10 - 3$ degrees of freedom



Add twelve edges - *Pseudo-Triangulation*

2. Properties

Theorem 1 (Streinu - 2000)
The following are equivalent

- *G is a pseudo-triangulation with the minimum number of edges.*
- *G is a pointed pseudo-triangulation*
- *G is a pseudo-triangulation with exactly $2n - 3$ edges*
- *G is non-crossing, pointed, and has $2n - 3$ edges*
- *G is non-crossing, pointed, and maximal with this property*

Corollary 1 *If any of the above conditions are satisfied, then G is generically minimally rigid in the plane and any realization of G as a pseudo-triangulation is 1'st order rigid.*

Theorem 2 *Every planar graph which is generically minimally rigid has a realization as a pointed pseudo-triangulation.*

Proof 1 uses an inductive construction together with topological information.

Proof 2 uses linear algebra - Tutte's approach to drawing a graph.

3. Definition of CPPT

A *combinatorial pointed pseudo-triangulation* (cppt) is an assignment of labels, *big* and *small*, to the angles of a plane graph such that

- every vertex has exactly one big angle,
- every interior face as exactly three small angles
- the outside face has only big angles.

G has

$-n$ vertices,
$-e$ edges and
$-f$ faces.

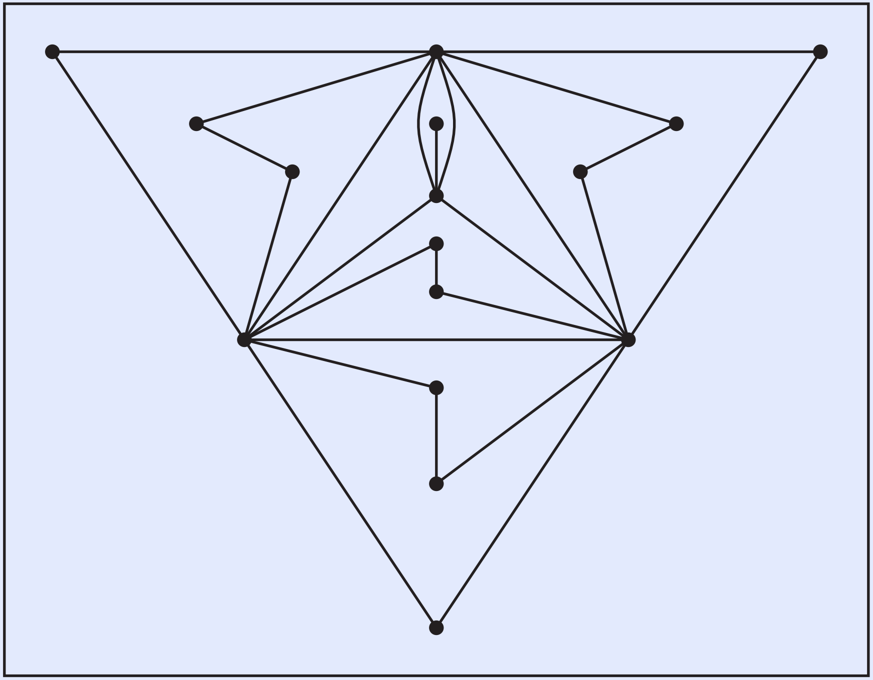
Necessary condition for the existence of a cppt:

$$e = 2n - 3$$

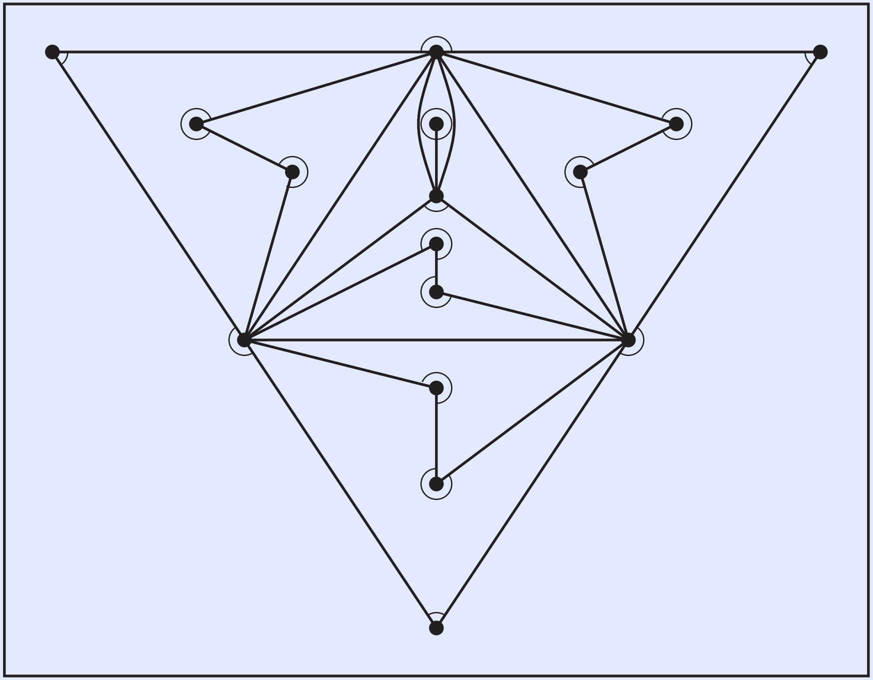
(Since $n - e + f = 2$ and $3f - 3 + n = 2e$.)

4. Combinatorial CPPT

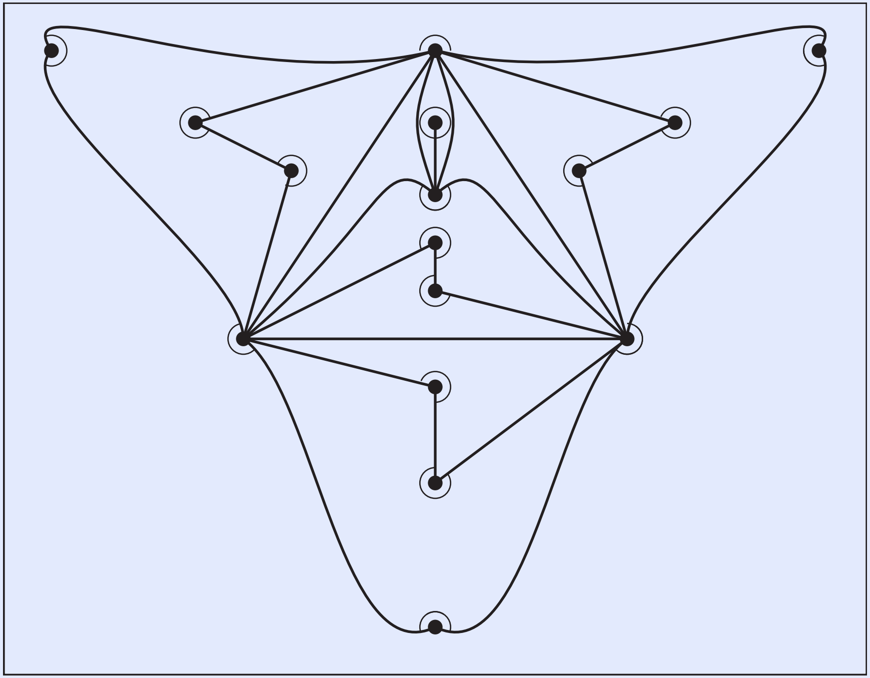
A graph in the plane



A combinatorial pseudo-triangulation

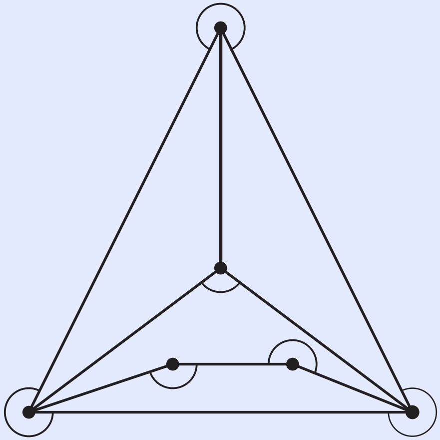


A topological realization

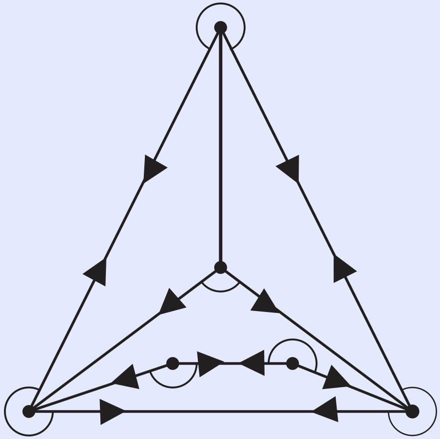


5. Examples

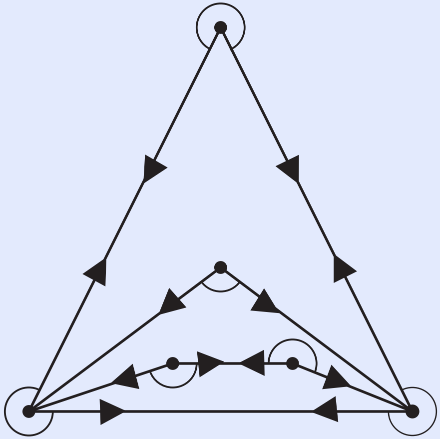
A Combinatorial Pseudotriangulation



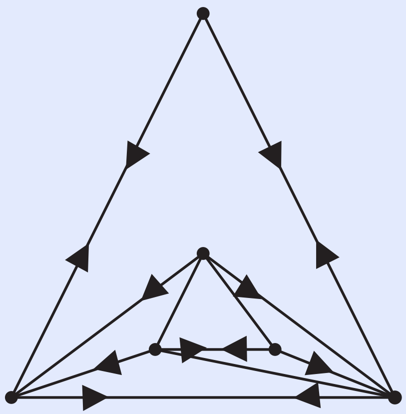
Orient edges away from the pointed vertex



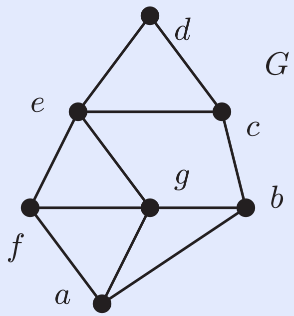
Delete all non-oriented edges



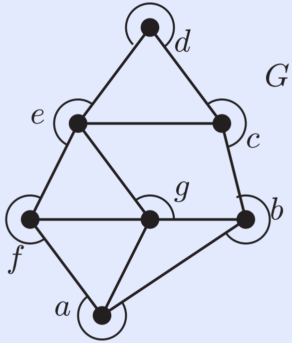
Triangulate the pseudotriangles



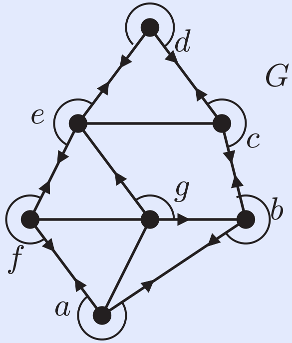
Start again



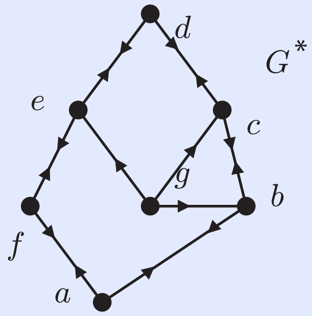
Add combinatorial angles



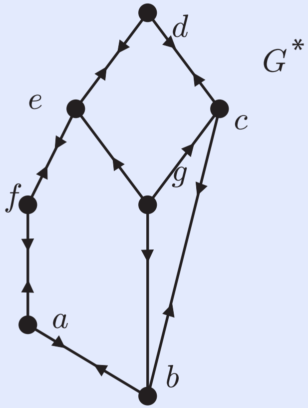
Orient away from the large angles.



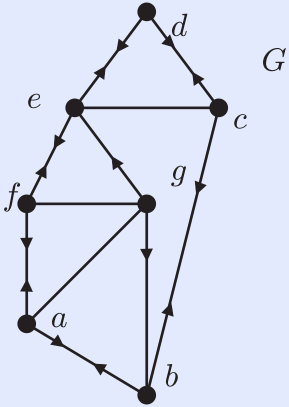
Form G^*



convexify



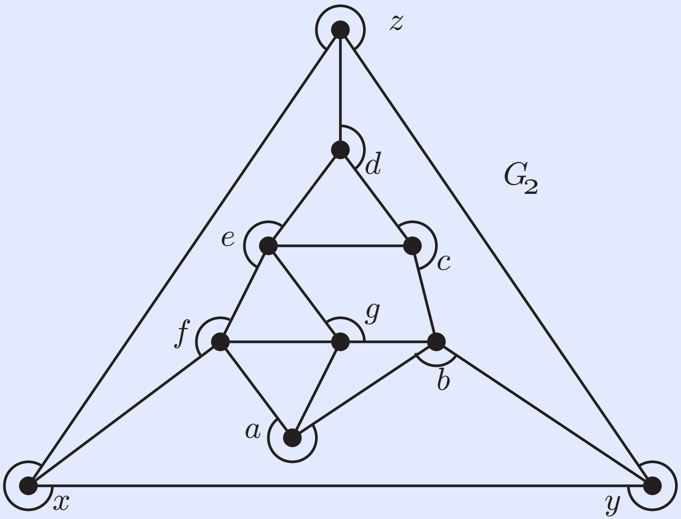
Back to G.

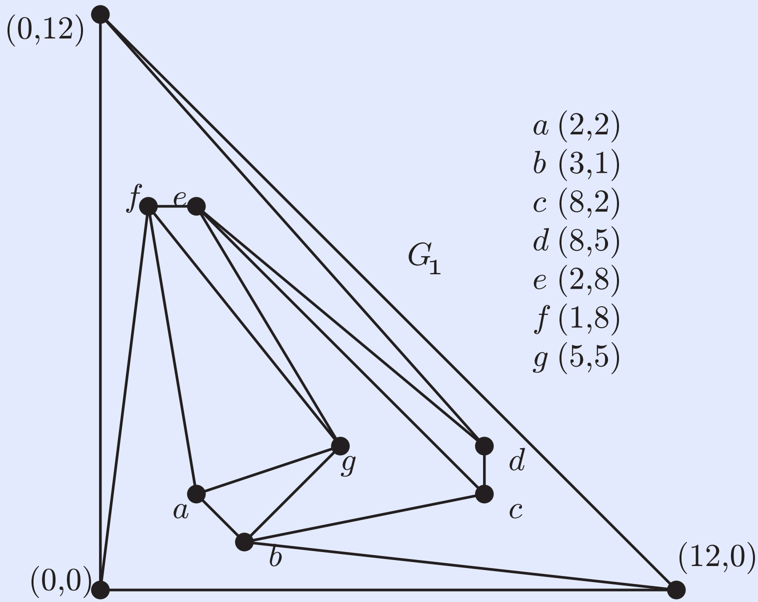


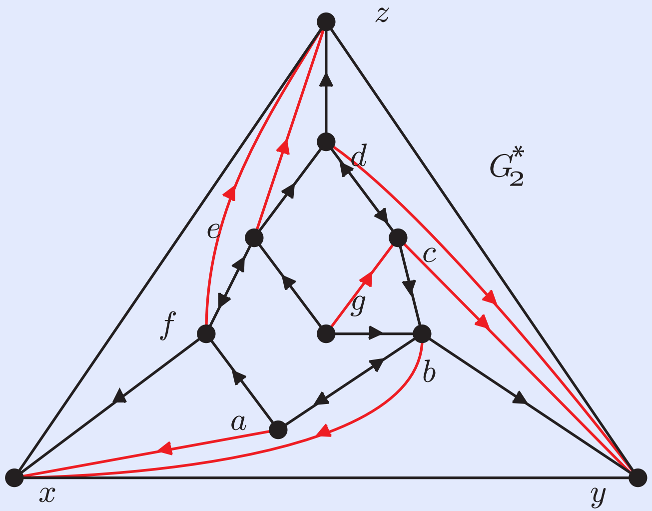
6. Directed Tutte method

Theorem 3 *From every interior vertex of* $G*$ *there are three vertex disjoint paths to the boundary, consequently* $G*$ *can be drawn with straight non-crossing lines in the plane in such a way that a given positive stress on all directed edges is resolved.*

7. Schnyder trees.




(0,12)
f
e
G_1
g
d
a
c
b
(0,0)
(12,0)
a (2,2)
b (3,1)
c (8,2)
d (8,5)
e (2,8)
f (1,8)
g (5,5)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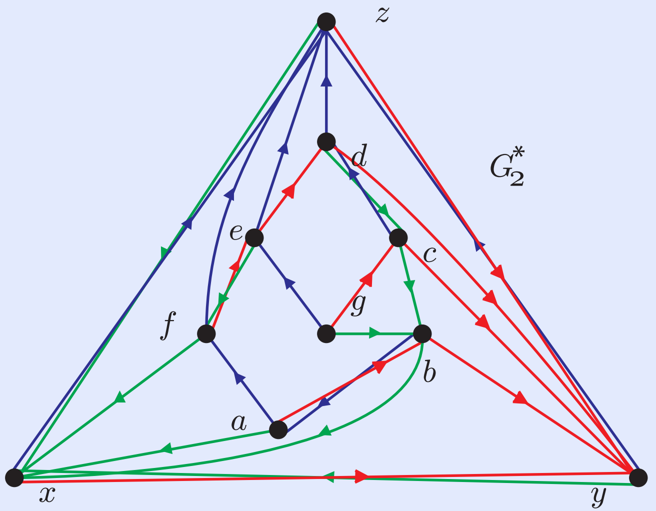
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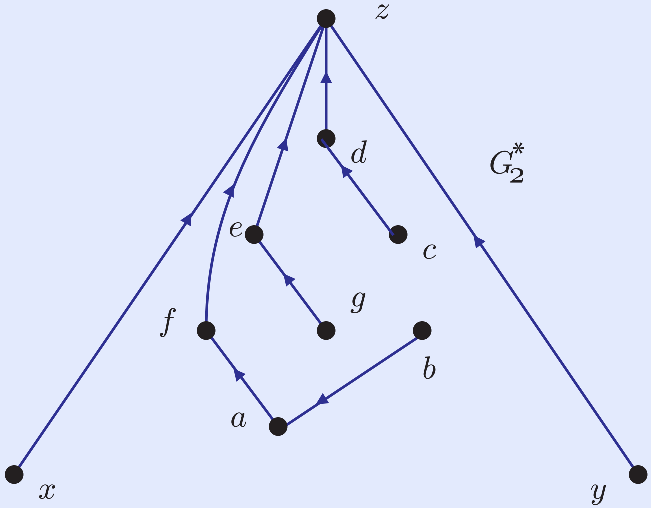

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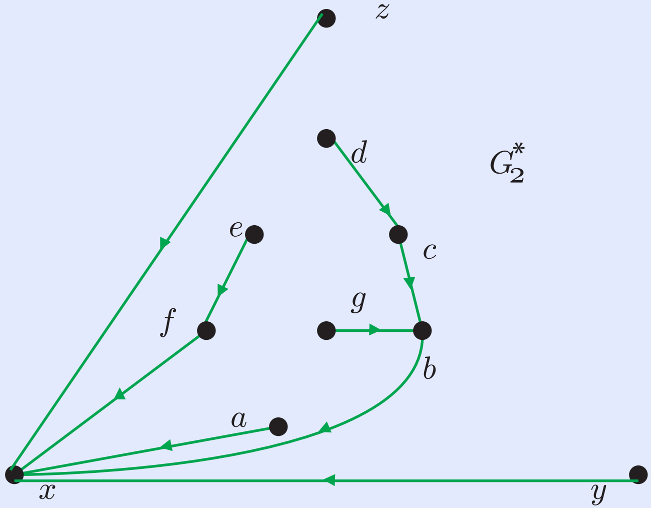

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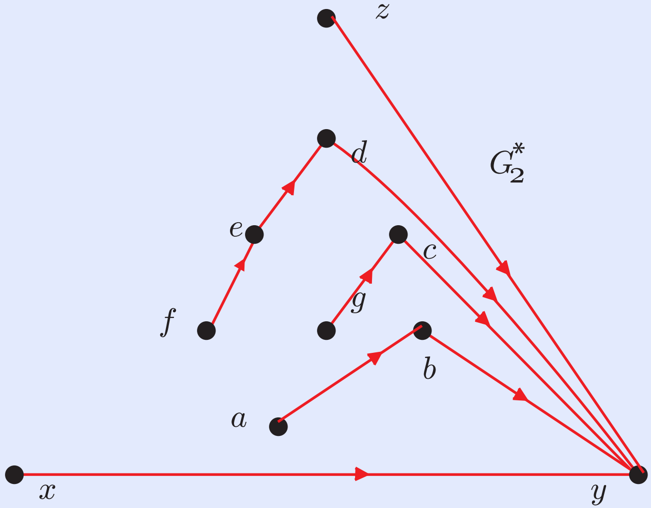
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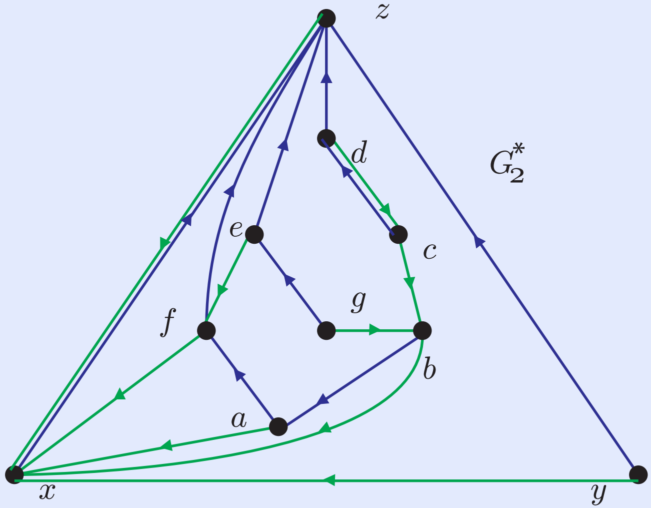
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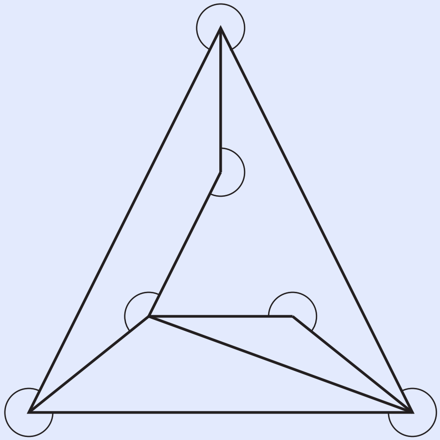
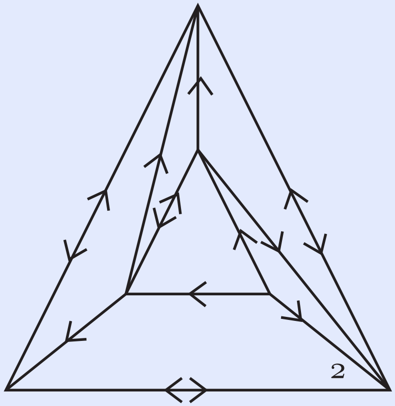
8. A bad example

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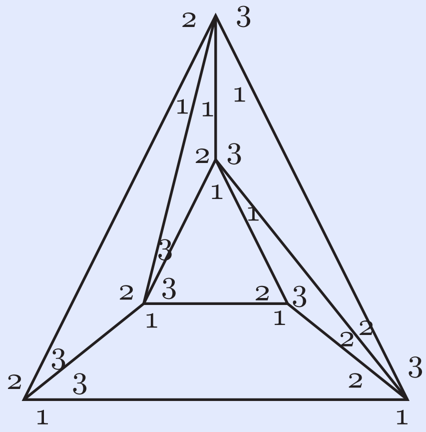


9. Generalizations.

10. Reciprocal Figures

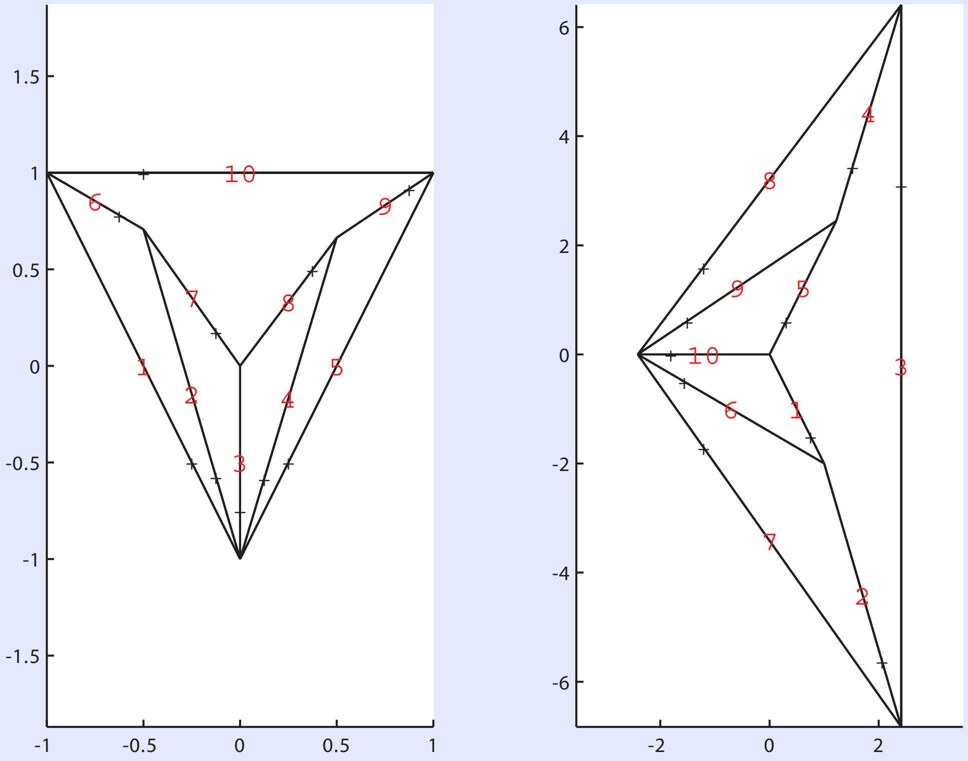
We want to draw the geometric dual using the same edge directions.

Construction

Use a framework with a resolvable stress, non-zero on every edge, for example a cycle in the rigidity matroid. Such a cycle

corresponds to a pseudo-triangulation with one non-pointed vertex.

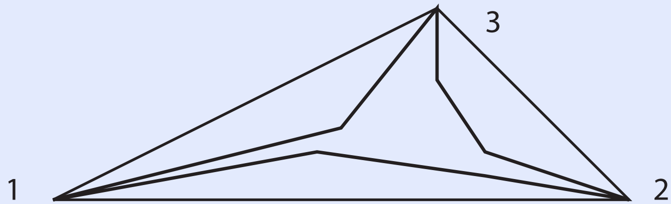
A Wheel and Its Reciprocal

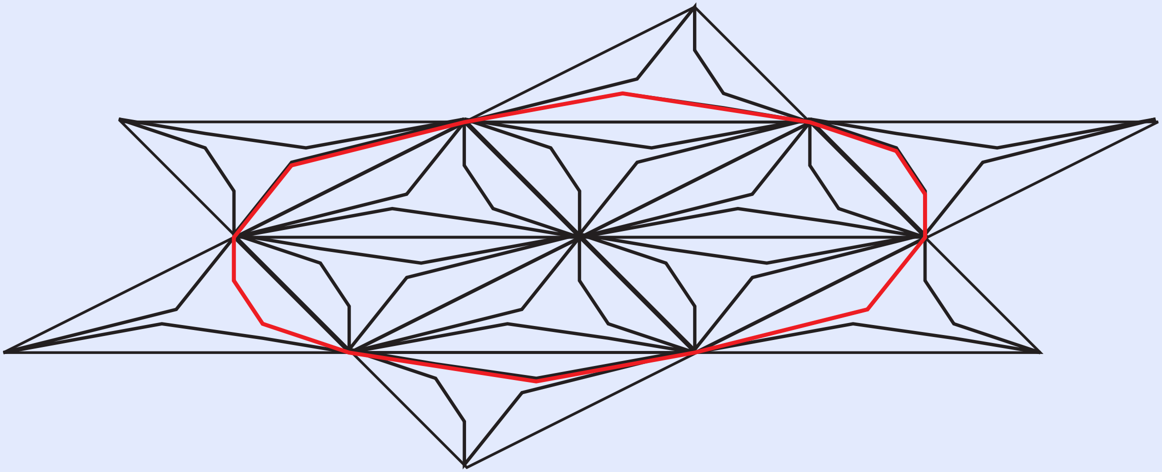


Theorem 4 *If a generic 2-cycle is realized as a pseudo-triangulation, then the reciprocal diagram is also a pseudo-triangulation.*

Lemma 1 *There is, up to rotation, a one to one correspondence between the set of pseudo-triangles T and the set of PTC cells, $C(T)$, such that the vector paths between the distinguished vertices on the boundary of $C(T)$ are translations and half-turns of the pseudo-arcs of T.*

proof





Lemma 2 *Given a framework, together with a resolvable stress s and a pseudo-triangle T. Suppose that T is a face in the rotation system which governs the reciprocal. The following are equivalent:*

1. *The cyclic ordering around the reciprocal vertex is the reverse of the cyclic ordering around T.*
2. *The reciprocal figure is pointed at the vertex corresponding to T.*
3. *There is exactly one improper sign change on the stresses as one reads around T.*

11. Some Examples:

Seven Wheel 1:

Seven Wheel 2:

Center segment revolves:

Graph not in a plane embedding.

Center segment revolves and rotates:

Graph not in a plane embedding.

A non-planar reciprocal: