



# Pseudotriangulations and Rigidity

Brigitte Servatius

University of Ljubljana  
Worcester Polytechnic Institute

*Pseudo-Triangulating*

*Properties*

*Definition of CPPT*

*Combinatorial CPPT*

*Examples*

*Directed Tutte method*

*Schnyder trees.*

*A bad example*

*Generalizations.*

*Reciprocal Figures*

*Some Examples:*

*Home Page*

*Title Page*



*Page 1 of 55*

*Go Back*

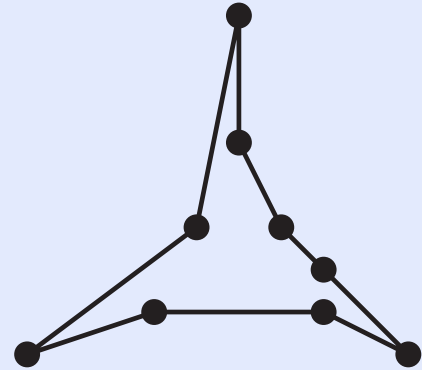
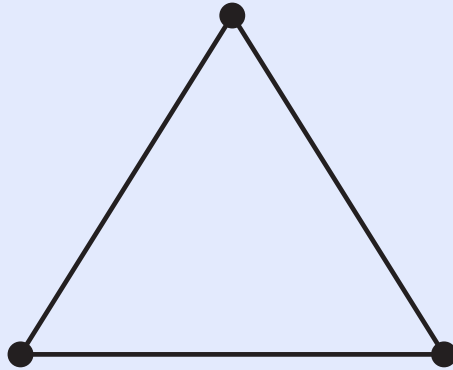
*Full Screen*

*Close*

*Quit*



- [Pseudo-Triangulating](#)
- [Properties](#)
- [Definition of CPPT](#)
- [Combinatorial CPPT](#)
- [Examples](#)
- [Directed Tutte method](#)
- [Schnyder trees.](#)
- [A bad example](#)
- [Generalizations.](#)
- [Reciprocal Figures](#)
- [Some Examples:](#)



Collaborators:

- Ruth Haas
- David Orden
- Günter Rote
- Francisco Santos
- Herman Servatius
- Ileana Streinu
- Walter Whiteley

Research supported by NSF Grant CCR0203224

[Home Page](#)

[Title Page](#)

◀ | ▶

◀ | ▶

Page 2 of 55

[Go Back](#)

[Full Screen](#)

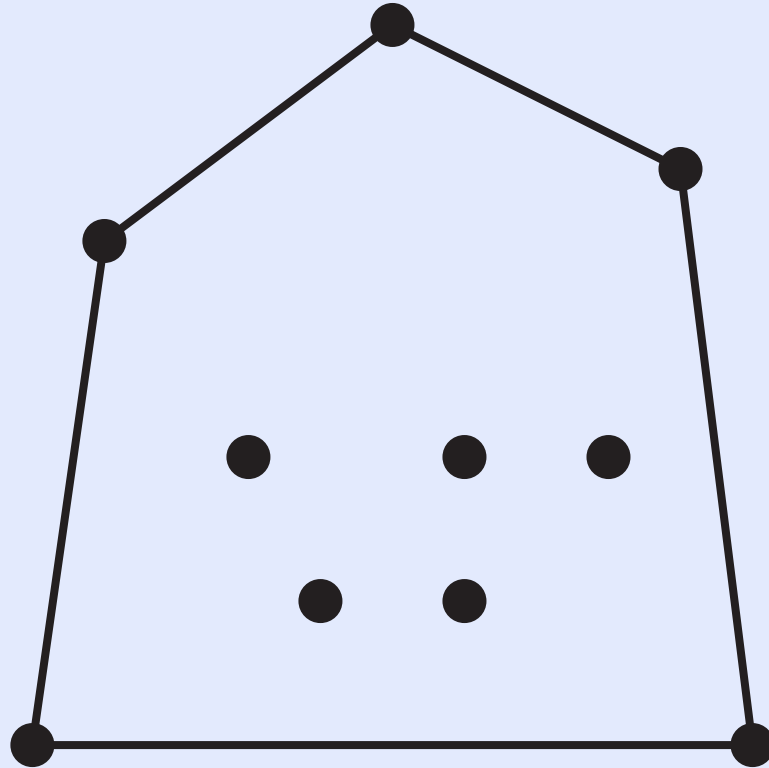
[Close](#)

[Quit](#)



# 1. Pseudo-Triangulating

Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add edges...

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 3 of 55

Go Back

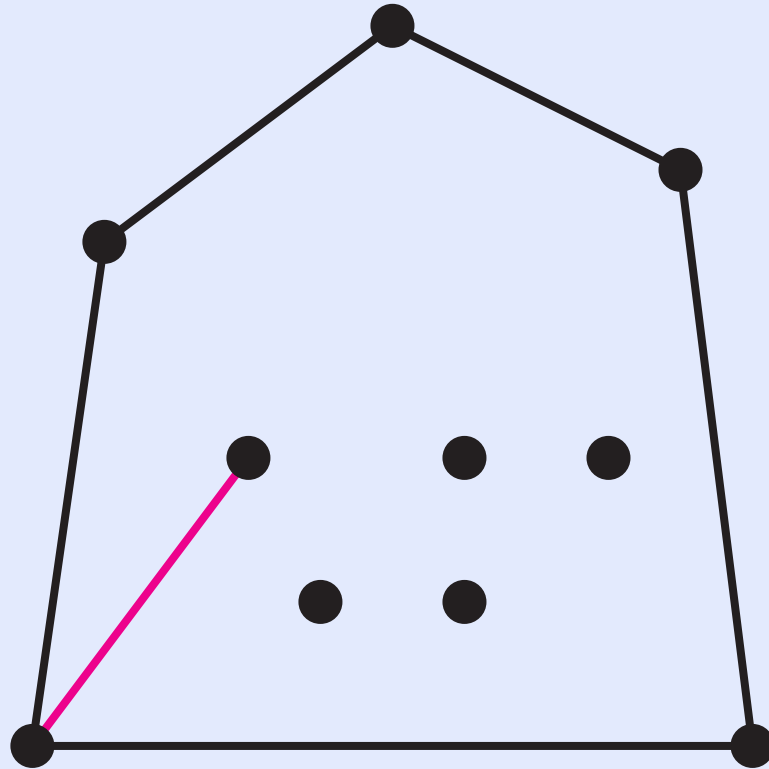
Full Screen

Close

Quit



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add one edge

- Pseudo-Triangulating
- Properties
- Definition of CPPT
- Combinatorial CPPT
- Examples
- Directed Tutte method
- Schnyder trees.
- A bad example
- Generalizations.
- Reciprocal Figures
- Some Examples:

Home Page

Title Page

◀ ▶

◀ ▶

Page 4 of 55

Go Back

Full Screen

Close

Quit



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



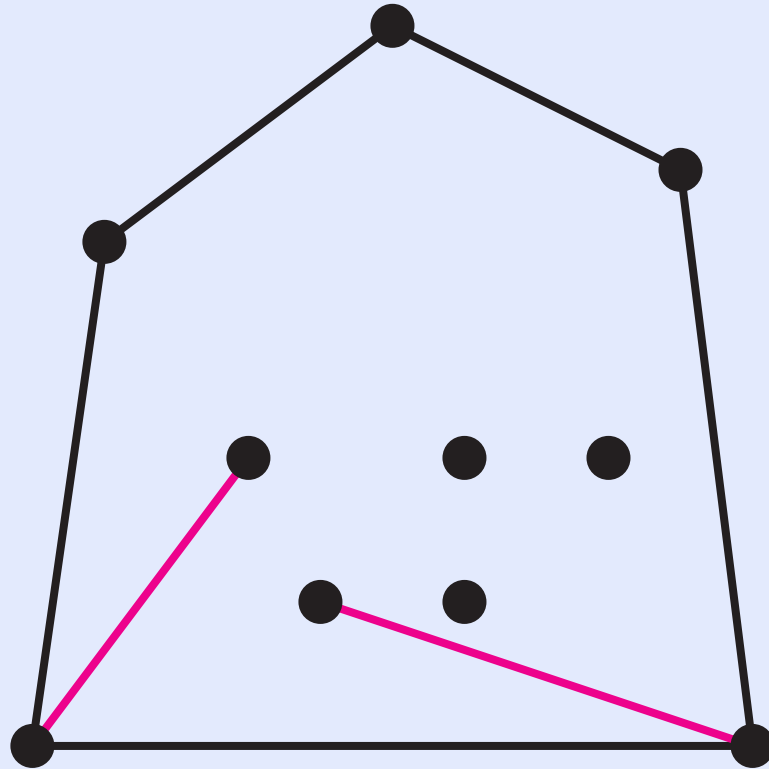
Page 5 of 55

Go Back

Full Screen

Close

Quit



Add two edges



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



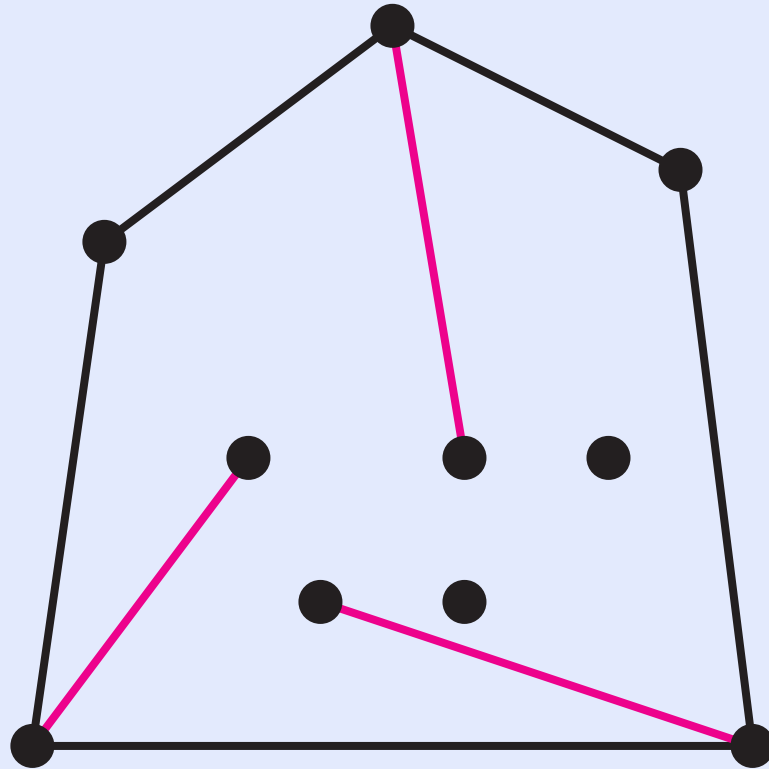
Page 6 of 55

Go Back

Full Screen

Close

Quit



Add three edges



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



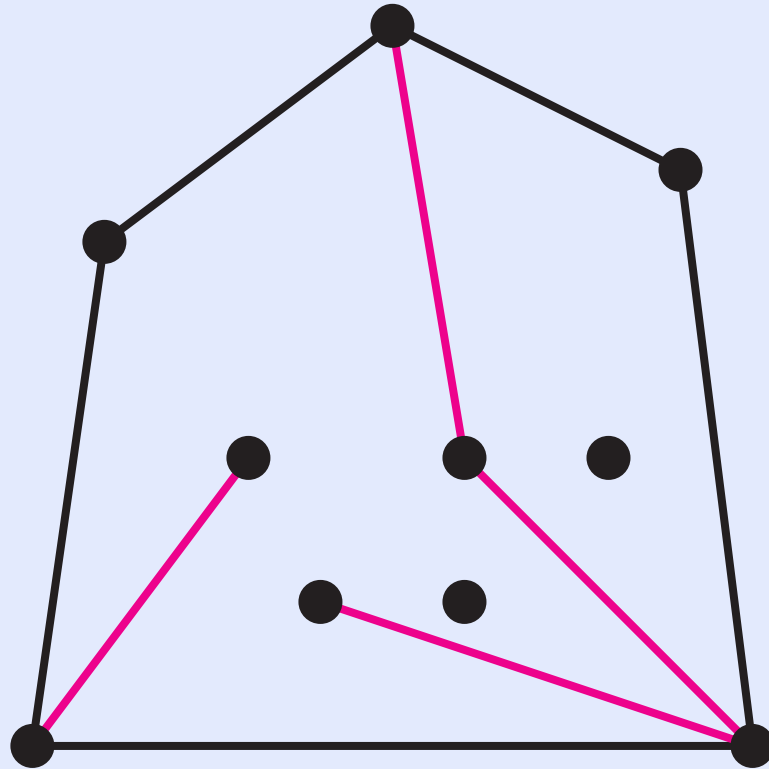
Page 7 of 55

Go Back

Full Screen

Close

Quit



Add four edges



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



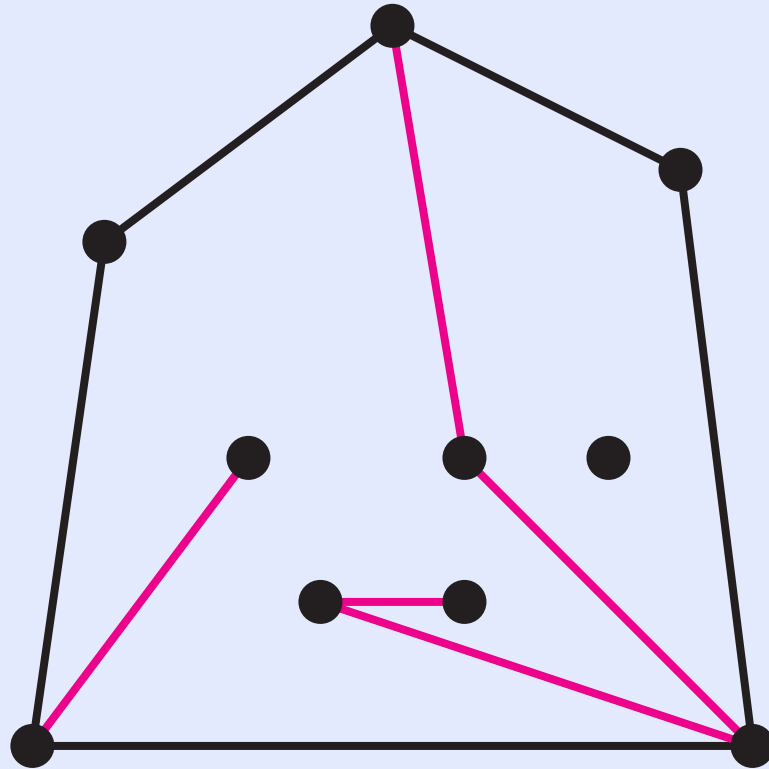
Page 8 of 55

Go Back

Full Screen

Close

Quit

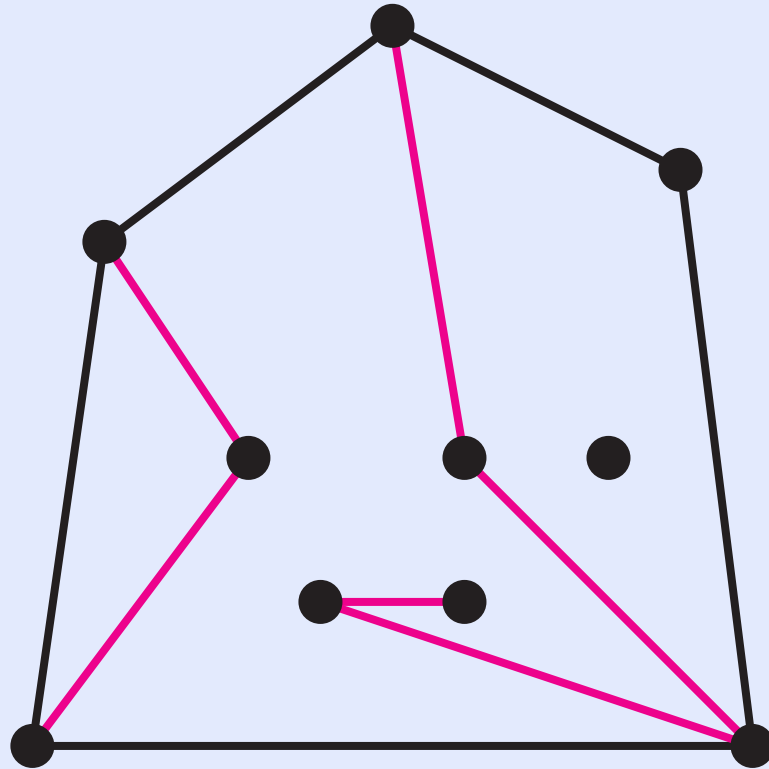


Add five edges





Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add six edges

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 9 of 55

Go Back

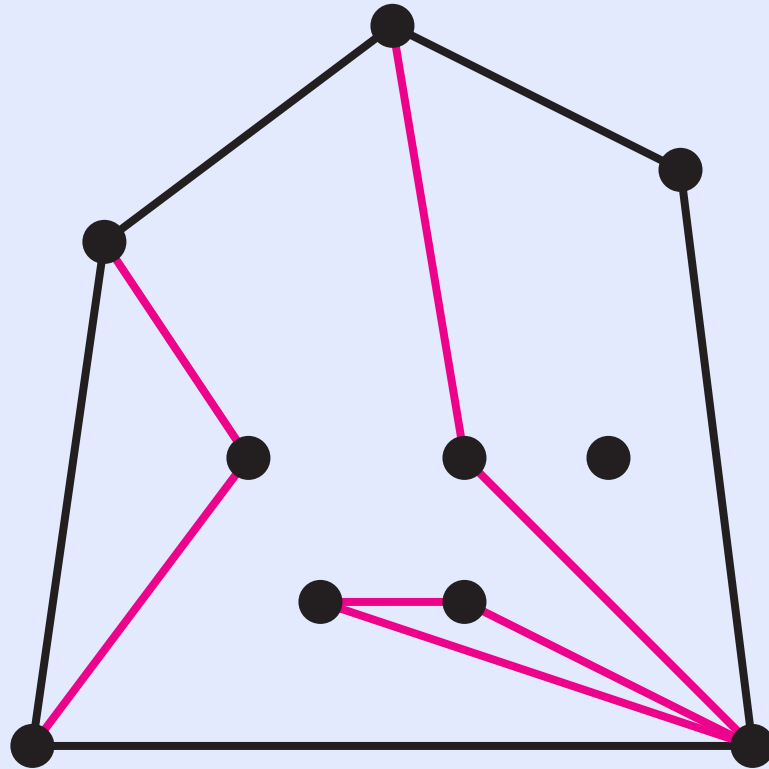
Full Screen

Close

Quit



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add seven edges

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 10 of 55

Go Back

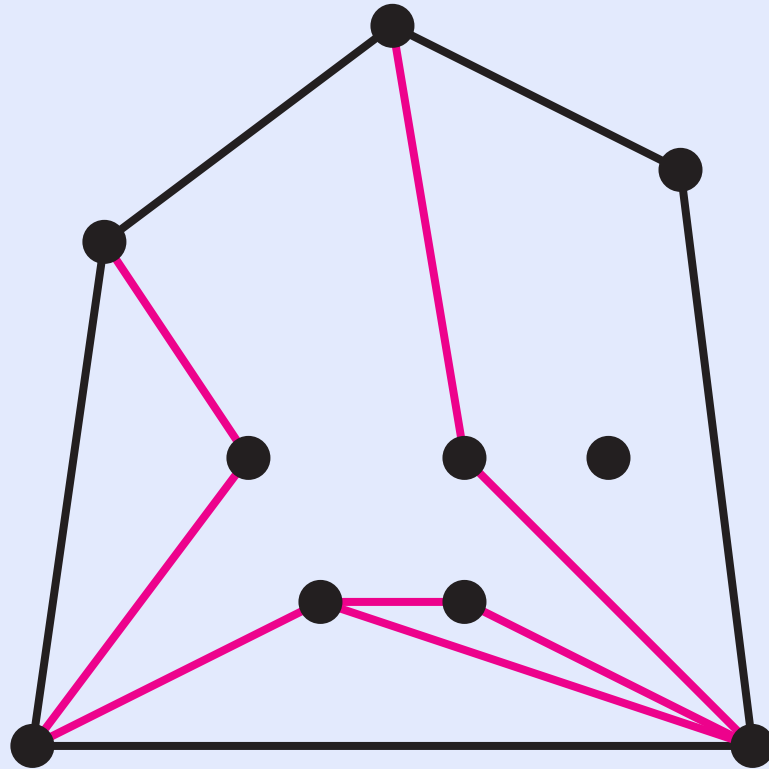
Full Screen

Close

Quit



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add eight edges

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 11 of 55

Go Back

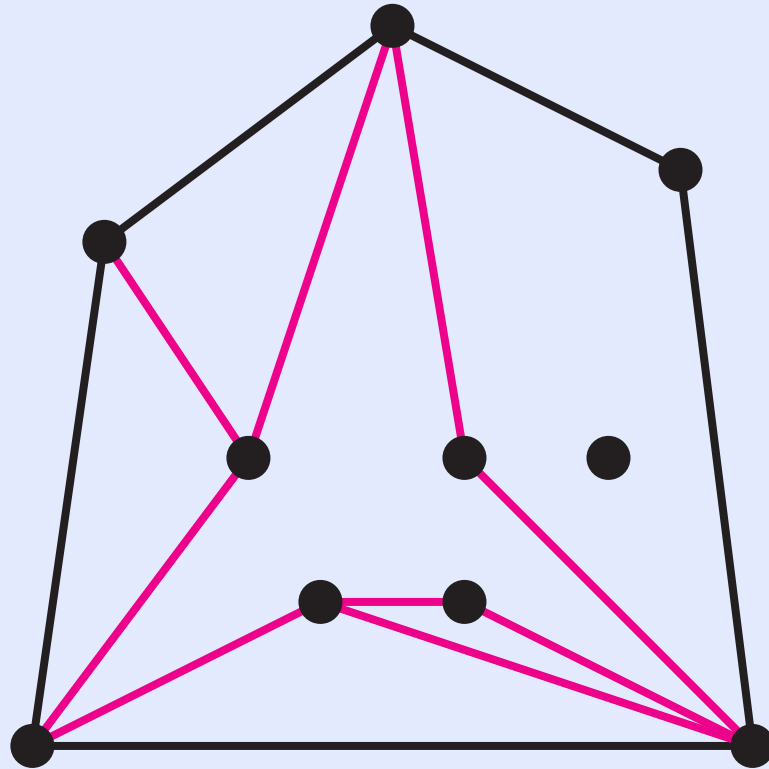
Full Screen

Close

Quit



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add nine edges

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 12 of 55

Go Back

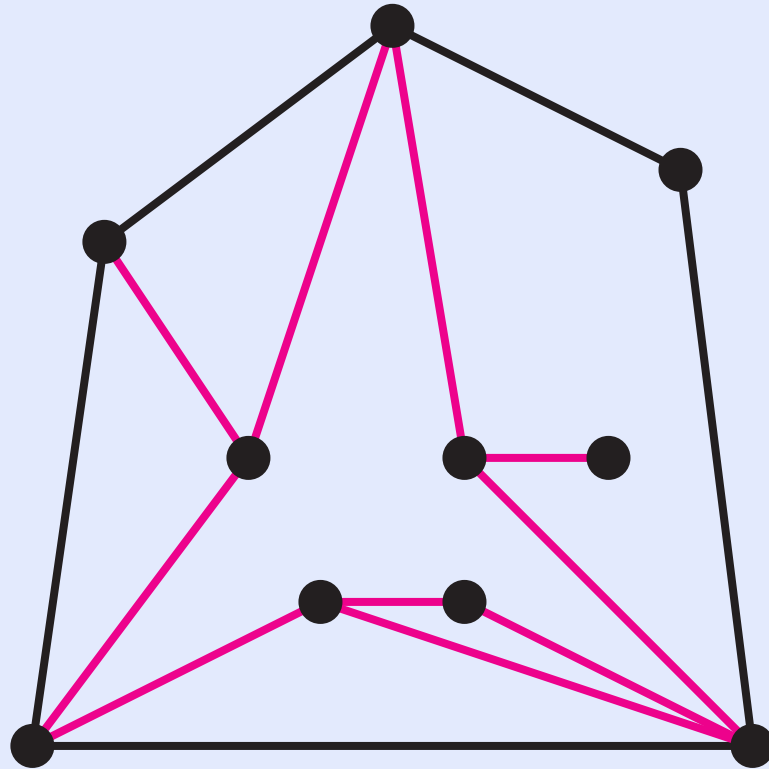
Full Screen

Close

Quit



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add ten edges

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 13 of 55

Go Back

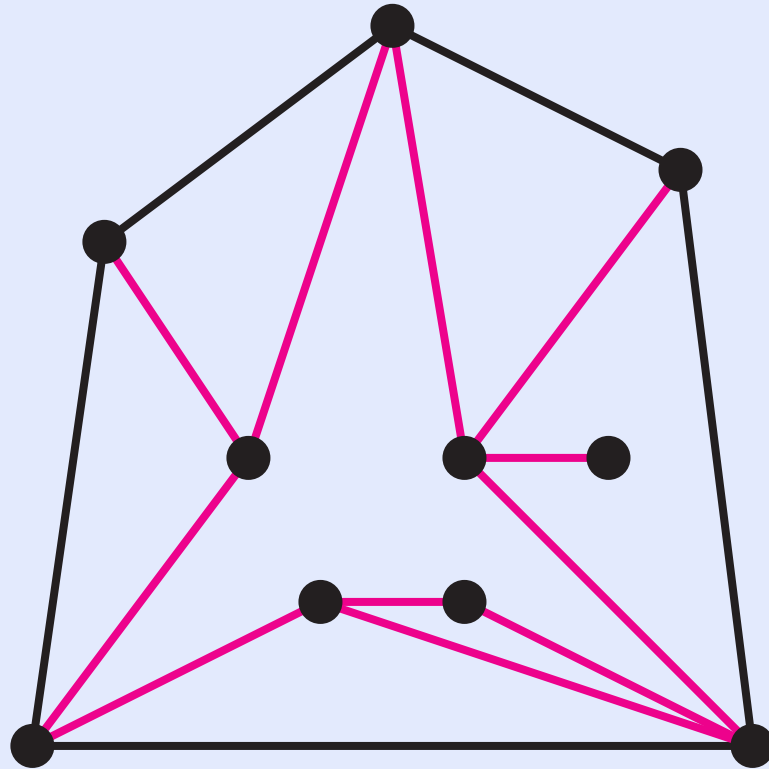
Full Screen

Close

Quit



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add eleven edges

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 14 of 55

Go Back

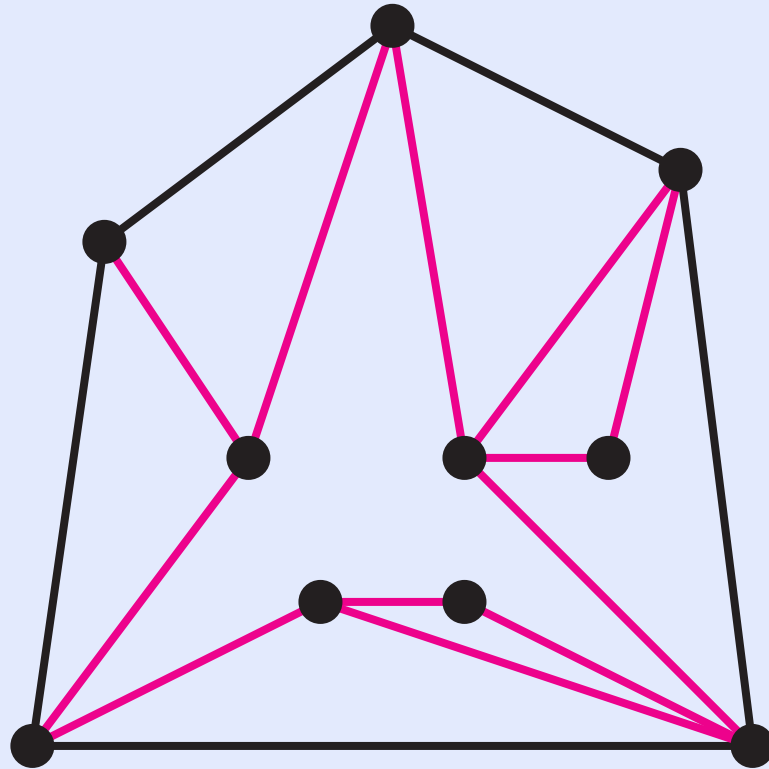
Full Screen

Close

Quit



Start with a point set... form the convex hull  
10 vertices:  $2 \cdot 10 - 3$  degrees of freedom



Add twelve edges - *Pseudo-Triangulation*

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 15 of 55

Go Back

Full Screen

Close

Quit



## 2. Properties

### Theorem 1 (Streinu - 2000)

*The following are equivalent*

- *$G$  is a pseudo-triangulation with the minimum number of edges.*
- *$G$  is a pointed pseudo-triangulation*
- *$G$  is a pseudo-triangulation with exactly  $2n - 3$  edges*
- *$G$  is non-crossing, pointed, and has  $2n - 3$  edges*
- *$G$  is non-crossing, pointed, and maximal with this property*

Home Page

Title Page



Page 16 of 55

Go Back

Full Screen

Close

Quit





**Corollary 1** *If any of the above conditions are satisfied, then  $G$  is generically minimally rigid in the plane and any realization of  $G$  as a pseudo-triangulation is 1'st order rigid.*

**Theorem 2** *Every planar graph which is generically minimally rigid has a realization as a pointed pseudo-triangulation.*

Proof 1 uses an inductive construction together with topological information.

Proof 2 uses linear algebra - Tutte's approach to drawing a graph.

[Home Page](#)

[Title Page](#)



Page 17 of 55

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)



### 3. Definition of CPPT

A *combinatorial pointed pseudo-triangulation* (cppt) is an assignment of labels, *big* and *small*, to the angles of a plane graph such that

- every vertex has exactly one big angle,
- every interior face as exactly three small angles
- the outside face has only big angles.

$G$  has

- $n$  vertices,
- $e$  edges and
- $f$  faces.

Necessary condition for the existence of a cppt:

$$e = 2n - 3$$

(Since  $n - e + f = 2$  and  $3f - 3 + n = 2e$ .)

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 18 of 55

Go Back

Full Screen

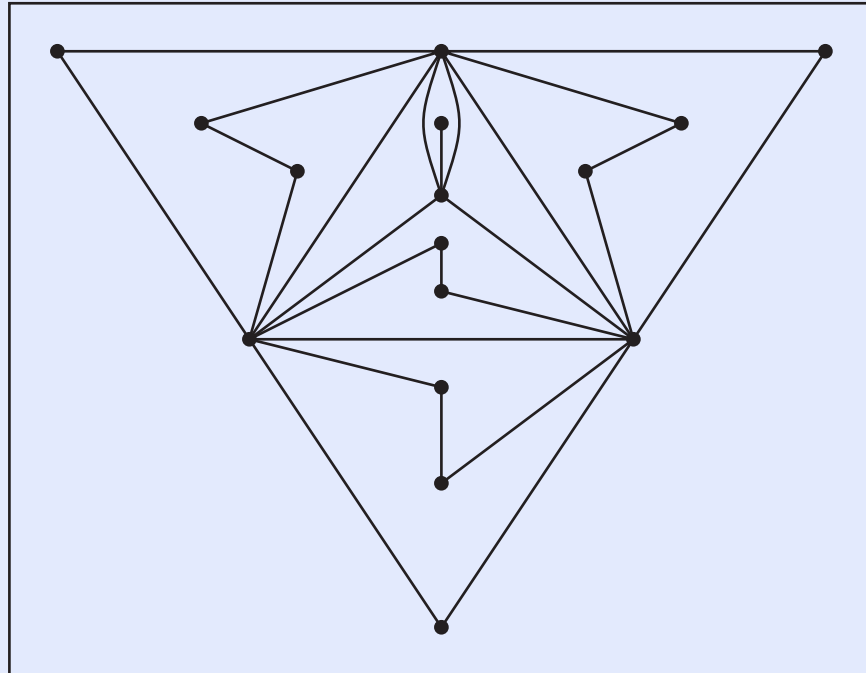
Close

Quit



# 4. Combinatorial CPPT

A graph in the plane



Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 19 of 55

Go Back

Full Screen

Close

Quit



Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 20 of 55

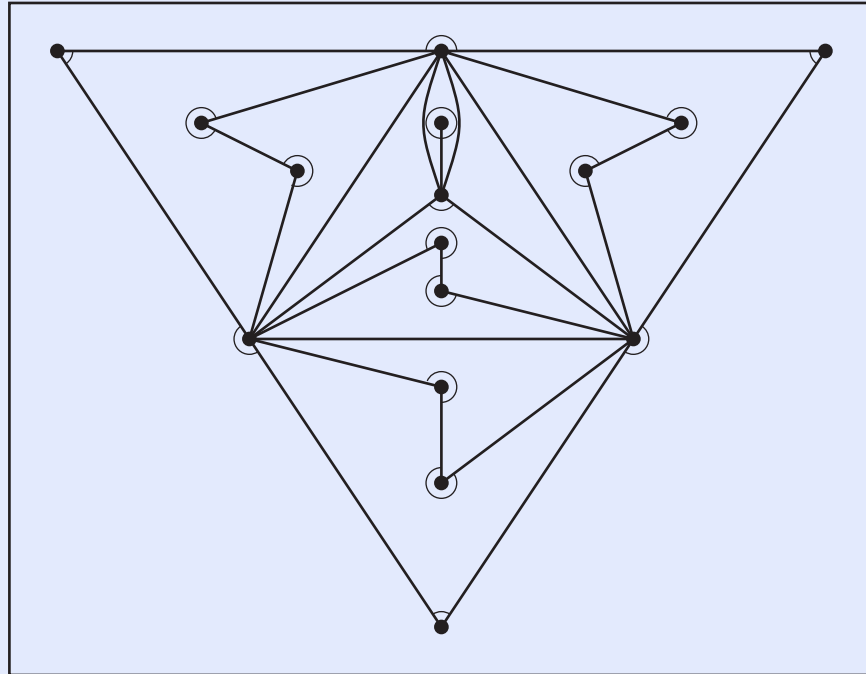
Go Back

Full Screen

Close

Quit

## A combinatorial pseudo-triangulation





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 21 of 55

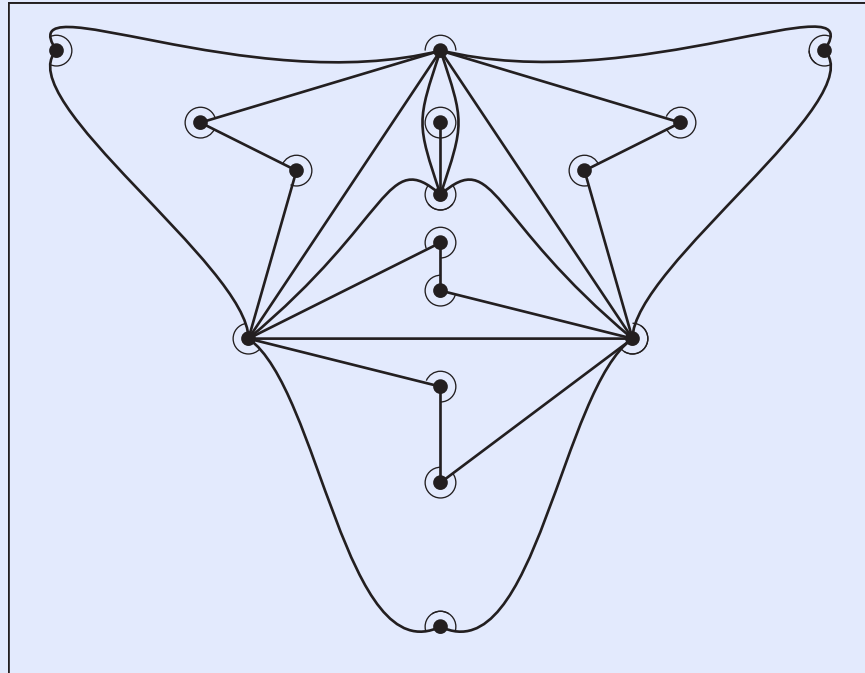
Go Back

Full Screen

Close

Quit

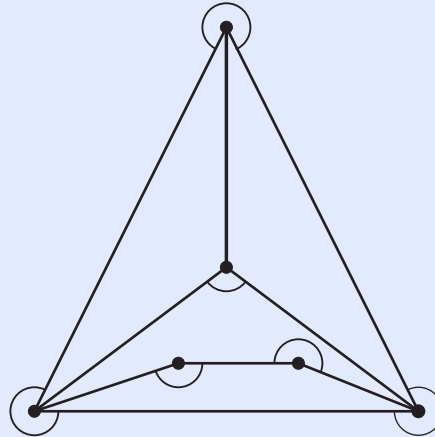
## A topological realization





# 5. Examples

## A Combinatorial Pseudotriangulation



- Pseudo-Triangulating
- Properties
- Definition of CPPT
- Combinatorial CPPT
- Examples**
- Directed Tutte method
- Schnyder trees.
- A bad example
- Generalizations.
- Reciprocal Figures
- Some Examples:

Home Page

Title Page

◀▶

◀▶

Page 22 of 55

Go Back

Full Screen

Close

Quit



Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 23 of 55

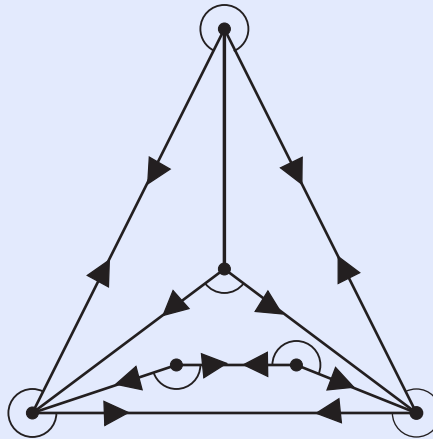
Go Back

Full Screen

Close

Quit

Orient edges away from the pointed vertex





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 24 of 55

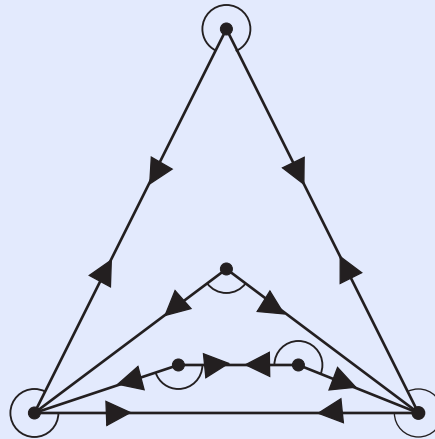
Go Back

Full Screen

Close

Quit

# Delete all non-oriented edges







Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 25 of 55

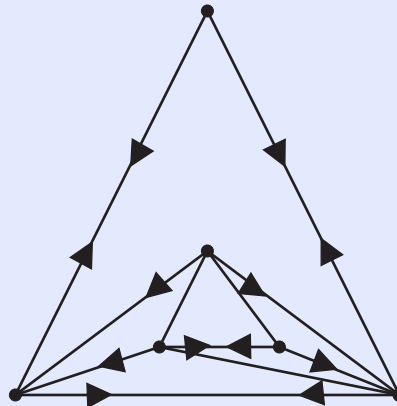
Go Back

Full Screen

Close

Quit

## Triangulate the pseudotriangles





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 26 of 55

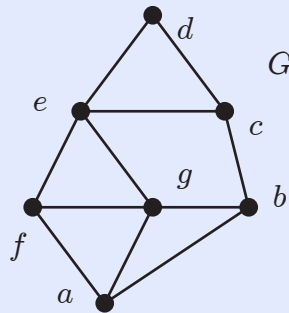
Go Back

Full Screen

Close

Quit

Start again





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 27 of 55

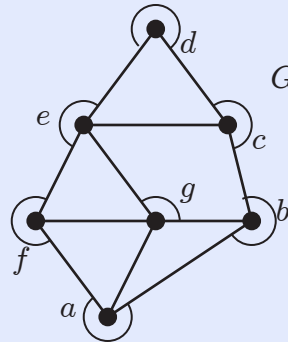
Go Back

Full Screen

Close

Quit

## Add combinatorial angles





- Pseudo-Triangulating
- Properties
- Definition of CPPT
- Combinatorial CPPT
- Examples**
- Directed Tutte method
- Schnyder trees.
- A bad example
- Generalizations.
- Reciprocal Figures
- Some Examples:

Home Page

Title Page

◀▶

◀▶

Page 28 of 55

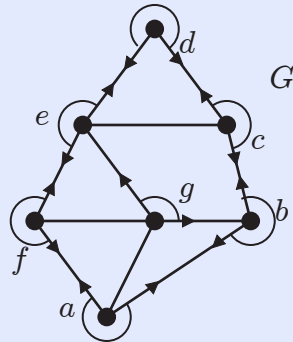
Go Back

Full Screen

Close

Quit

Orient away from the large angles.





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 29 of 55

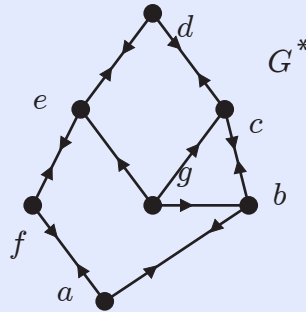
Go Back

Full Screen

Close

Quit

Form  $G^*$





- Pseudo-Triangulating
- Properties
- Definition of CPPT
- Combinatorial CPPT
- Examples**
- Directed Tutte method
- Schnyder trees.
- A bad example
- Generalizations.
- Reciprocal Figures
- Some Examples:

Home Page

Title Page

◀▶

◀▶

Page 30 of 55

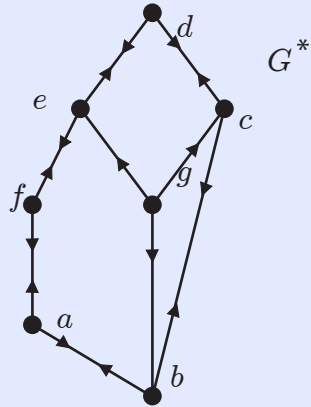
Go Back

Full Screen

Close

Quit

convexify





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 31 of 55

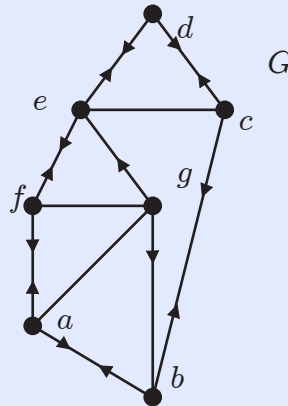
Go Back

Full Screen

Close

Quit

Back to  $G$ .





## 6. Directed Tutte method

**Theorem 3** *From every interior vertex of  $G^*$  there are three vertex disjoint paths to the boundary, consequently  $G^*$  can be drawn with straight non-crossing lines in the plane in such a way that a given positive stress on all directed edges is resolved.*

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 32 of 55

Go Back

Full Screen

Close

Quit





# 7. Schnyder trees.

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



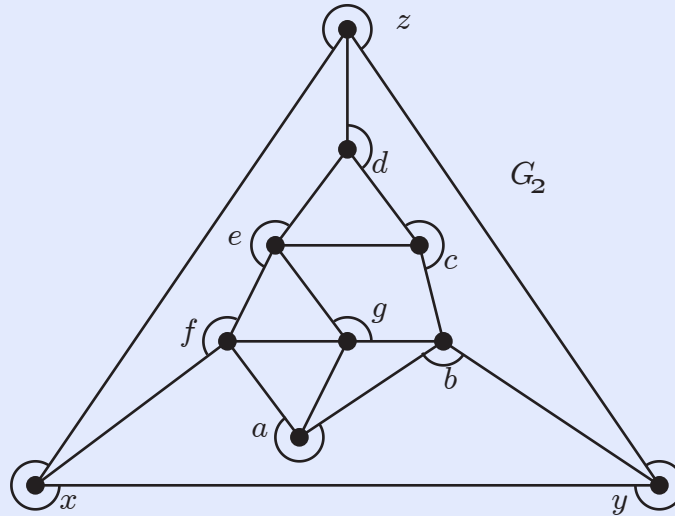
Page 33 of 55

Go Back

Full Screen

Close

Quit





- Pseudo-Triangulating
- Properties
- Definition of CPPT
- Combinatorial CPPT
- Examples
- Directed Tutte method
- Schnyder trees.**
- A bad example
- Generalizations.
- Reciprocal Figures
- Some Examples:

Home Page

Title Page

◀ ▶

◀ ▶

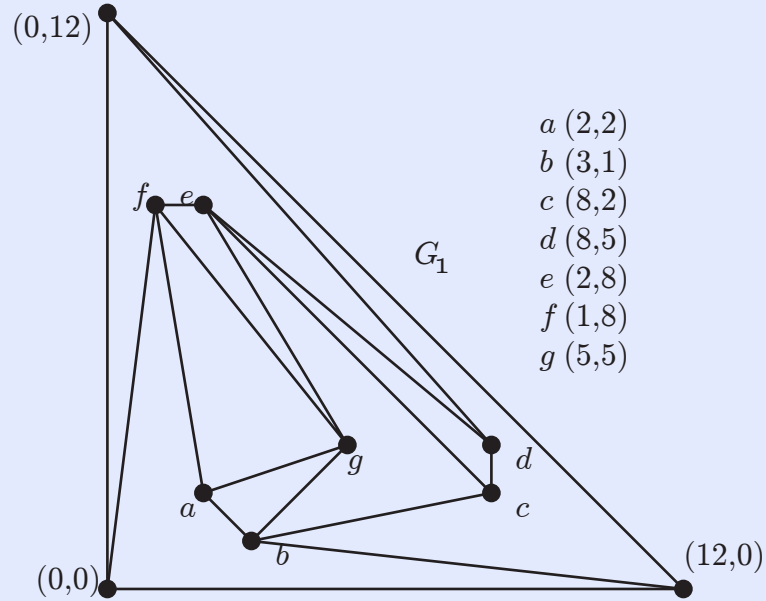
Page 34 of 55

Go Back

Full Screen

Close

Quit





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



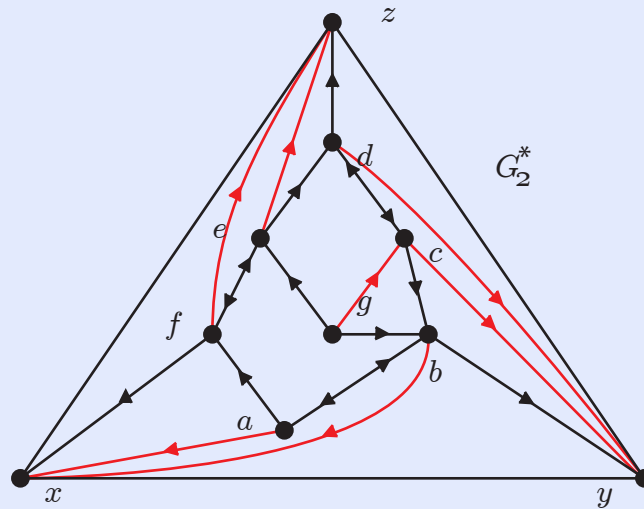
Page 35 of 55

Go Back

Full Screen

Close

Quit





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



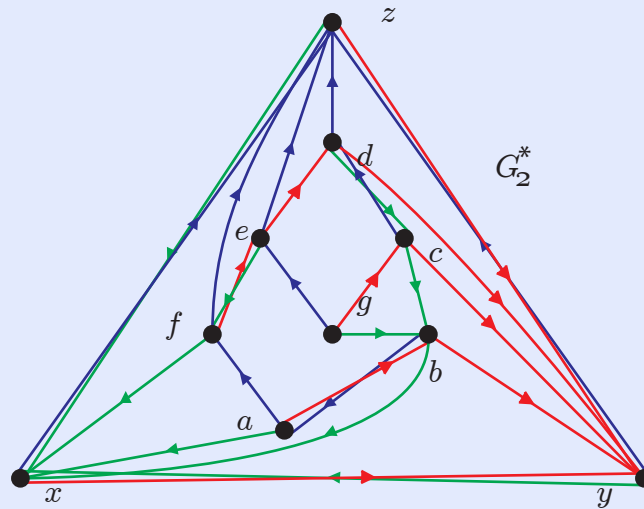
Page 36 of 55

Go Back

Full Screen

Close

Quit





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



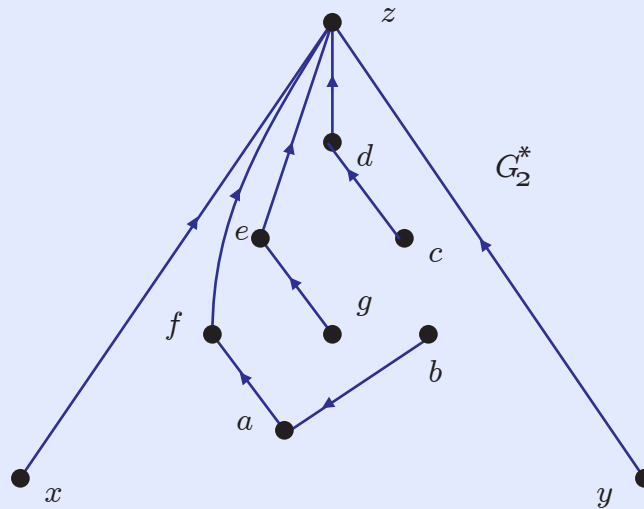
Page 37 of 55

Go Back

Full Screen

Close

Quit





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



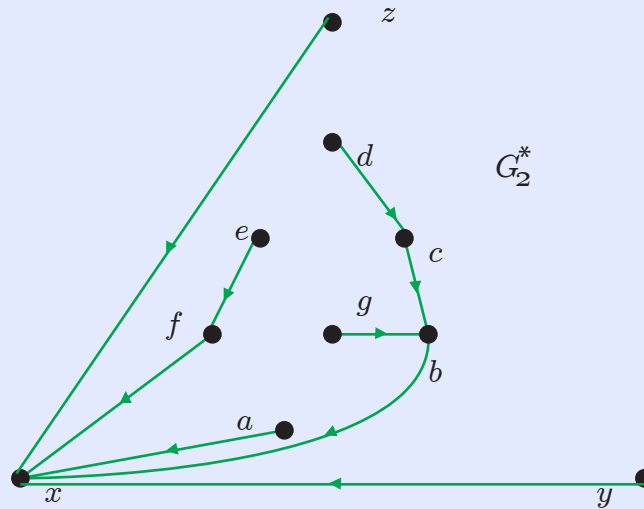
Page 38 of 55

Go Back

Full Screen

Close

Quit





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



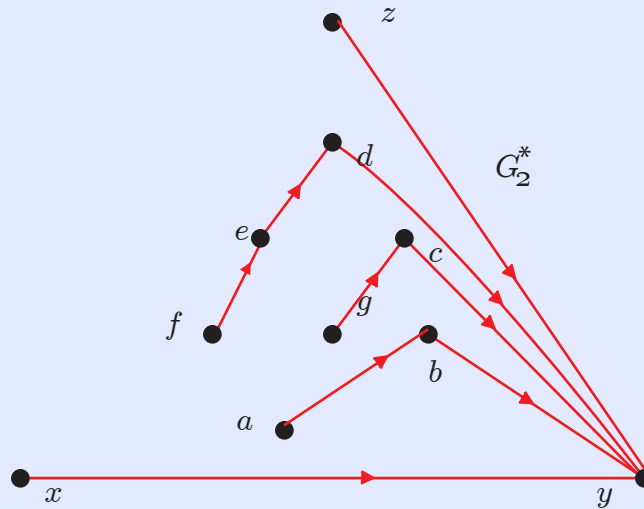
Page 39 of 55

Go Back

Full Screen

Close

Quit





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



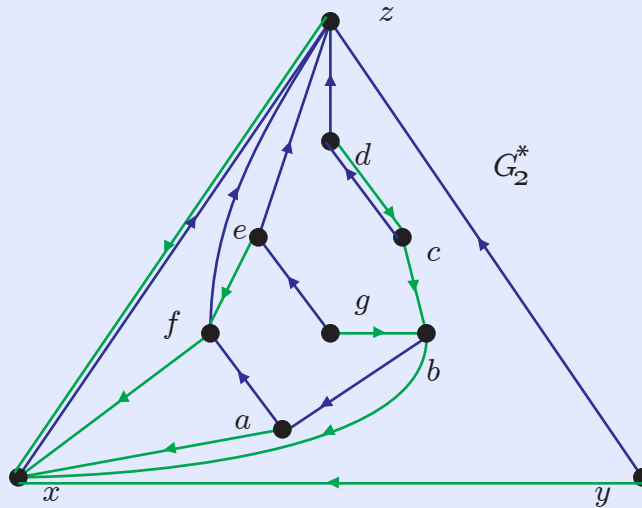
Page 40 of 55

Go Back

Full Screen

Close

Quit







# 8. A bad example

- Pseudo-Triangulating
- Properties
- Definition of CPPT
- Combinatorial CPPT
- Examples
- Directed Tutte method
- Schnyder trees.
- A bad example**
- Generalizations.
- Reciprocal Figures
- Some Examples:

Home Page

Title Page



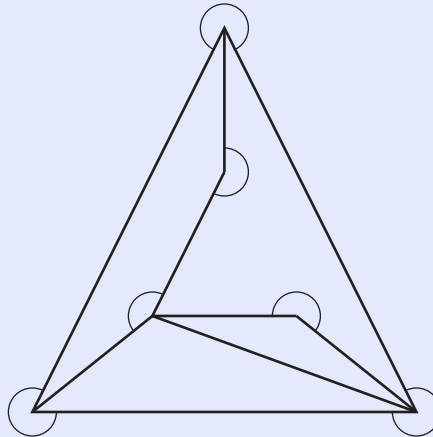
Page 41 of 55

Go Back

Full Screen

Close

Quit





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



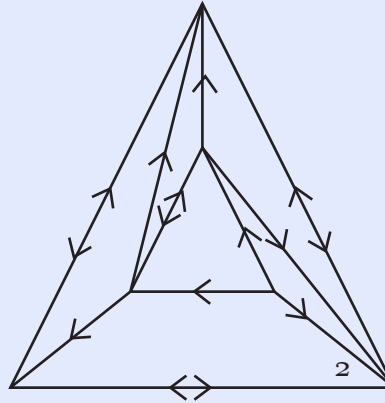
Page 42 of 55

Go Back

Full Screen

Close

Quit





Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

**A bad example**

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



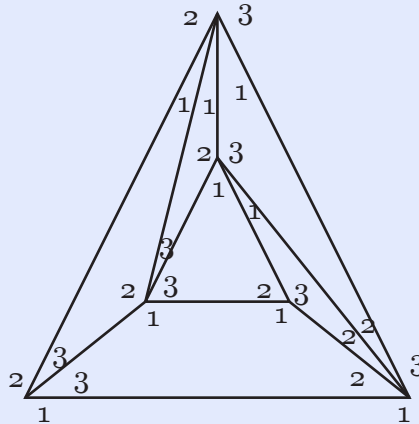
Page 43 of 55

Go Back

Full Screen

Close

Quit





# 9. Generalizations.

*Pseudo-Triangulating*

*Properties*

*Definition of CPPT*

*Combinatorial CPPT*

*Examples*

*Directed Tutte method*

*Schnyder trees.*

*A bad example*

*Generalizations.*

*Reciprocal Figures*

*Some Examples:*

*Home Page*

*Title Page*



*Page 44 of 55*

*Go Back*

*Full Screen*

*Close*

*Quit*



# 10. Reciprocal Figures

We want to draw the geometric dual using the same edge directions.

## Construction

Use a framework with a resolvable stress, non-zero on every edge, for example a cycle in the rigidity matroid. Such a cycle corresponds to a pseudo-triangulation with one non-pointed vertex.

[Home Page](#)

[Title Page](#)



Page 45 of 55

[Go Back](#)

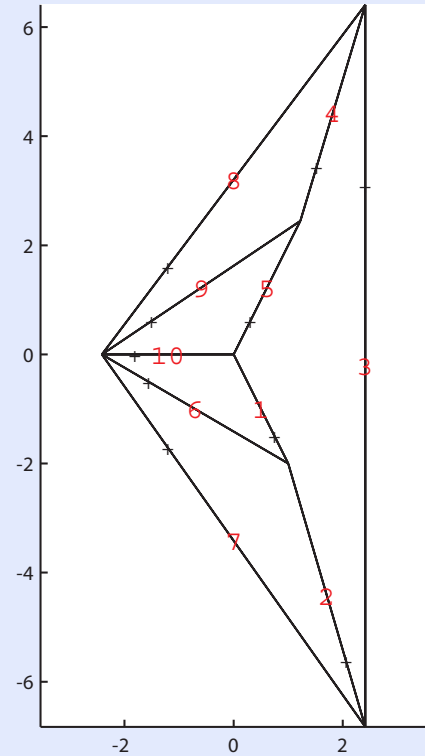
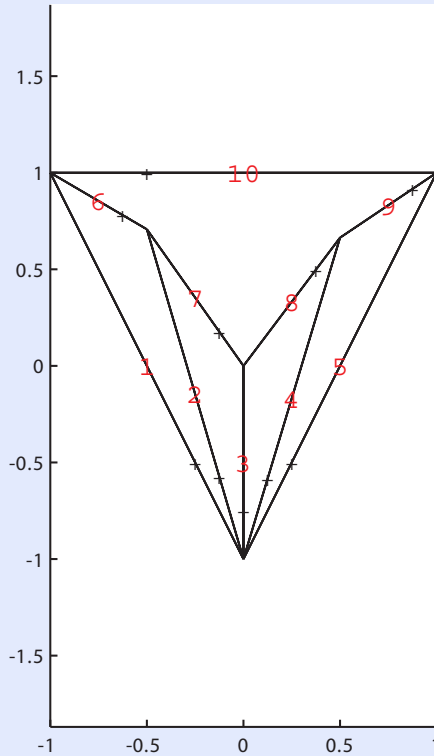
[Full Screen](#)

[Close](#)

[Quit](#)



# A Wheel and Its Reciprocal



**Theorem 4** *If a generic 2-cycle is realized as a pseudo-triangulation, then the reciprocal diagram is also a pseudo-triangulation.*

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 46 of 55

Go Back

Full Screen

Close

Quit



**Lemma 1** *There is, up to rotation, a one to one correspondence between the set of pseudo-triangles  $T$  and the set of PTC cells,  $C(T)$ , such that the vector paths between the distinguished vertices on the boundary of  $C(T)$  are translations and half-turns of the pseudo-arcs of  $T$ .*

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 47 of 55

Go Back

Full Screen

Close

Quit



# proof

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



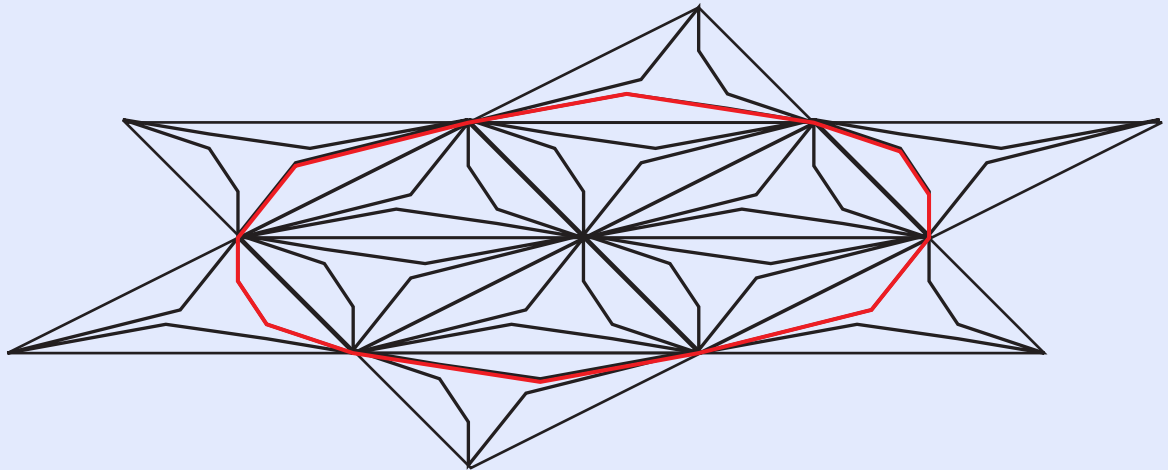
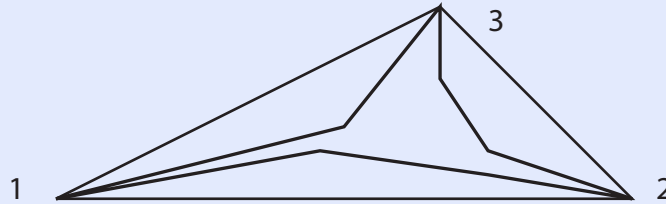
Page 48 of 55

Go Back

Full Screen

Close

Quit







**Lemma 2** *Given a framework, together with a resolvable stress  $s$  and a pseudo-triangle  $T$ . Suppose that  $T$  is a face in the rotation system which governs the reciprocal.*

*The following are equivalent:*

- 1. The cyclic ordering around the reciprocal vertex is the reverse of the cyclic ordering around  $T$ .*
- 2. The reciprocal figure is pointed at the vertex corresponding to  $T$ .*
- 3. There is exactly one improper sign change on the stresses as one reads around  $T$ .*

Pseudo-Triangulating

Properties

Definition of CPPT

Combinatorial CPPT

Examples

Directed Tutte method

Schnyder trees.

A bad example

Generalizations.

Reciprocal Figures

Some Examples:

Home Page

Title Page



Page 49 of 55

Go Back

Full Screen

Close

Quit



# 11. Some Examples:

*Pseudo-Triangulating*

*Properties*

*Definition of CPPT*

*Combinatorial CPPT*

*Examples*

*Directed Tutte method*

*Schnyder trees.*

*A bad example*

*Generalizations.*

*Reciprocal Figures*

*Some Examples:*

*Home Page*

*Title Page*



*Page 50 of 55*

*Go Back*

*Full Screen*

*Close*

*Quit*



# Seven Wheel 1:

*Pseudo-Triangulating*

*Properties*

*Definition of CPPT*

*Combinatorial CPPT*

*Examples*

*Directed Tutte method*

*Schnyder trees.*

*A bad example*

*Generalizations.*

*Reciprocal Figures*

*Some Examples:*

*Home Page*

*Title Page*



*Page 51 of 55*

*Go Back*

*Full Screen*

*Close*

*Quit*



## Seven Wheel 2:

*Pseudo-Triangulating*

*Properties*

*Definition of CPPT*

*Combinatorial CPPT*

*Examples*

*Directed Tutte method*

*Schnyder trees.*

*A bad example*

*Generalizations.*

*Reciprocal Figures*

*Some Examples:*

*Home Page*

*Title Page*



*Page 52 of 55*

*Go Back*

*Full Screen*

*Close*

*Quit*



## Center segment revolves:

*Pseudo-Triangulating*

*Properties*

*Definition of CPPT*

*Combinatorial CPPT*

*Examples*

*Directed Tutte method*

*Schnyder trees.*

*A bad example*

*Generalizations.*

*Reciprocal Figures*

*Some Examples:*

*Home Page*

*Title Page*



*Page 53 of 55*

*Go Back*

*Full Screen*

*Close*

*Quit*

Graph not in a plane embedding.



## Center segment revolves and rotates:

*Pseudo-Triangulating*

*Properties*

*Definition of CPPT*

*Combinatorial CPPT*

*Examples*

*Directed Tutte method*

*Schnyder trees.*

*A bad example*

*Generalizations.*

*Reciprocal Figures*

*Some Examples:*

*Home Page*

*Title Page*



*Page 54 of 55*

*Go Back*

*Full Screen*

*Close*

*Quit*

Graph not in a plane embedding.



# A non-planar reciprocal:

*Pseudo-Triangulating*

*Properties*

*Definition of CPPT*

*Combinatorial CPPT*

*Examples*

*Directed Tutte method*

*Schnyder trees.*

*A bad example*

*Generalizations.*

*Reciprocal Figures*

*Some Examples:*

*Home Page*

*Title Page*



*Page 55 of 55*

*Go Back*

*Full Screen*

*Close*

*Quit*