Matroids on graphs

Brigitte Servatius

Worcester Polytechnic Institute

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1. Matroids

Whitney [?] defined a matroid M on a set E:

$M = (E, \mathcal{I})$
  E is a finite set
  $\mathcal{I}$ is a collection of subsets of E such that

I1 $\emptyset \in \mathcal{I}$;

I2 If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$

I3 If I_1 and I_2 are members of $\mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element e in $I_2 - I_1$ such that $I_1 + e$ is a member of $\mathcal{I}$.

Bases

Because of condition [I2], all of the maximal independent sets have the same cardinality. These maximal independent sets are called the *bases* of the matroid. The bases may be described directly: Let E be a finite set, a nonempty collection $\mathcal{B}$ of subsets of E is called a *basis system* for M if

B1 $\mathcal{B} \neq \emptyset$

B2 For all $B_1, B_2 \in \mathcal{B}$, $|B_1| = |B_2|$

B3 For all $B_1, B_2 \in \mathcal{B}$ and $e_1 \in B_1 - B_2$, there exists $e_2 \in B_2 - B_1$ such that $B_1 - e_1 + e_2 \in \mathcal{B}$.

Condition [B3] is sometimes called the *exchange axiom.* It also has a slightly different but equivalent formulation:

B3′ For all $B_1, B_2 \in \mathcal{B}$ and $e_2 \in B_2 - B_1$, there exists $e_1 \in B_1 - B_2$ such that $B_1 - e_1 + e_2 \in \mathcal{B}$.

Complements of bases also satisfy [B3], these complements are bases of the dual matroid.

Every matroid M has a dual M^*.

Rank

Let M be a matroid on E with independent sets $\mathcal{I}$ and define $r_{(\mathcal{I})}$, a function from the power set of E into the nonnegative integers by $r_{(\mathcal{I})}(S) = \max\{|I| : I \in \mathcal{I}, I \subseteq S\}$. The function $r = r_{\mathcal{I}}$ is called the *rank function of* M.
In general, let E be a finite set and r a function from the power set of E into the nonnegative integers so that

R1 $r(\emptyset) = 0$;

R2 $r(S) \leq |S|$;

R3 if $S \subseteq T$ then $r(S) \leq r(T)$;

R4 $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$;

then r is called a *rank function on* E. If r is a rank function on E we define $\mathcal{I}(r) = \{I \subseteq E \mid r(I) = |I|\}$
Condition [R4] is called the *submodular inequality.*

Cycles

Given a finite set E, we call a collection $\mathcal{C}$ a *cycle system* [?] for E, if the following three conditions are satisfied:

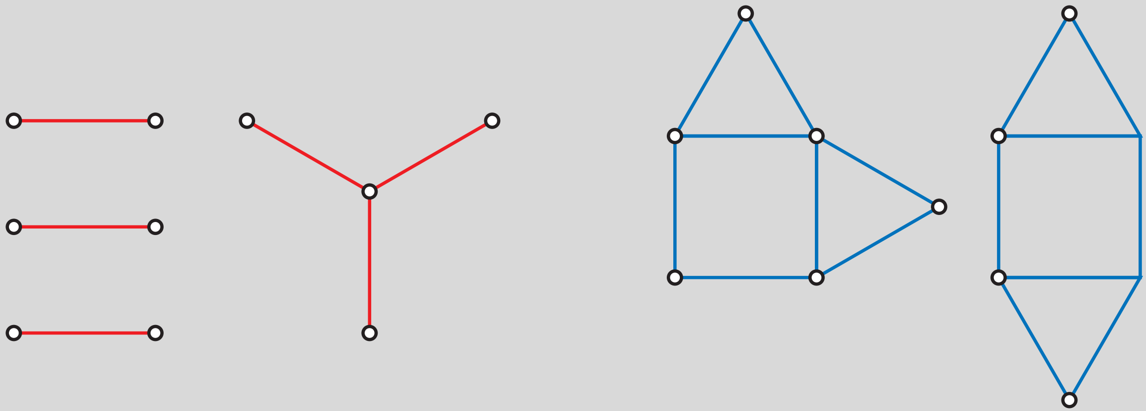
Z1 If $C \in \mathcal{C}$ then $C \neq \emptyset$

Z2 If C_1 and C_2 are members of $\mathcal{C}$ then $C_1 \not\subseteq C_2$

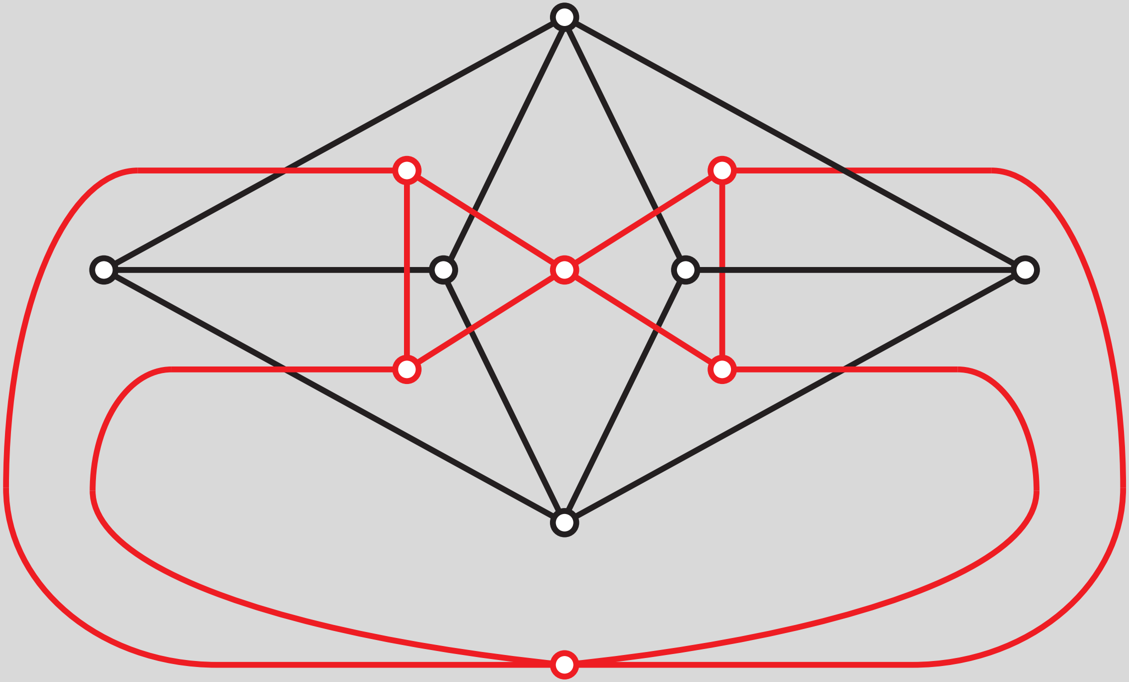
Z3 If C_1 and C_2 are members of $\mathcal{C}$ and if e is an element of $C_1 \cap C_2$ then there is an element $C \in \mathcal{C}$, such that $C \subseteq (C_1 \cap C_2 - e)$.

2. Graphs

A matroid is *graphic* if it is isomorphic to the *cycle matroid* on the edge set E of a graph $G = (V, E)$. Non-isomorphic graphs may have the same cycle matroid, but 3-connected graphs are uniquely determined by their matroids.



M is co-graphic if M^* is graphic.
M is graphic as well as co-graphic if and only if G is planar.
Map duality (geometric duality) agrees with matroid duality.
The facial cycles generate the cycle space.



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Euler's formula

If $G(V, E)$ is planar and connected, its cycle matroid has rank $|V| - 1$, its co-cycle matroid has rank $|F| - 1$, so $|V| - 1 + |F| - 1 = |E|$, i.e.

$$|V| - |E| + |F| = 2$$

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3. Rigidity

framework (in m-space)

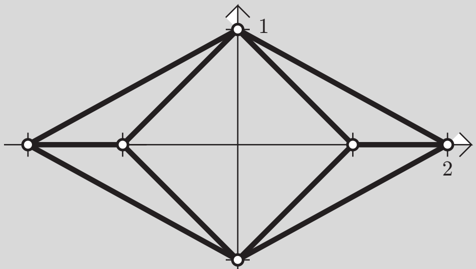
a triple $(V, E, \overrightarrow{p})$,

(V, E) is a graph

$$\overrightarrow{p} : V \longrightarrow \mathbb{R}^m$$

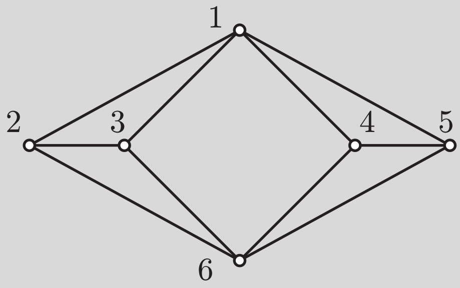
rigid framework

if all solutions to the corresponding system of quadratic equations of length constraints for the edges in some neighborhood of the original solution (as a point in mn-space) come from congruent frameworks.

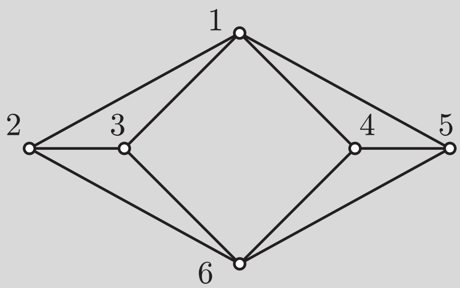


Rigidity Matrix

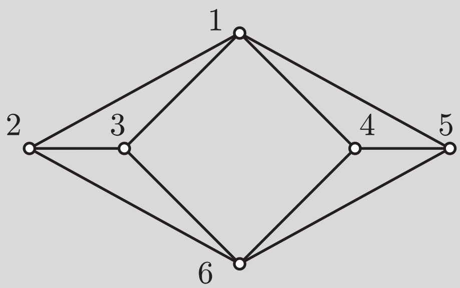
Jacobian of the system



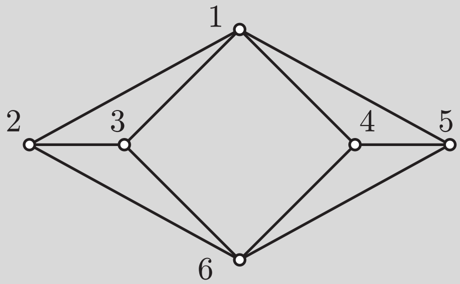
	$\mathbf{p}_1=(0,1)$	$\mathbf{p}_2=(-2,0)$	$\mathbf{p}_3=(-1,0)$	$\mathbf{p}_4=(1,0)$	$\mathbf{p}_5=(2,0)$	$\mathbf{p}_6=(0,-1)$
(1,2)	$(-2,-1)$	$(2,1)$				
(1,3)	$(-1,-1)$		$(1,1)$			
(1,4)	$(1,-1)$			$(-1,1)$		
(1,5)	$(2,-1)$				$(-2,1)$	
(2,6)		$(-2,1)$			$(-2,1)$	$(2,-1)$
(3,6)			$(-1,1)$			$(1,-1)$
(4,6)				$(1,1)$		$(-1,-1)$
(5,6)					$(2,1)$	$(-2,-1)$
(2,3)		$(-1,0)$	$(1,0)$		$(2,1)$	
(4,5)				$(-1,0)$	$(1,0)$	



	$p_{1,x}$	$p_{2,x}$	$p_{3,x}$	$p_{4,x}$	$p_{5,x}$	$p_{6,x}$	$p_{1,y}$	$p_{2,y}$	$p_{3,y}$	$p_{4,y}$	$p_{5,y}$	$p_{6,y}$
(1,2)	-2	2					-1	1				
(1,3)	-1		1				-1		1			
(1,4)	1			-1			-1			1		
(1,5)	2				-2		-1				1	
(2,6)		-2			-2	2		1			1	-1
(3,6)			-1			1			1			-1
(4,6)				1		-1				1		-1
(5,6)					2	-2					1	-1
(2,3)		-1	1		2			0	0		1	
(4,5)				-1	1					0	0	



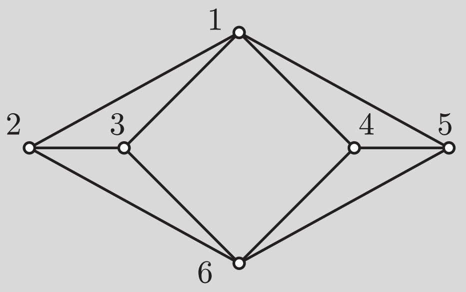
	$p_{1,x}$	$p_{2,x}$	$p_{3,x}$	$p_{4,x}$	$p_{5,x}$	$p_{6,x}$	$p_{1,y}$	$p_{2,y}$	$p_{3,y}$	$p_{4,y}$	$p_{5,y}$	$p_{6,y}$
(1,2)	1	1					-1	-1				
(1,3)	1		1				-1		-1			
(1,4)	1			1			-1			-1		
(1,5)	1				1		-1				-1	
(2,6)		1			1	1		-1			-1	-1
(3,6)			1			1			-1			-1
(4,6)				1		1				-1		-1
(5,6)					1	1					-1	-1
(2,3)		1	1		1			-1	-1		-1	
(4,5)				1	1					-1	-1	



	$p_{1,x}$	$p_{2,x}$	$p_{3,x}$	$p_{4,x}$	$p_{5,x}$	$p_{6,x}$	$p_{1,y}$	$p_{2,y}$	$p_{3,y}$	$p_{4,y}$	$p_{5,y}$	$p_{6,y}$
(1,2)	1	1					1	1				
(1,3)	1		1				1		1			
(1,4)	1			1			1			1		
(1,5)	1				1		1				1	
(2,6)		1			1	1		1			1	1
(3,6)			1			1			1			1
(4,6)				1		1				1		1
(5,6)					1	1					1	1
(2,3)		1	1		1			1	1		1	
(4,5)				1	1					1	1	

Kirchhoff's matrix-tree theorem

Let A be the incidence matrix of a graph G on n vertices. The determinant of an $(n-1)\times(n-1)$ minor of A^TA (the Laplacian matrix of G) counts the number of spanning trees in G.



$$A^TA = \begin{pmatrix} 4 & 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \\ 1 & 1 & 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 & 1 & 1 \\ 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \quad \det \begin{pmatrix} 3 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix} = 192$$

Oldest characterization of 2d-rigidity

Hilda Pollaczek-Geiringer [?] (1927)

In a rigidity matrix with $2k-3$ rows and $2k$ columns no $2k-3$ sub-determinant is identically zero if and only if there is no p-set of columns ($p < 2k-3$) where all elements are zero which these p columns have in common with more than $(2k-3)-p$ rows.

Frobenius [?]

A determinant of order n some of whose elements are zero and the others independent variables is identically equal to zero if and only if there exists at least a group of p rows in which more than $n-p$ columns contain all zeros.

Frobenius didn't think highly of graph theory:

negativ, so verschwinden alle Elemente von C, demnach alle Elemente der $p$ten Spalte, und mithin ist $s = 0$.

Die Theorie der Graphen, mittels deren Hr. KÖNIG den obigen Satz abgeleitet hat, ist nach meiner Ansicht ein wenig geeignetes Hilfsmittel für die Entwicklung der Determinantentheorie. In diesem Falle führt sie zu einem ganz speziellen Satze von geringem Werte. Was von seinem Inhalt Wert hat, ist in dem Satze II ausgesprochen.

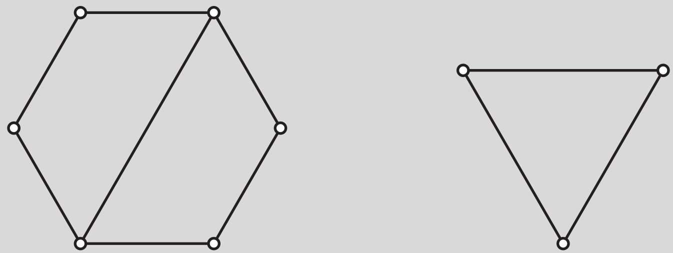
From Whitney's original paper [?]

THEOREM 28. Let H be a hyperplane through the origin in E^n, of dimension r, and let H' be the orthogonal hyperplane through the origin, of dimension $n - r$. Let M and M' be the associated matroids. Then M and M' are duals.

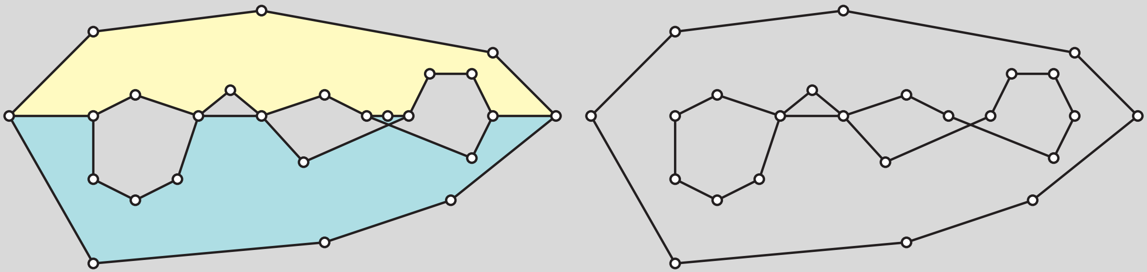
Problem 2 on page 27 of [?]

Given an arbitrary collection $\mathcal{D}$ of incomparable subsets of E does there exist a matroid M which has a circuit set

$$\mathcal{C}(M) \supseteq \mathcal{D}?$$



Cycle axioms for graphs

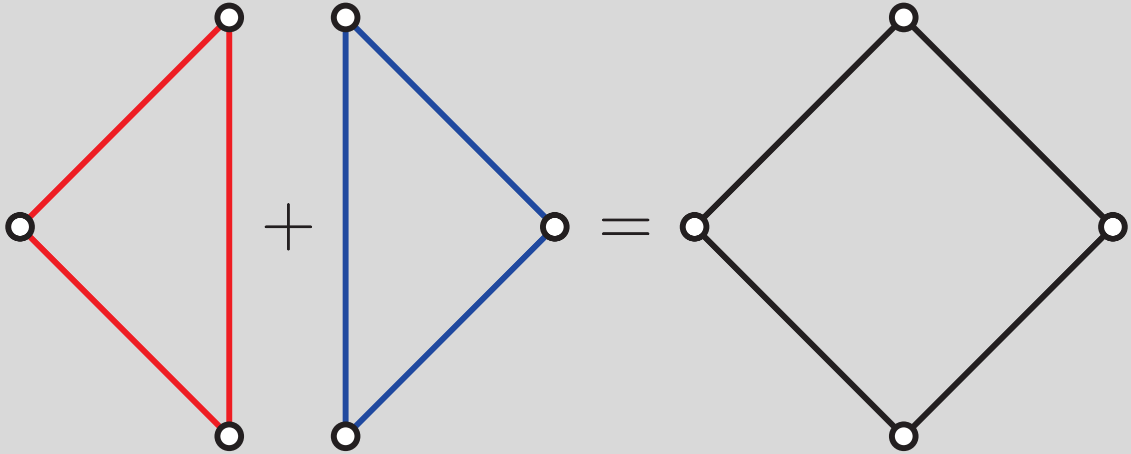


$C_1 \triangle C_2$ is the edge disjoint union of cycles.

4. Matroids on K_n

Wanted:

The matroid of largest possible rank that contains a specified set of graphs as dependent sets.



Theorem 1 *The unique maximal matroid on K_n containing all triangles as cycles is the cycle matroid.*

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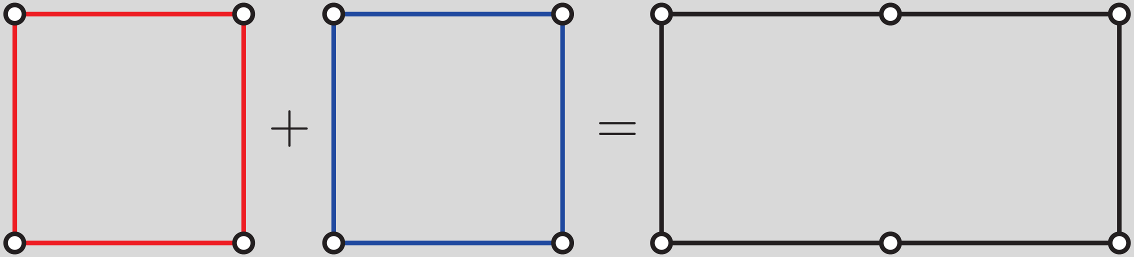
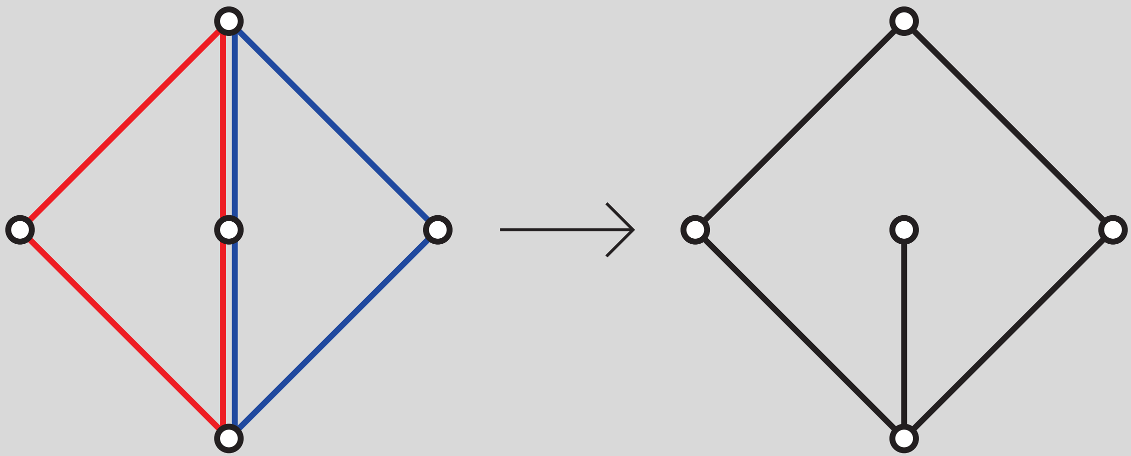
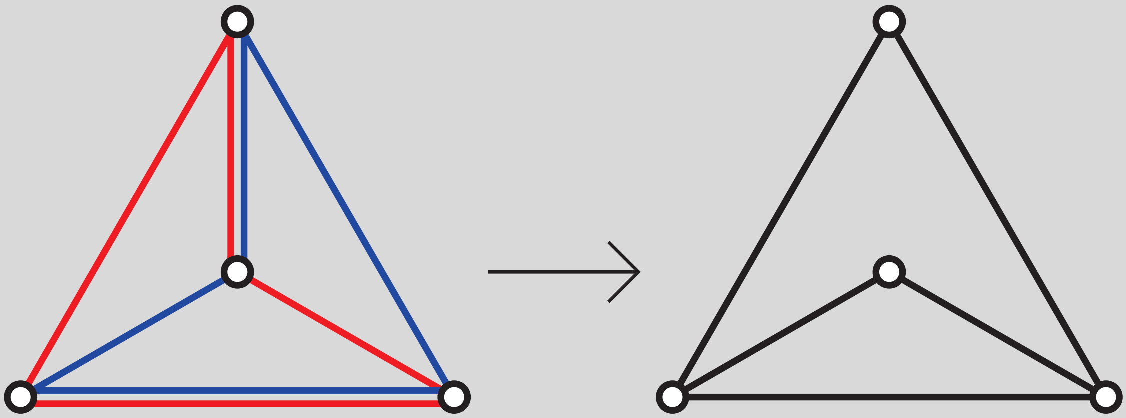
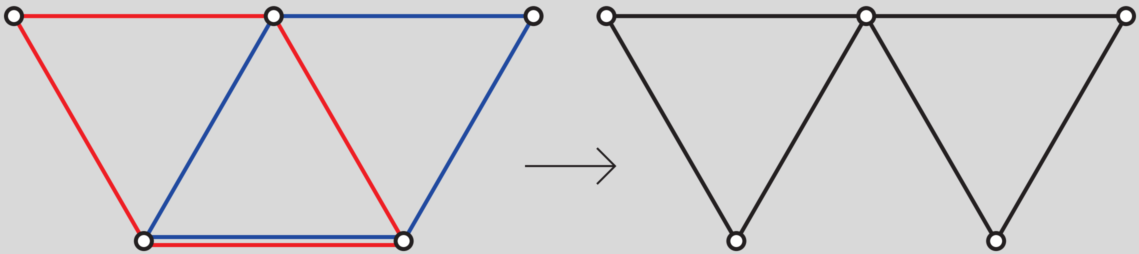
What do we get if we want the maximal matroid on K_n containing all 4-gons?









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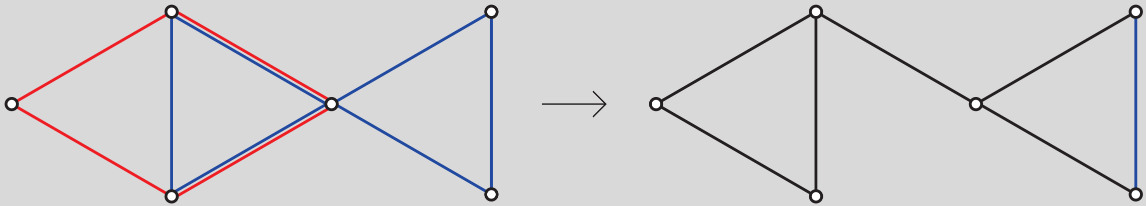
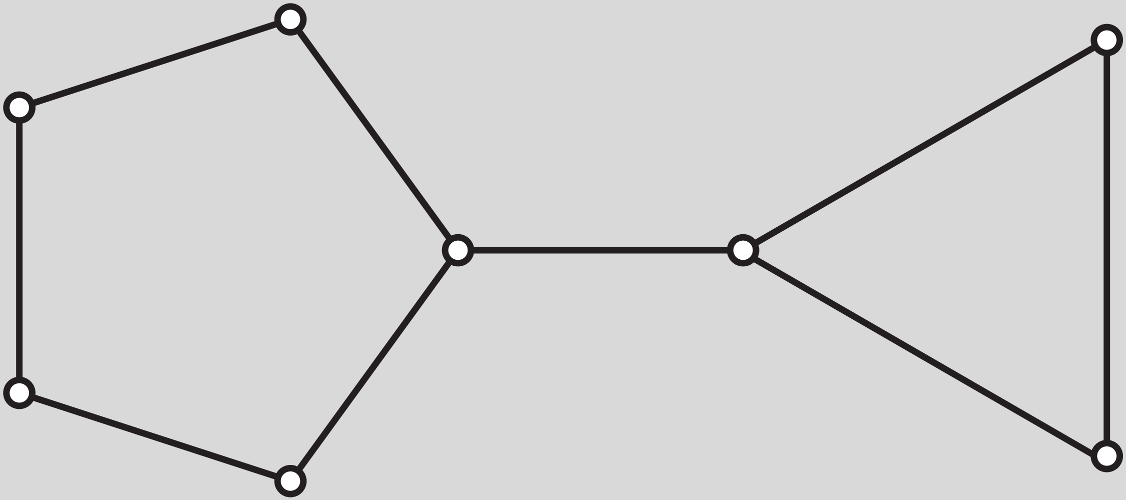
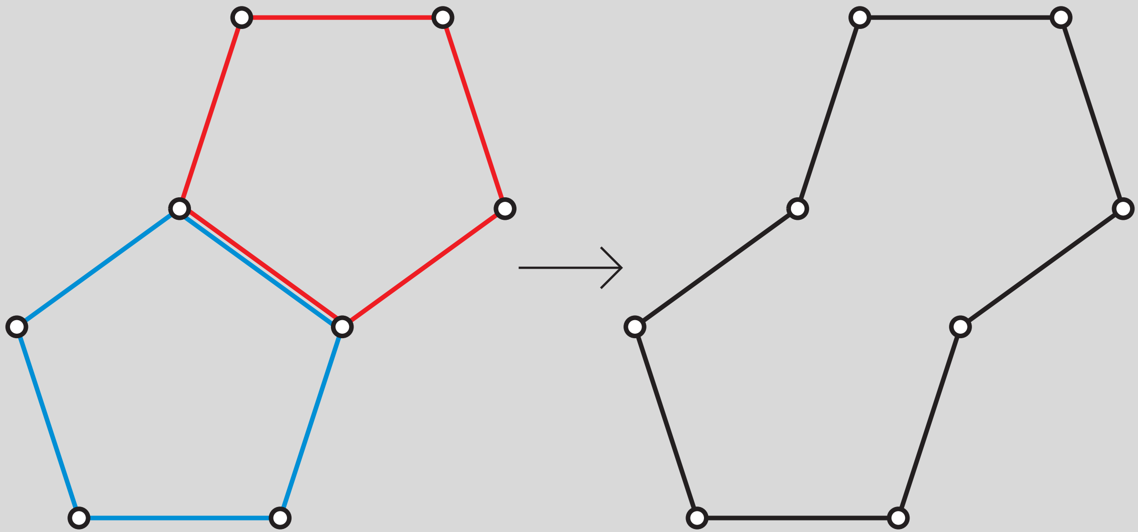
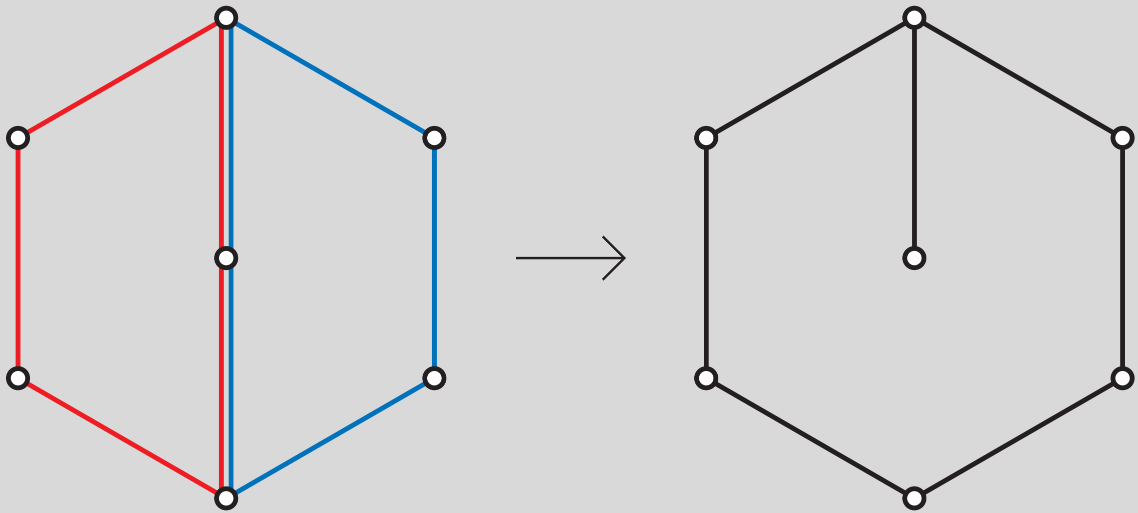


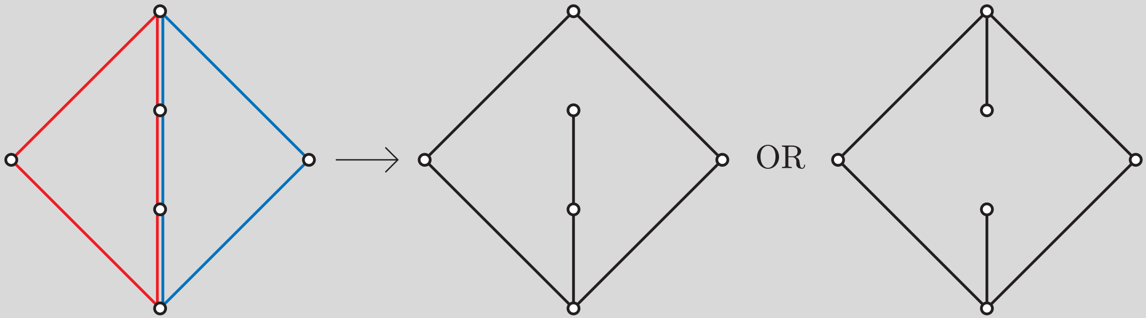


The set of cycles consists of all even cycles and odd dumbbells.

What do we get from pentagons?



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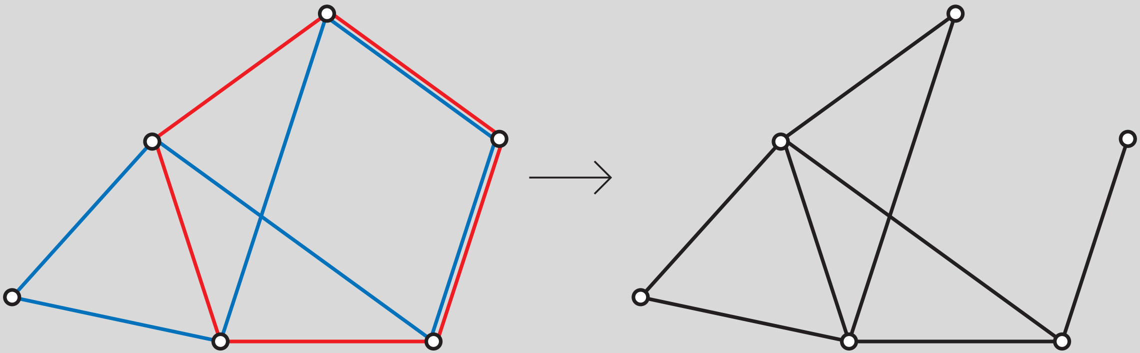

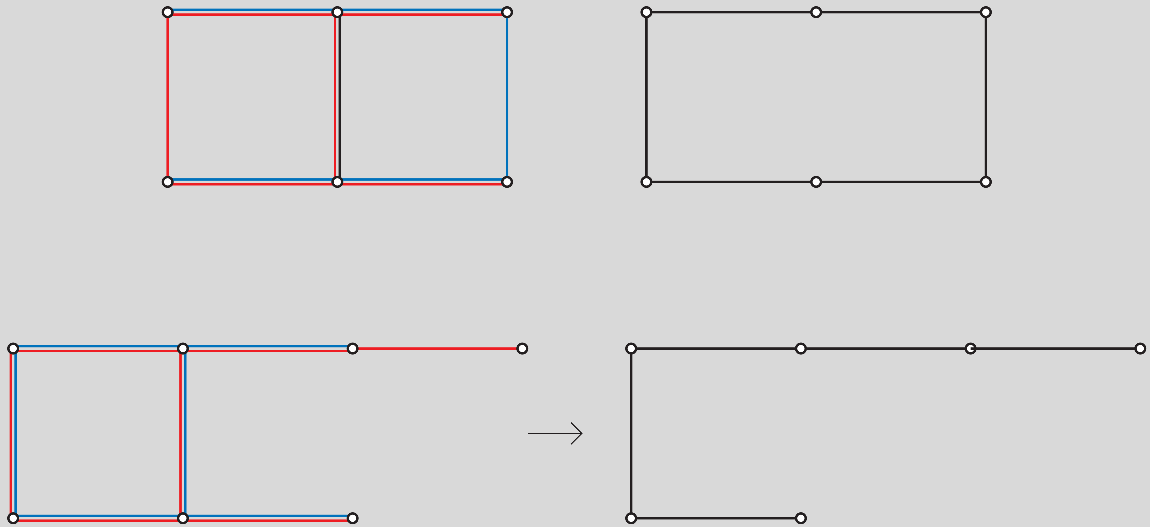

A hexagon is dependent. A path of length 6 is dependent. The rank is bounded by 7!

Theorem 2 *The unique maximal matroid on K_n containing all tetrahedra as cycles is the 2d-rigidity matroid.*

Conjecture 1 *The unique maximal matroid on K_n containing all K_5's as cycles is the 3d-rigidity matroid.*

5. Geometry

Oxley [?] emphasizes matroids coming from geometries.

Oriented matroids [?] come from hyperplanearrangements. Interesting new applications are plentiful [?].

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