On the Rigidity of Ramanujan Graphs

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1 Introduction

A graph G is generically [4] rigid in dimension one if and only if it contains a spanning tree, that is, a spanning subgraph assembled by inductively joining 1-simplices along 0-simplices. The analogous property is sufficient but not necessary for the generic rigidity of graphs in higher dimensions, that is, a generically rigid graph in $\mathbb{R}^n$ need not contain a spanning subgraph consisting of n-simplices joined along $(n-1)$-simplices, see Figure 1a. Indeed, a graph which is generi-

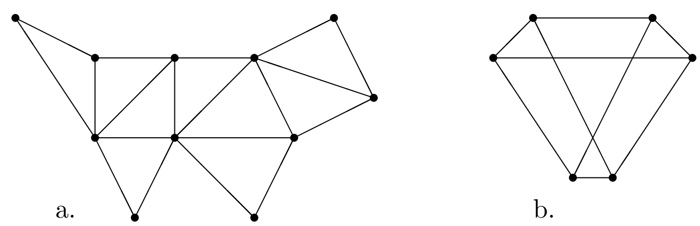

Figure 1:

cally rigid in the plane need not contain any triangles at all. For example the graph in Figure 1b, $K_{3,3}$, is generically isostatic in $\mathbb{R}^2$, and its shortest cycle is of length four. Observe further that the graph in Figure 1a behaves tree-like with respect to rigidity in the sense that the removal of any single edge cuts the graph into two rigid components. In sharp contrast, the removal of any edge of $K_{3,3}$ produces a graph of degree of freedom one in which any edge can move nontrivially relative to any other edge, which is to say that the set of maximal rigid subgraphs equals the edge set.

The two graphs in Figure 1 also behave quite differently with respect to the addition of a single edge. For the graph in Figure 1a the addition of an edge yields minimally dependent sets of various sizes depending on where it is placed. On the other hand, the addition of any edge to $K_{3,3}$ produces a single minimally dependent edge set comprising all 10 edges, i.e. the graph can be globally reinforced by the addition of a single edge.

It is our aim to construct rigid graphs of large girth and show that they possess the featured properties of $K_{3,3}$.

2 The Ramanujan Graph $X^{p,q}$

The length of the shortest cycle in a graph is called the *girth* of the graph. If we fix the number of vertices and try to construct an edge maximal graph of large girth, we expect the connectivity to be low which tends to produce non-rigidity. A graph theoretic concept that might be more intimately related to rigidity than connectivity is toughness. A graph is *t-tough* if the removal of at least tx vertices is necessary to disconnect the graph into x connected components (where $x > 1$). Note that t-toughness implies 2-t connectivity but the reverse implication is not true.

We now describe the construction of a class of Cayley graphs given in [7]: Let p and q be primes, $p \equiv q \equiv 1 (\mathrm{mod}\ 4)$. $X^{p,q}$ will be a $(p+1)$-regular graph, namely the Cayley graph of $\mathrm{PSL}(2,q)$ if $(\frac{p}{q}) = 1$, (where $(\frac{p}{q})$ is the Legendre symbol) and $\mathrm{PGL}(2,q)$ if $(\frac{p}{q}) = -1$. The generators correspond to the $p+1$ ways of presenting p as a sum of four squares under the following normalizing conditions: $p = a_0^2 + a_1^2 + a_2^2 + a_3^2$ (with $a_0 > 0$, a_0 odd and a_j even for $j \in \{1,2,3\}$.)

The number of representations of integers by certain quarternary quadratic forms is needed in the construction and in the proofs that the constructions work. Progress on one of Ramanujan's conjectures was a necessary ingredient in the work of Lubotzky, Phillips and Sarnak, [7] hence the name Ramanujan graph was chosen by them. Ramanujan graphs possess, among other nice extremal properties large chromatic number, incidence number and girth, $g > 2\log_p(q)$, and good expansion properties. The second largest eigenvalue of their adjaceny matrix equals $2\sqrt{p}$.

In [1] an explicit proof is given that the toughness t of $X^{p,q}$ satisfies

$$t > \frac{1}{3}\left(\frac{(p+1)^2}{\sqrt{p}(p+1)+p} - 1\right)$$

Therefore we can choose p large enough so that $t > 3$. Choose q large enough so that $2\log_p(q) \geq g$. Then $X^{p,q}$ will be 3-tough, therefore 6-connected, hence generically rigid in $\mathbb{R}^2$ by [6], and of girth at least g by the bounds in [7]. We have proved the following.

Theorem 2.1 *Given a natural number g, there exists a graph which is generically rigid in the plane and has girth at least g.*

While upper and lower bounds of $X^{p,q}$, see [2, 7], are quite close, the bound on the toughness is not tight. Looking for a triangle free rigid graph in the plane using these bounds we would need to construct $X^{401,q}$ where q is a prime number larger than $401^3 = 64,481,201$, so the number of vertices is on the order of 10^{23}, (Avogadro's number.) Note that $K_{3,3}$ does the job with only 6 vertices. Thus it would be of great interest to study the rigidity properties of the Ramanujan graphs directly.

3 An Example: $X^{5,13}$

We now construct the Ramanujan graph $X^{5,13}$. There are $8(p{+}1) = 48$ solutions to $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 5$, with 6 of them having the property that $a_0 > 0$ and a_1, a_2, a_3 even and a_0 odd. To each of these solutions α we associate a matrix $\tilde{\alpha}$ in PGL(2,q) as follows:

$$\tilde{\alpha} = \begin{bmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{bmatrix}$$

where $i^2 \equiv -1 (\mathrm{mod}\ 13)$.

$$\alpha_1 = (1, 0, 0, -2) \qquad \tilde{\alpha}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\alpha_2 = (1, 0, -2, 0) \qquad \tilde{\alpha}_2 = \begin{bmatrix} 1 & 11 \\ 2 & 1 \end{bmatrix}$$

$$\alpha_3 = (1, -2, 0, 0) \qquad \tilde{\alpha}_3 = \begin{bmatrix} 4 & 0 \\ 0 & 11 \end{bmatrix}$$

$$\alpha_4 = (1, 0, 0, 2) \qquad \tilde{\alpha}_4 = \begin{bmatrix} 1 & 10 \\ 10 & 1 \end{bmatrix}$$

$$\alpha_5 = (1, 0, 2, 0) \qquad \tilde{\alpha}_5 = \begin{bmatrix} 1 & 2 \\ 11 & 1 \end{bmatrix}$$

$$\alpha_6 = (1, 2, 0, 0) \qquad \tilde{\alpha}_6 = \begin{bmatrix} 11 & 0 \\ 0 & 4 \end{bmatrix}$$

The six matrices $\tilde{a_i}$ are the generators for $X^{5,13}$, the Cayley graph of the group PGL(2,13). This Cayley graph is bipartite and has $n = q(q^2 - 1) = 2,184$ vertices. It is 6-regular and hence has 6,552 edges. The rigidity matrix in $\mathbb{R}^3$ is a square matrix of size 6,552. It follows that $X^{5,13}$ is generically dependent in $\mathbb{R}^3$.

In [9], $X^{5,13}$ was randomly embedded in $\mathbb{R}^3$ and the rank of the corresponding rigidity matrix was computed to be 6,546, which shows that $X^{5,13}$ is generically rigid in dimension three. The girth was computed to be 8. (The theory of cages [8] yields that the girth is at most 10, the bounds from [7] and [2] imply that 6 and 8 are the only possible values.) $X^{5,13}$ is not only rigid, it remains rigid even after the removal of any two vertices, or after the removal of six "random" edges.

4 Open Problems

Is $X^{5,q}$ rigid (vertex birigid) for all q? If one could show that they are Hamiltonian, in fact, possibly even the union of three disjoint Hamiltonian cycles, one might be able to use the 6T3 decompositions obtained by deleting 6 of the edges of the graphs (avoiding the removal of of more than 3 incident with one vertex, to show rigidity.

Is there a realization of $X^{5,q}$ in $\mathbb{R}^3$ such that the ratio of the longest to shortest edge is small, and the ratio of the diameter to the length of the longest edge is large?

An example of an embedding of a regular vertex birigid graph in 2-space is the following: $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, $E = \{(i, (i+3) \bmod n\} \cup \{(i, (i+1) \bmod n\}$. If the vertices are embedded on a regular polygon, the graph is realized with two edge lengths and, as n approaches infinity, the ratio of the diameter to either of these edge lengths approaches infinity also, while the ratio of the longest to shortest length approaches 3. These graphs are the edge disjoint union of two Hamiltonian cycles, as indicated by the thick and thin edges of Figure 2. One can use this partition to quickly get a 3T2 decomposition of the

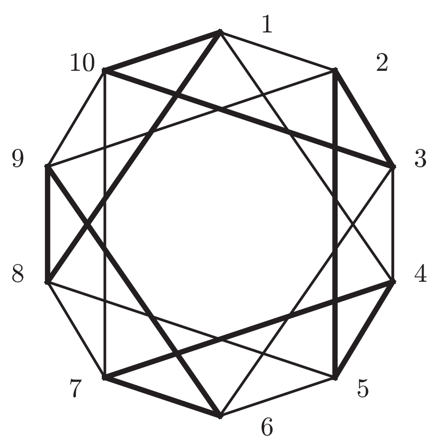

Figure 2: A Decagon

graph (after the deletion of three non-mutually-incident edges).

Recentley, in [5], an algorithm was published which generates random k-regular graphs on n vertices quickly. Is a random 6 regular graph rigid with probability converging to 1 as n goes to infinity? Given a random embedding of a 6-regular graph, can anything be said about the proportion of long edges to short edges as described in the previous problem?

Given t and g, one can construct a graph which is $2t$-connected (in fact t-tough) and has girth at least g. For $t = 3$ this provides a class of rigid graphs in $\mathbb{R}^2$ which has arbitrarily large girth. Is $t = 3$ best possible for $\mathbb{R}^2$? What t works for the same result in $\mathbb{R}^3$?

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