Planar Rigidity

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Contents

Basic notation

Set *union*, set *intersection* and set *inclusion* are denoted by $\cup$ and $\cap$ and $\subseteq$ respectively. Some,times we also use $+$ instead of $\cup$.

The *empty set* is denoted by $\emptyset$.

$A - B$ denotes the set difference of A and B.

$A \Delta B$ denotes the symmetric difference of A and B.

$$A \Delta B = (A - B) \cup (B - A).$$

Sometimes when it is clear from the context that we are referring to a set rather than an element. We abbreviate $\{x\}$ to x. For example $X - x$ means $X - \{x\}$.

$|A|$ denotes the cardinality of A.

$P[A]$ denotes the collection of subsets of A.

$\binom{n}{m}$ denotes the *binomial coefficient* "n choose m".

$\times$ denotes the cartesian product.

Introduction

Graph Theory

A *graph* G consists of a finite, non-empty set $V = V(G)$ of *vertices* together with a set $E = E(G)$ of *edges*, E disjoint from V, together with a function $f = E \to V \bar{\times} V$, where $V \bar{\times} V$ is the set obtained from $V \times V$ by identifying every pair (a, b) with (b, a). We will denote elements of $V \bar{\times} V$ by (a, b). noting that $(a, b) = (b, a)$, and identify an edge $e \in E$ with its image under f, writing $e = (u, v)$ instead of $f(e) = (u, v)$. The vertices u and v are called the *endpoints* of e. An edge of the form $e = (x, x)$ is called a *loop*. Edges with common endpoints which are not loops are called *parallel*. Two edges are said to be *incident* if they have a common endpoint. Two vertices u and v are said to be adjacent if they are the endpoints of an edge.

A *simple graph* is a graph without loops or parallel edges.

The *valence* of a vertex v is the number of edges of G having v as an endpoint. The set of edges having v as an endpoint is denoted by $\mathrm{star}(v)$.

A vertex u of G is called *isolated* if its valence is zero. The edge (u, v) is called *pendant* if exactly one of u or v has valence one.

Two graphs G and G' are *isomorphic* if there is a bijection $\phi : V(G) \to V(G')$ between their vertex sets which preserves adjacency.

If $G = (V, E)$ and F is a subset of E, then the *support* of F, denoted by $\sigma(F)$, is the set of endpoints of edges in F.

A subgraph of a graph $G = (V, E)$ is a pair (U, F), where U is contained in V and F is contained in E, such that $\sigma(F)$ is contained in U. We call $(\sigma(F), F)$ the subgraph *generated* by F. We shall often simply write F for the subgraph generated by F, when it is clear from the context.

If $H = (\sigma(F), F)$ is the subgraph of G generated by the subset F of E, then $G - H$ will denote the subgraph $(\sigma(E-F),(E-F))$. Using the above convention we sometimes write $G - F$, or $G - e$, where e is an edge.

For any subset U of V, the maximal subgraph of $G = (V, E)$ on the vertex set U is called the subgraph *induced* by U. If v is a vertex of V, $G-v$ will denote the subgraph of G induced by $V - v$. $G - v$ is also the subgraph generated by $E - \mathrm{star}(v)$. We therefore sometimes write $G - \mathrm{star}(v)$ instead of $G - v$. If U is a subset of $V(G)$, we let $G - U$ denote the subgraph of G generated by $V - U$.

A *path* on G is a finite sequence of distinct edges of the form

$$(v_0, v_1), (v_1, v_2), \ldots (v(n-1), v(n)).$$

The *length* of this path is n and it is said to *connect* v_0 and $v(n)$. A *cycle* is a path in which $v_i \neq v_j$ for $i \neq j$, except that $v_0 = v(n)$.

If we define a relation # on $V(G)$ by setting $x\#y$ if either $x = y$ or there is a path in G joining x and y, then it is easy to show that # is an equivalence relation on $V(G)$. The induced subgraphs on the distinct equivalence classes are called the *connected components* of G. G is *connected* if there is only one connected component. A subgraph of G is a *spanning subgraph* if it has vertex set $V(G)$. A tree is a connected graph which has no cycles. A *spanning tree* is a spanning subgraph which is a tree. It is easy to prove that G is connected if and only if it has a spanning tree, and that a spanning tree must have $|V(G)|-1$ edges. A connected graph is n*-connected* for some positive integer n if there exists no subset U of vertices, with $|U| < n$, such that $G - U$ is disconnected or a single vertex.

A vertex u of a connected graph G, such that $G - u$ is disconnected is called a *cut vertex*. A subset U of the vertex set of G is called a *cutset* if $G - U$ is disconnected. A *cocycle* is a minimal cutset.

If F is a subset of edges, the subgraph generated by F will also be called the *restriction* of G onto F, denoted by $G \mid F$, or just F if it is clear from the context.

We say that a graph G is *contractible* to a graph H if H can be obtained from G by a finite sequence of elementary contractions, where an *elementary contraction* is either one of the operations:

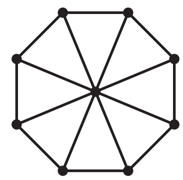
(a) deleting a loop, or

(b) identifying the endpoints of an edge which is not a loop.

Finally, we want to list special names for frequently encountered graphs.

A *complete* graph is a simple graph in which each pair of vertices is adjacent. The complete graph on n vertices is denoted by $K(n)$. $K(3)$ is also called a *triangle*. $K(4)$ a tetrahedron.

A *wheel* $W(n)$ of order $n \geq 3$ is obtained from the circuit of length n, called the *rim*, by adjoining a new vertex, called the *hub*, and then setting every vertex of the rim adjacent to the hub by a single edge called a *spoke*.

$W(8) =$ 

Matroid Theory

Matroid theory has exactly the same relationship to linear algebra as does point set topology to the theory of real variables. That is, it postulates certain subsets to be "independent" and develops a fruitful theory from a set of independence axioms.

— **Axiom systems for a matroid.** A *matroid* M is a finite set S together with a collection $I[M]$ of subsets of S, called *independent* sets, such that the following are satisfied:

(I1) $\emptyset \in I[M]$,

(I2) if $x \in I[M]$ and $Y \subseteq X$, then $Y \subseteq I[M]$, and

(I3) if $U, V \in I[M]$ and $|U| = |V| + 1$, then there exists an element $x \in U - V$ such that $V \cup \{x\} \in I[M]$.

A subset of S not belonging to $I[M]$ is called *dependent*.

If $\{x\}$ is dependent, it is called a *loop*. If $\{x, y\}$ is dependent and x and y are not loops, x and y are called *parallel*. A *simple matroid* is a matroid without loops or parallel elements.

A *base* of M is a maximal independent subset of S. The collection of bases is denoted by $B[M]$.

The *rank function* of a matroid is a function $r : P[S] \to \mathbb{Z}$ defined by

$$r(A) = \max\{|X| : X \subseteq A, X \in I[M]\}$$

The *rank of the matroid* M, denoted by $r(M)$, is the rank of the set S.

A subset $A \subseteq S$ is *closed* if for all $x \in S - A$, $r(A + x) = r(A) + 1$. If for $x \in S$ and $A \subseteq S$ we have $r(A + x) = r(A)$, then we say that x *depends* on A.

A closed subset H of S with $r(H) = r(S) - 1$ is called a *hyperplane*.

We define the *closure operator* of the matroid M to be the function $c : P[S] \to P[S]$, where $c(A)$ is the set of elements of S which depend on A.

A subset X is *spanning* in n if it contains a base.

A *circuit* of M is a minimal dependent set. We denote the collection of circuits by $c[n]$.

The knowledge of bases, circuits, rank function, or closure operator is sufficient to uniquely determine the matroid. There exist axiom systems for a matroid in terms of each of these concepts.

- Base Axioms

(**B1**) If $B, B' \in B[M]$ and $x \in B - B'$, then there exists $y \in B - B'$ such that $(B + y) - x \in B[M]$.

- Rank Axioms

(**R1**) $r(\emptyset) = 0$.
(**R2**) $r(X) \leq r(X + y) \leq r(X) + 1$.
(**R3**) If $r(X + y) = r(X + x) = r(X)$, then $r(X + x + y) = r(X)$.

- Closure Axioms

(**C1**) $X \subseteq c(X)$,
(**C2**) If $Y \subseteq X$, then $c(Y) \subseteq c(X)$,
(**C3**) $c(X) = c(c(X))$,
(**C4**) If y is not an element of $c(X)$, but $y \in c(X + x)$, then $x \in c(X + y)$.

- Circuit Axioms

(**D1**) If C and C' are distinct circuits in $C[M]$, then C is not contained in C'.
(**D2**) If C and C' are distinct circuits in $C[M]$ and z is an element of both C and C', then there exists $D \in C[M]$ such that $D \subseteq [(C \cup C') - z]$.

The proof that all these axiom systems are equivalent can be found in Welsh [1976], [**26**].

We shall often use the sub modular inequality

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y),$$

which can easily be deduced from the rank axioms.

From the circuit axioms it is easy to deduce that, if B is a base of M on S and $x \in S - B$, then there exists a unique circuit containing x in $B + x$, the so-called *fundamental circuit* of x in the base B.

— **Duality.** Most of the concepts introduced in section 1 were familiar from linear algebra. The concept of matroid duality is less familiar, but is of fundamental importance in the applications of matroids to combinatorial theory.

If $\{B_i \mid i \in J\}$, where J is some index set, is the set of bases of a matroid M on S, then $\{S - B_i \mid i \in J\}$ is the set of bases of a matroid M^* on S, Whitney [1935], [**28**]. We call M^* the *dual matroid* of M.

The following properties of M^* follow immediately from its definition:

(1) $(M^*)^* = M$.

(2) A subset X of S is independent in M^* if and only if $S - X$ is spanning in M.

(3) An element $x \in S$ is a loop of M if and only if x belongs to every base of M^*.

(4) The rank of M^* is $|S| - r(M)$.

Since M^* determines M uniquely, we have the obvious result that a matroid is uniquely determined by its *cobases*, *cocircuits*, etc. Moreover, to every statement about a matroid. there is a dual statement, which we denote by an asterisk. For example:

3^* An element $x \in S$ is a coloop of M if and only if x belongs to every cobase of M^*.

The next two theorems link M and M^*. Proofs may be found in Welsh [1976] [**26**].

Theorem. *A subset X of S is a basis of a matroid M iff X has non-empty intersection with every circuit of M and is minimal with respect to that property. C^* is a circuit of M^* iff $S - C^*$ is a hyperplane of M.*

Theorem. *A set $C[M]$ of subsets of S is the set of circuits of a matroid M on S iff the members of $C[M]$ are minimal nonempty subsets of S such that $|C \cap C^*|1$ for every cocircuit C^* of M.*

— **Minors.** A matroid M on S induces two matroids on a subset T of S which, we shall see, correspond in a natural way to subgraphs of a graph obtained by the operations of deletion and contraction of edges. If $I[M]$ is the set of independent sets of M on S and T is a subset of S, let

$$I[M \mid T] = \{X \mid X \subseteq T, X \in I[M]\},$$

$I[M \mid T]$ is the set of independent sets of a matroid on T. We denote this matroid by $M \mid T$ and call it the *restriction* of M to T. The following properties of $M \mid T$ are easily checked:

(1) The rank function of $M \mid T$ is the restriction of r to T.

(2) $M \mid T$ has as its circuits all circuits of M which are contained in T.

The second matroid on T, $M \cdot T$, the *contraction* of M to T, is obtained as follows:

Define $I[M \cdot T]$ to be those subsets X of T such that there exists a maximal independent subset Y of $S - T$ in M such that $X + Y \in I[M]$.

From the definition of $M \cdot T$ we easily get

(1) $r \cdot (A) = r(A + (S - T)) - r(S - T)$. Here $r\cdot$ denotes the rank function in $M \cdot T$. In particular, the rank of the matroid $M \cdot T$ is $r(S) - r(S - T)$.

(2) The circuits of $M \cdot T$ are the minimal non-empty intersections of circuits of M and T.

— **A Decomposition Theorem.** If C is a circuit of a matroid M on S, then $r(C) = |C| - 1$. If C_1 and C_2 are two circuits, then the *submodular inequality* gives an upper bound for the rank of the union of C_1 and C_2:

$$r(C_1 \cup C_2) + r(C_1 \cap C_2) \leq |C_1| + |C_2| - 2. \tag{1}$$

If C_1 and C_2 are distinct, $r(C_1 \cap C_2) = |C_1 \cap C_2|$, since proper subsets of circuits are independent, and we can rework (1) as:

$$r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 2. \tag{2}$$

In case equality holds in (2), there is a nice description of the circuits of $M \mid (C_1 \cup C_2)$.

Theorem 1. *If C_1 and C_2 are distinct circuits of a matroid M on S such that*

$$r(C_1 \cup C_2) = |C_1 \cup C_2| - 2,$$

there exists a partition of $C_1 \cup C_2 = T$, $T = \bigcup T_i$, $i = 1, \ldots, m$ such that $\{T - T_i\}$ is the collection of circuits of $M \mid T$.

Proof. If C_1 and C_2 are disjoint, then $T = C_1 \cap C_2$ is the desired decomposition.

If the intersection of C_1 and C_2 is non-empty, then by the second circuit axiom, there exists for each x in the intersection of C_1 and C_2 a circuit C_x which is contained in $C_1 \cup C_2$ and which does not contain x. Let $T_x = T - C_x$. We have to show that if T_x and T_y are not identical, then their intersection is empty. Then the collection $\{T(x)\}$ together with $C_1 - C_2$ and $C_2 - C_1$ will give the desired decomposition of T.

Consider the circuits $C_x = T - T_x$ and $C_y = T - T_y$. The symmetric difference $C_1 \Delta C_2$ is contained in the intersection of C_x and C_y, since T_x and T_y are subsets of $C_1 \cap C_2$. Pick an element z in $C_1 \Delta C_2$, w.l.o.g. $z \in C_1$. Note that $T - z$ contains only one circuit, namely C_2. Since $z \in C_x \cap C_y$, there is a circuit C_z contained in $C_x \cup C_y$ which does not contain z. By the above observation, this circuit must be C_2. By symmetry, C_1 also must be contained in $C_x \cup C_y$. Hence $C_x \cup C_y$ is T, which implies that $T_x \cap T_y$ is empty. $\square$

— **The Representability Problem.** A matroid M on a set S is *representable* over the field k if there exists a vector space V over k and a map $\psi : S \to V$ which preserves rank. ψ is called a *representation* of M, and we describe a matroid M as representable if it is representable over some field.

Clearly a matroid M on S is representable over k if and only if there exists a matrix A with entries in k such that if $C = \{C_1, \ldots, c_n\}$ denotes the column vectors of A and $S = \{x_1, \ldots, x_n\}$, then the map $\psi : S \to C$ defined by $\psi(x_i) = c_i$ is a representation of M over k.

In 1935, Whitney [**28**] posed the so-called *Representability Problem*: Give necessary and sufficient condition for a matroid to be representable over some field. So far, no one has even suggested a reasonable set of such conditions. Conditions for representability were given by Tutte [1965], [**24**], Ingleton [1971], [**13**], and Vamos [1971], [**25**], but these conditions are very unwieldy.

A matroid M is called *binary* if it is representable over k_2, the field of order 2. Let M be a binary matroid on S, $|S| = n$, and let V be the vector space of rank n over k_2. The *circuit space* of M is the subspace of V generated by the incidence vectors of the circuits in M. Similarly, the *cocircuit space* of M is the subspace of V generated by the cocircuits of M.

We have that the rank of the circuit space of a binary matroid is $r(M^*)$ and the incidence vectors of the set of fundamental circuits in any base M form a base of the circuit space.

There are well known conditions for a matroid to be binary. The following are equivalent:

(1) M is binary.

(2) For any circuit C and cocircuit C^*, $|C \cap C^*|$ is even.

(3) The symmetric difference of any collection of distinct circuits of M is the union of disjoint circuits of M.

(4) If C_1 and C_2 are distinct circuits of M, the symmetric difference $C_1 \Delta C_2$ contains a circuit.

If k_n is a field of order n, let us denote the class of matroids which are representable over k_n by $M[k_n]$. In the following, we give a necessary condition for a matroid to belong to $M[k_n]$.

Theorem 2. *For every $M \in M[k_n]$ and any circuits C_1, C_2 of M with $r(C_1 \cup C_2 = |C_1 \cup C_2| - 2$, the number of circuits in $C_1 \cup C_2$ is bounded by a constant which in only dependent upon n.*

Proof. If M is representable over k, so is any minor of n (see Welsh [1976], [**26**]. Restrict M to $T = C_1 \cup C_2$ and use Theorem 1 to decompose T into $\cup T_i$, $i = 1, \ldots, m$, such that $\{T - T_i\}$ is the collection of circuits of $M|T$. Now, pick an edge i in every T_i and contract $M|T$ to M' on $S = \{1, 2, \ldots, m\}$. The circuits of M' are given by $C_i = S - i$. If $\psi : S \to V$ is a representation, and $\psi(i) = x_i$, we get a linear equation

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,m}x_m = 0,$$

$a_{i,j} \in k_n$, with $a_{i,i} = 0$ for every C_i. If $m = an$, (n is the order of the field k) then at least a of the coefficients of the x_i's in every equation must be the same. Now it is clear that if m is large enough, we can find two equations among the m equations and a pair (i, j) of indices such that x_i and x_j have the same coefficients in both equations. So we can simultaneously eliminate x_i and x_j, contradicting the requirement that ψ preserves rank. □

— **Matroid Connectivity.** A matroid M on S is said to be *connected* if for every pair of distinct elements x and y of S, there is a circuit of M containing x and y.

Let us define a relation R between the elements of S by xRy if and only if $x = y$ or, for x and y distinct, there is a circuit C of the matroid M which contains both x and y. R is an equivalence relation and we call the equivalence classes induced by R the *(connected) components* of M.

The following theorems will be useful. For proofs see Welsh [1976], [**26**].

Theorem 1. *A matroid M is connected if and only if its dual M^* is connected.*

Theorem 2. *The components of a matroid M on S are disjoint subsets of S and coincide with the components of the dual matroid M^*.*

Theorem 3. *A matroid M on S is not connected if and only if there exists a proper subset A of S such that $r(A) + r(S - A) = r(S)$.*

— **The Connectivity Matroid of a Graph.** Let $G = (V, E)$ be a graph. We define a matroid $M(G)$ on the edge set $E(G)$ by calling a subset F of E independent if and only if F does not contain a cycle. With this definition, the circuits of $M(G)$ are clearly the cycles of G. From now on, we will write cycle for both the cycle in G and the corresponding circuit in $M(G)$.

By definition a maximal subgraph of G which contains no cycles is a spanning forest. It is well known that the number of edges in a forest F is the number of vertices minus the number of connected components. Hence we can list the following basic properties of $M(G)$:

(1) If G is a (dis)connected graph, the bases of $M(G)$ are the spanning (forests) trees of G.

(2) For any subset F of $E(G)$ the rank of F in $M(G)$ is given by $r(F) = |\sigma(F)| - \alpha(F)$, where $\alpha(F)$ is the number of components of $(\sigma(F), F)$.

(3) Let F be a set of edges of G and $e \in E(G) - F$. Then e belongs to the closure of A in $M(G)$ if and only if there is a cycle C of G with $e \in C \subseteq F + e$.

Clearly $e \in E(G)$ is a loop in $M(G)$ if and only if e is a loop of the graph, and similarly, two edges e and f are parallel in $M(G)$ if and only if they are parallel edges in G.

The main benefit of a matroid treatment in graph theory seems to be a much more natural understanding of dual concepts, such as the structure of the set of cocycles, or the effect of contraction of a set of edges. We list some related results.

(1) If G is a connected graph and $C^*(G)$ denotes the set of cocycles of G, then $C^*(G)$ is the set of circuits of a matroid $M^*(G)$ on $E(G)$, (recall that a cocycle is a minimal cutset.) We call $M^*(G)$ the *cocycle matroid* of G.

(2) For any graph G, the cycle and cocycle matroids of G are dual matroids.

(3) If G is a planar graph and H^* is the geometric dual of G, then the cycle matroid of G and the cycle matroid of $H^*(G)$ are dual matroids.

(4) Let G be a planar graph and H^* the geometric dual of G. If C is the set of edges of a cycle (cocycle) in G, then the edges in H^* corresponding to C are the edges of a cocycle (cycle) in H^*.

To demonstrate the power of matroid theory as a tool in graph theory, we prove Euler's formula:

Let G be a connected planar graph. Then $r(M(G)) = |V(G)| - 1$. $r(M^*(G))$ is equal to $|F(G)| - 1$ by (3), where $|F(G|$ is the number of faces of G. We know that $r(M(G)) + r(M^*(G)) = E(G)$, for this is basic property (4) of M^*. This immediately gives

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

On the other hand, many problems in graph theory cannot even be posed in matroid language, since there is no simple exact counterpart of a vertex in a matroid. Also, non-isomorphic graphs may have isomorphic connectivity matroids, for example all trees with the same number of edges have isomorphic connectivity matroids.

If M is a disconnected graphic matroid, it will in general be the connectivity matroid for several graphs G. To see this, consider two components M_1 and M_2 of M and graphs H_1 and H_2 such that $M_i = M(H_i)$ for $i = 1, 2$. Then the disjoint union of H_1 and H_2, or the graph obtained from H_1 and H_2 by identifying an arbitrary vertex of H_1 with a vertex of H_2, will have the same connectivity matroid, which is a restriction of M.

Thus a necessary condition that M uniquely determines G is that M be connected.

Recall that a matroid M is connected if every two elements of E are contained in a cycle. Hence M is connected if and only if G is biconnected.

While there are non-isomorphic biconnected graphs with the same connectivity matroid. Whitney, [1932], [**27**], proved that a matroid without loops which is the cycle matroid of a 3-connected graph G, uniquely determines G up to isolated vertices.

Tutte, [1958], [**23**], has characterized graphic matroids in what is certainly the deepest theorem on this subject: A matroid which is representable over every field is graphic if and only if it does not contain the cocycle matroid of either K_5 or $K_{3,3}$ as a minor. This theorem can be regarded as a complete abstract generalization of Kuratkowski's Theorem.

The Rigidity Matroid of a Graph

— **The Physical Problem.** Let $G = (V, E)$ be a simple graph whose vertices are points in the plane. Let us visualize the edges of G as rigid bars, pin jointed at each vertex. Such a graph will be called a *plane structure*. Let $|V| = n$, $|E| = m$, and (x_i, y_i) be the cartesian coordinates of the vertex v_i.

A *mechanical motion* of G is a parameterized family $\{G(t) \mid a \le t \le b\}$ of plane structures, all embeddings of the same graph G, such that the position $(x(t), y(t))$ of each point of G is a differentiable function of t, and

(1) $$(x_i - x_j)^2 + (y_i - y_j)^2 = \text{constant},$$

for every edge (v_i, v_j) in G. Differentiating (1) by t gives

(2) $$(x_i - x_j)(\dot{x}_i - \dot{x}_j) + (y_i - y_j)(\dot{y}_i - \dot{y}_j) = 0.$$

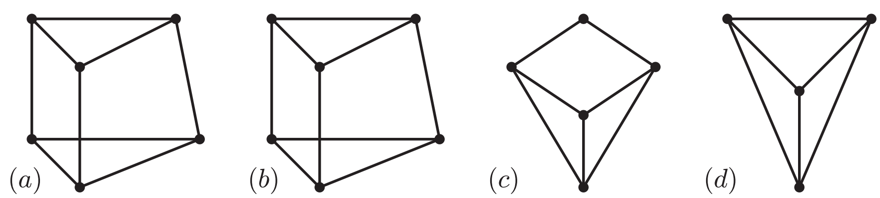
where the dot denotes differentiation by t. Equation 2 implies that the relative velocity of the endpoints of an edge is perpendicular to the edge. Writing down such an equation for every edge (v_i, v_j) in E we obtain a system of linear equations

(3) $$H\mathbf{w} = \mathbf{0}$$

where H is an m by $2n$ matrix and $\mathbf{w}$ is a column vector $(\dot{x}_1, \dot{y}_1, \ldots, \dot{x}_n, \dot{y}_n)$. A $\mathbf{w}$ satisfying (3) is called an *infinitesimal motion* of G. The infinitesimal motions of G form a linear subspace of $2n$ dimensional Euclidean space. The rigid motions of G yield a 3-dimensional subspace of this linear space.

The codimension of this subspace of rigid motions in the space of all infinitesimal motions is called the *degree of freedom* of G. G is called *rigid* if its degree of freedom is 0, or, equivalently, if the infinitesimal motions of G form a 3-dimensional linear space. This definition of rigidity is due to Laman [1970], [**14**].

The rigidity of a structure depends on the coordinates of the vertices in the plane. In the following figure, (a) and (b) have the same underlying graph but (a) is rigid whereas (b) is not.



Similarly (c) is rigid but (d) is not. An infinitesimal motion does not always correspond to an actual movement of the structure. For example, structure (b) deforms mechanically whereas (d) does not. But if the vertices of a plane structure are not in mutually special positions, every infinitesimal motion of G is the velocity of some mechanical motion.

The vertices of a plane structure are in *generic position* if $x_i, y_i, \ldots, x_n, y_n$ are algebraically independent over the rational field. This highly non-mechanical assumption means that a subdeterminant of H is 0 if and only if it is identically zero when we consider $x_1, y_1, \ldots, x_n, y_n$ as variables. Therefore, if the vertices are in generic position, the linear dependence of the equation (2) depends only on the underlying graph, and consequently the rigidity depends on the graph only.

A graph $G = (V, E)$ is called *generically rigid* if there is a generic embedding of G in the plane which is rigid.

Laman's Theorem [1970]. *Let $G = (V, E)$ be a graph. G is generically rigid if and only if there is a subset F of E such that $|F| = 2|V| - |3|$, and $|F'| \leq 2|\sigma(F')| - 3$ for all subsets F' of F.*

If we are given an abstract graph, we can define the generic independence of a subset of edges as the independence of this set of edges in a generic embedding of the graph in the plane.

From now on, we only talk about generic rigidity and suppress the adjective generic. So, rigidity, independence, and degree of freedom stand for generic rigidity, generic independence and generic degree of freedom.

Using Laman's Theorem, we can now define a subset F of edges in a graph $G = (V, E)$ to be *independent* if

$$|F'| \leq 2|\sigma(F')| - 3$$

holds for all non-empty subsets F' of F.

It is well known (see for example Crapo [1979], [**3**]) that the independent subsets of E form the independent sets of a matroid, $R(G)$, the *rigidity matroid* of the graph G.

— **The Axiomatic Approach.** Suppose that we are given a finite set V, $|V| \geq 2$, and a matroid on the edge set of $K(V)$ whose closure operator satisfies, in addition to the closure axioms $C(1), \ldots, C(4)$ of a matroid, also

(**C5**) For $E, F \subseteq K$ with $|\sigma(E) \cap \sigma(F)| < 2$ we have $c(E \cup F) \subseteq K(\sigma(E)) \cup K(\sigma(F))$.

(**C6**) For $E, F \subseteq K$, with $c(E) = K(\sigma(E))$ and $c(F) = K(\sigma(F))$, if $|v(E) \cap v(F)| \geq 2$ then $c(E \cup F) = K(\sigma(E \cup F))$.

(**C7**) For all $E \subseteq K$ and all $v \in \sigma(E)$ such that $|\text{star}(v)| = 3$ and not all three edges between the endpoints of $\text{star}(v)$ not equal to v are in the closure of $E - \text{star}(v)$, then $r(E) = r(E - \text{star}(v)) + 3$ (where r is the rank function)

is called the 2-*dimensional generic rigidity matroid* for V and is denoted by $R(V)$.

We define an edge set $E \subseteq K(V)$ to be rigid if $c(E) = K(\sigma(E))$.

It is easy to check (see Graver [1984], [**8**]) that the rank of $R(V)$ is equal to $2|V| - 3$ and that a subset F of edges of $K(V)$ is independent in $R(V)$ if and only if

$$|F'| \leq 2|v(F')| - 3$$

holds for every nonempty subset F' of F. Given a graph $G = (U, E)$, G can be thought of as a subgraph of $K(V)$ if U is a subset of V. Also $R(G)$ can be thought of as the restriction of $R(V)$ to the edge set U.

Note that, as an immediate consequence of C(5), rigidity of G implies biconnectivity of G. Clearly, the reverse implication is false. (Take for example a cycle of length greater than three.)

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Chapter 1

Rigidity and Connectivity

1. Introduction

Motivated by the trivial fact that, in the case of one dimensional structures, rigidity and connectivity are equivalent, we will explore some of the analogous properties of the rigidity matroid $R(G)$ and the connectivity matroid $M(G)$ of a graph G.

We are in particular looking for an analogue to the fact that a graph G is (vertex) two-connected if and only it the cycle matroid $M(G)$ is a connected matroid. Our main result is that (vertex) birigidity of G implies that the rigidity matroid $R(G)$ is connected, and the connectivity of $R(G)$ implies that G is edge birigid. We also show that the reverse implications are false.

2. Bases

If G is connected, then it has a spanning tree, T. T has vertex set V and $|V| - 1$ edges and, moreover, every non-empty subset, F, of the edge set of T satisfies the condition

$$|F| \leq |\sigma(F)| - 1.$$

It follows that T has at least two vertices of valence 1. Moreover, the edge set of T is a basis for the connectivity matroid $M(G)$.

If G is rigid, Laman's theorem tells us that there is a subset B of E such that

(1) $|B| = 2|V| - 3$, and

(2) $|F| \leq 2|\sigma(F)| - 3$ for all non-empty subsets F of B.

It follows from $\sigma(B) = V$ that B has at least three vertices of valence at most three and that every vertex in B has valence at least two unless $|B| = 1$. Moreover, B is a basis for the matroid $R(G)$.

3. R-Components

Connectivity induces an equivalence relation on the vertex set as well as the edge set of G by decomposing G into its connected components.

Rigidity induces an equivalence relation on the edge set of G, since condition **C6** implies that the maximal rigid subgraphs, the *rigid components* of G, have at most one vertex in common.

This leads us to compare rigidity to biconnectivity, since biconnectivity also induces an equivalence relation on the edge set E into *blocks*, the maximal biconnected subgraphs of G. Blocks have at most one vertex in common.

4. Edge Birigidity

A graph $G(V, E)$ is a called *edge birigid* if $c(E - e) = K(V)$ for all $e \in E$. We are of course interested in characterizing edge minimal graphs with that property.

Let us first look at edge biconnected graphs and then generalize our findings to rigidity.

A circuit in $M(G)$ is a cycle in G, and if we consider only graphs without loops, then cycles are the simplest edge minimally edge biconnected graphs. We find that

Proposition 1. *Circuits in $R(G)$ are minimally edge birigid.*

Proof. Let C be a subset of E which is a circuit in $R(G)$. C is minimally dependent in $R(G)$, i.e., for each proper subset F of C we have

(a) $$|F| \leq 2|\sigma(F)| - 3.$$

The dependency of C implies $|C| > 2|\sigma(C)| - 3$ and we conclude that

(b) $$|C - e| = 2|\sigma(C)| - 3$$

for each e in C. Equations (a) and (b) imply that C is rigid, and that $C - e$ is independent for all e in C. □

We can characterize edge minimal edge biconnected graphs as follows:

Theorem 1. *A graph G is minimally edge biconnected if and only if the relation R*

$$eRf \iff G - e - f \text{ is disconnected, } e, f \in E(G),$$

is transitive, symmetric and irreflexive, and is such that for all $f \in E$, the set $\{c \mid cRf\}$ is nonempty.

Proof. The irreflexivity of R implies that G is biconnected and conversely. A biconnected graph is minimally biconnected if and only if the removal of any edge f of G destroys this property, which is to say that there is a $g \in E(G)$ such that gRf. □

It is useful to reformulate Theorem 1 in terms of cocycles. We observe that gRf if $\{g, f\}$ is a cocycle.

Theorem 1*. *A connected graph G is minimally edge biconnected if and only if $M^*(G)$ has no loops and every edge of G is contained in a cocycle of cardinality two.*

The partition induced on the edges of G by the relation in Theorem 1 are cycles or paths in G, since the restriction to such a class in $M^*(G)$ is connected, which means that the contraction of $M(G)$ onto this class is connected, hence the contraction of G to this class is biconnected and such that the deletion of any pair of edges disconnects the contracted graph. An example of this partition is given in Figure 1. In $R(G)$ we obtain

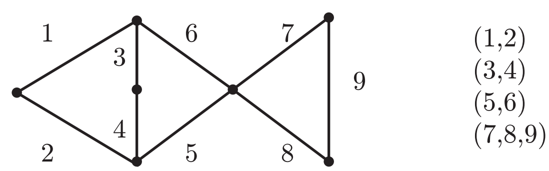


Figure 1. The partition induced by the relation R

Theorem 2. *A graph G is minimally edge birigid if and only if the relation S,*

$$eSf \iff G - e - f \text{ is not rigid, } e, f \in E(G),$$

is transitive, symmetric and irreflexive, and is such that for all $f \in E$ the set $\{g \mid fSg\}$ is non-empty.

The proof of this theorem is the same as in theorem 1.

In chapter 4, where we discuss the structure of $R^*(G)$, we shall see that we have also

Theorem 2*. *A rigid graph G is minimally edge birigid if and only if $R^*(G)$ has no loops and every edge of G is contained in a cocycle of cardinality two.*

An example of the partition S is illustrated in Figure 2. It is now

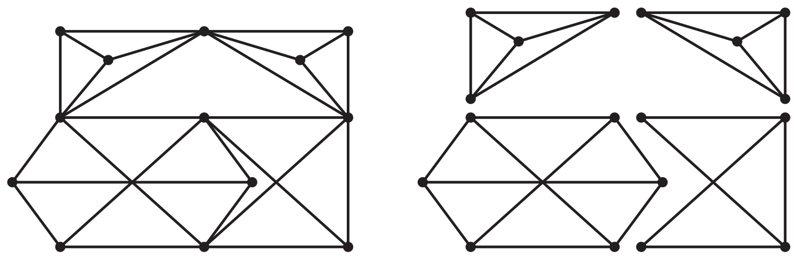


Figure 2. Partition induced by the relation S.

tempting to define r-paths to be the classes induced on E by the relation S which are not circuits in $R(G)$. R-paths will be studied in more detail in chapter 3.

Theorems 1 and 2 indicate two easy ways to obtain minimally biconnected (resp. birigid) graphs:

(1) consider an independent connected (rigid) graph and replace all its edges by cycles (circuits), or, more generally, by minimally biconnected (birigid) graphs.

(2) combine paths ("r-paths") of length greater than one by identifying their endpoints.

Figures 3 and 4 illustrate these procedures.

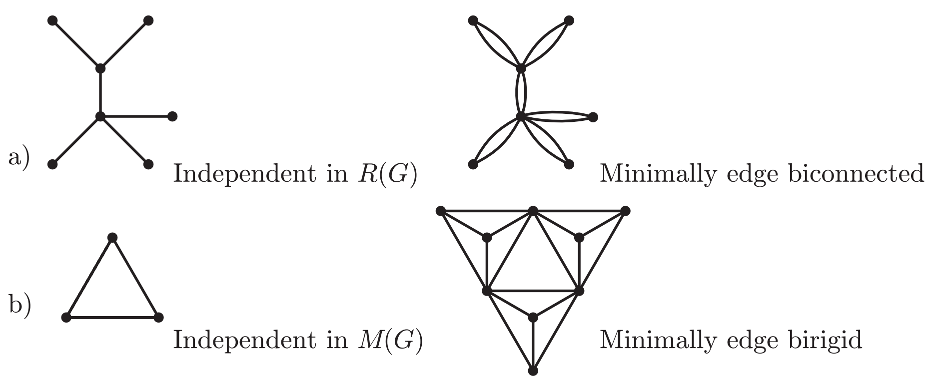


Figure 3.

Given two edge minimally biconnected graphs A and B, with $a \in V(A)$, $b \in V(B)$, we can construct an edge minimally biconnected graph from A

and B by identifying a and b. Since a tree always contains a pendant vertex, procedure 1 can be thought of as a series of such one point unions.

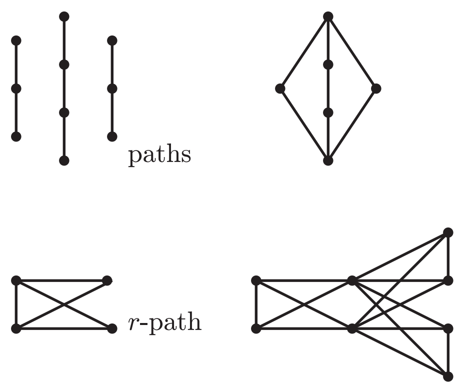


Figure 4.

Given two minimally rigid graphs, their one point union will not be rigid and their two point union will not be minimal. Hence procedure 1, where the underlying graph is the rigidity analogue of a tree (an r-tree) is the natural generalization of the idea of one point union for connected graphs. The concept of r-tree is developed in chapter 3.

It is still an open question if all minimally biconnected (birigid) graphs are obtained by 1 and 2.

Finally we want to examine the relationship between the connectivity of the matroid $R(G)$ and edge birigidity.

Proposition 2. *If* $|V(G)| \geq 2$ *and* $R(G)$ *is connected, then* G *is edge birigid but not conversely.*

Proof. Assume that there is an edge e such that $G - e$ is not rigid. Then

$$r(E - e) = 2|V| - 4$$

and

$$r(E - e) + r(e) = r(E).$$

The last equation contradicts the connectivity of $R(G)$.

The converse is not true: Let G_0 be minimally rigid, having n_0 vertices and $2n_0 - 3$ edges. We attach to each edge e_i a circuit C_i, $1 \leq i \leq 2n_0 - 3$, with C_i having n_i vertices, by identifying one edge of each C_i with one edge of G_0. Then the resulting graph is clearly rigid and hence has rank $2n - 3$, where

$$n = \sum_{i=1}^{2n_0-3} (n_i - 2) + n_0,$$

the rank of each C_i is $2n_i - 3$.

If we sum over the ranks, we get

$$\sum_{i=1}^{2n_0-3}(2n_i - 3) = 2\left[\sum n_i\right] - 3(2n_0 - 3) = 2n + 6n_0 - 12 - 6n_0 + 9 = 2n - 3.$$

So, by theorem 3 on matroid connectivity in the introduction, $M(G)$ is not connected. On the other hand, G is clearly edge birigid. □

Figure 3b serves as an example.

5. Vertex Birigidity

A graph G is called *birigid* if

$$c(E - \text{star}(v) = K(V - v)$$

for each vertex $v \in V$.

We recall that G is biconnected if and only if $M(G)$ is connected.

Connectivity of G implies edge connectivity of G, and we are naturally led to believe that birigidity implies edge birigidity. Instantly we find a counterexample: If G is a triangle, G is birigid since the removal of the star of any vertex leaves a single edge. On the other hand the removal of a single edge from G leaves a nonrigid graph. However. the next proposition shows that this is the only counterexample.

Proposition 3. *A birigid graph on more than three vertices is edge birigid.*

Proof. First we show that G does not contain a vertex of valence two. For if G did contain a vertex of valence two, we could remove one of its neighbors and obtain a graph with a vertex of valence one, which is rigid if and only if it is a single edge.

Now, let G be birigid on more than three vertices, let e be any edge of G, and v one of its endpoints. $G - \text{star}(v)$ is rigid and $\text{star}(v)$ contains more than two edges, so $G - e$ is rigid. □

Proposition 4. *A cutset in a birigid graph on more than three vertices has cardinality at least three.*

Proof. The removal of any vertex from G leaves a rigid graph which is 2-connected, hence G is 3-connected. □

Theorem 3. *If G is birigid then $R(G)$ is connected but not conversely.*

Proof. Assume that G is birigid and that $R(G)$ is not connected. Consider the connected components R_i of $R(G)$. Then there is a partition of E,

$$E = E_1 \cup E_2 \cup \ldots \cup E_k,$$

such that

$$R(G) = R_i \oplus R_2 \oplus \ldots \oplus R_k, \tag{1}$$

where $R_i = R(G_i)$, with $G_i = G(\sigma(E_i), E_i)$. Every edge of G is contained in a circuit of $R(G)$, therefore every R_i contains a circuit and is thus rigid by proposition 2. So we obtain from (1) the equation

$$2|V| - 3 = \sum_{i=1}^{k} [2|\sigma(E_i)| - 3]. \tag{2}$$

Observe that

$$|\sigma(E_i) \cap \sigma(E_j)| \leq 1 \text{ for all } i \neq j, \tag{3}$$

since, if (3) does not hold, then there are two vertices, a and b, belonging to both $\sigma(E_i)$ and $\sigma(E_j)$. Let e be an edge from a to b. There is a circuit C_i contained in $E_i \cup e$ and a circuit C_j contained in $E_j \cup e$. $C_i \Delta C_j$ contains a circuit intersecting both R_i and R_j, a contradiction.

Furthermore, we have that

$$|\sigma(E_i) \cap \left[\bigcup_{i \neq j} \sigma(E_j) \right] | \geq 3, \tag{4}$$

since every cutset in a birigid graph has cardinality at least 3. Let us define N_i, n_i, N, and n by the following equations:

$$\begin{aligned} N_i &= |\sigma(E_i) - \{\bigcup_{i \neq j} \sigma(E_j)\}| \\ |\sigma(E_i)| &= n_i + N_i, \\ N &= \sum_{j=1}^{k} N_i, \text{ and} \\ |V| &= n + N. \end{aligned}$$

Rewriting (2) in this new notation we obtain

$$2n + 2N - 3 = \sum i = 1^k (2[n_i + N_i] - 3) \text{ or}$$

$$2n = \left[\sum_{i=1}^{k} 2n_i \right] - 3(k-1) \tag{5}$$

Equation (4) implies that

$$n_i \geq 3. \tag{6}$$

By the definition of n we have

$$n \leq (1/2)\sum_{i=1}^{k} n_i \tag{7}$$

so that (5) and (7) gives

$$[\sum_{i=1}^{k} 2n_i] - 3(k-1) \leq \sum_{i=1}^{k} n_i, \text{ or}$$

$$\sum_{i=1}^{k} n_i \leq 3(k-1). \tag{8}$$

Equations (8) and (6) give

$$3k \leq \sum_{i=1}^{k} n_i \leq 3(k-1),$$

a contradiction.

If $R(G)$ is connected, G need not be birigid: If G is a wheel, $R(G)$ consists of a single circuit and hence is connected. But the removal of the center vertex leaves a non-rigid graph if the number of spokes is larger than 3. □

6. The Symmetric Difference of Circuits

$M(G)$ is a binary matroid, i.e., if C and C' are distinct cycles in G, then their symmetric difference, $C\Delta C'$, contains a cycle. The symmetric difference of two cycles is itself a cycle if the two cycles have exactly one edge and two vertices in common.

The last statement holds true for circuits in $R(G)$ as well.

Proposition 5. *Let C_1 and C_2 be circuits in $R(G)$ such that they intersect in exactly one edge and $|\sigma(C_1) \cap \sigma(C_2)| = 2$. Then their symmetric difference, $C_1\Delta C_2$, is a circuit in $R(G)$.*

Proof. Assume C_i, $i = 1, 2$, has n_i vertices in its support and $2n_i - 2$ edges, and there is a proper subset E of $C_1\Delta C_2$ such that E is a circuit, $E = E_1 \cup E_2$, $E_i \subseteq C_i$, and $E_i \neq \emptyset$. If E_1 is contained in $C_1 - e$, then E_1 is non-rigid, since E_1 has to contain both endpoints of e (the endpoints of

e form a cutset of $C_1 \Delta C_2$ and if it was rigid, $E_1 \cup e$ would be a dependent subset of C_1. Therefore $|E_1| < 2|\sigma(E_1)| - 3$, since

$$|E_1| + |E_2| < 2|\sigma(E_1)| - 3 + 2|\sigma(E_2)| - 3 \leq 2|\sigma(E)| - 2.$$

So $|E| \leq 2|\sigma(E)| - 3$ for every proper subset of $C_1 \Delta C_2$ and $C_1 \Delta C_2$ is itself a circuit. □

The symmetric difference of two cycles of G is a cycle if and only if their intersection is connected. The symmetric difference of two circuits in $R(G)$ is not necessarily a circuit if their intersection is rigid, in fact it need not even contain a circuit as the following example shows:

Example: Let C_1 and C_2 be circuits in $R(G)$, such that $\sigma(C_2)$ is contained in $\sigma(C_1)$ and $C_2 - e$ is contained in C_1 for some edge $e \in C_2$. Then

$$|\sigma(C_1 \Delta C_2)| = |\sigma(C_1)| - |\sigma(C_2)| + 2$$

and

$$|C_1 \Delta C_2| = |C_1| - |C_2| + 1.$$

If $C_1 \Delta C_2$ contained a circuit C, C would have to contain e. But then $C \Delta C_2$ would have to contain a circuit, but $C \Delta C_2$ is contained in C_1 which is a contradiction. See Figure 5.

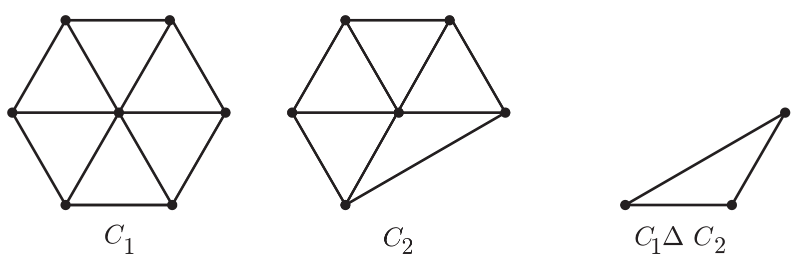


Figure 5.

Hence we have proved the following

Proposition 6. *$R(G)$ is not binary.*

We shall see later that, in fact, $R(G)$ is not representable over any finite field.

7. Minimum Valence

We have given a greatest lower and a least upper bound for the minimum valence in an (edge) minimal connected resp. rigid graph. In an edge minimal vertex birigid graph every vertex has valence at least three. Furthermore, we observe:

Proposition 7. *A minimally biconnected graph contains at least three vertices of valence two.*

Proof. We proceed by induction on the number of vertices. The smallest biconnected graph, the triangle, has three vertices of valence two. Assume the conclusion is true for every minimally biconnected graph with fewer than $n+1$ vertices and let G be such a graph on $n+1$ vertices. Choose a vertex v in G. From the minimality of G we conclude that for every edge e of $\text{star}(v)$, $G-e$ has a cut vertex. If $|\text{star}(v)| = k$, v is connected to k blocks, each of which satisfies the induction hypothesis and has at least three vertices of valence 2. One of them might be identical with the cut vertex of $G-e$ for $e \in \text{star}(v)$, another might be incident with e. So G has at least k vertices of valence two if $k \geq 3$, and at least $k+1$ if $k=2$, since in that case v is of valence 2. □

In general one can show along very similar lines that a minimally n-connected graph has at least $n+1$ vertices of valence n.

Lovasz and Yemini have shown that every six connected graph is rigid. It follows that every seven-connected graph is birigid. From our last remark we obtain seven as an upper bound for the minimum valence in an edge minimal vertex biconnected graph.

Proposition 8. *An edge minimal vertex birigid graph has at least one vertex of valence smaller than or equal to seven.*

Chapter 2

Minimally birigid graphs

1. Introduction

Every pair of edges in a biconnected graph is contained in a cycle. A cycle is an edge-minimal vertex-biconnected graph which has exactly one edge more than it needs to be connected. A biconnected graph can simply be thought of as a union of cycles.

It is natural to look for a rigid analogue: Given a birigid graph, can we write it as a union of birigid graphs of minimal excess, where the *excess* of a rigid graph $G(V, E)$ is defined to be $|E| - r(E)$.

First we have to ask if there are birigid graphs of excess one. In other words, are there circuits which are birigid? The answer is simple and somewhat disappointing: The only birigid graph of excess one is the tetrahedron, the complete graph on four vertices. To see this, we observe that the average valence in a rigid graph of excess one on n vertices is greater than or equal to $4 - (4/n)$. Therefore a birigid graph on more than four vertices contains a vertex of valence at least four. The removal of a vertex of valence four decreases the excess by two. Therefore a birigid graph on more than four vertices has to have excess at least two.

In section 2 we show that there are infinitely many birigid graphs of excess two.

In section 3 we show that the birigid graphs of excess two do not, unfortunately, fulfill the role of universal building blocks of birigid graphs.

2. Birigid Graphs of Excess Two

In this section we will restrict our attention to an edge minimal vertex birigid graph $G(V, E)$, which has exactly two edges more than it needs to be rigid, i.e.,

$$|E| - 2|V| - 1.$$

We first list some elementary properties of G.

Proposition 1. *Let G be a birigid graph of excess 2. Then*

(1) *G contains at least five vertices.*

(2) *If $e \in E(G)$, then $G - e$ is not birigid, and*

(3) *G has exactly two vertices of valence three and the remaining vertices each have valence four.*

Proof. (1) We have $2n - 1 = |E(G)| < \binom{n}{2} + 1$, which implies that $n > 4$.

(2) $G - e$ is not a complete graph. $G - e$ has excess one. Since the only birigid graph of excess one is K_4, $G - e$ is not birigid.

(3) Since G is rigid, it contains no vertex of valence less than two. Suppose that G had a vertex v of valence two. Let w be adjacent to v. Then $G - w$ contains a vertex of valence 1 and is not rigid. Now suppose G has a vertex v of valence k. Then $G - v$ has $n - 1$ vertices and $2(n - 1) - (k - 1)$ edges. Since $G - v$ is rigid, $k - 1 < 4$, which implies that $k < 5$.

Finally, if there are m vertices of valence three, we have $3m + 4(n - m) = 2(2n - 1)$, which gives $m = 2$. □

Next we examine the circuit structure of $R(G)$:

Theorem 1. *A graph on n vertices with $2n - 1$ edges is birigid if and only if there is a decomposition of the edge set E of G,*

$$E = E_1 \cup E_2 \cup \cdots \cup E_k$$

such that $E - E_i$ is a circuit in $R(G)$ for all i, and either

(1) *E_i is an edge for $3 \le i \le k$ and E_1 and E_2 are stars of two vertices of valence three,*

or

(2) *E_i is an edge for $2 \le i \le k$ and E_1 is the union of stars of two adjacent vertices of valence three.*

Proof. Assume that there exists such a decomposition. We have to show that G is birigid. To this end we show first that G has only vertices of valence 3 and 4 and then show that the removal of the star of a vertex leaves a rigid graph.

Consider a class E_i in the decomposition that contains exactly one element e. Then $E - e$ is a circuit in $R(G)$ and $G - e$ is a graph with minimum valence at least three. We conclude that the endpoints of e are of valence at least 4 in G and that G does not contain any vertex of valence smaller than 3. Hence G has exactly two vertices of valence three.

If there are n_i vertices of valence i, $i > 3$, then

$$\sum_i = 4^m i \cdot n_i + 6 = 4n - 2 \text{ and } \sum_i = 4^m n_i = n - 2.$$

So

$$\sum_i = 5^m (i - 4) n_i = 0.$$

which implies that $n_i = 0$ for all $i > 4$.

Depending on whether or not the two vertices of valence 3 are adjacent in G, conditions [1] and [2] imply that the removal of a vertex of valence three of G results in a circuit or in a circuit with a vertex of valence two attached, a rigid graph in both cases.

Consider a vertex v of valence four in G. Remove an edge e of $\text{star}(v)$ with endpoints of valence four. $E(G) - e$ is a circuit by assumption, and v has valence three in this circuit, so $E(G) - \text{star}(v)$ is rigid.

Conversely, assume that G is edge birigid on n vertices and $2n-1$ edges. By proposition 1, $n \geq 5$ and G has exactly two vertices of valence 3 and all other vertices are of valence 4. Therefore, G is edge birigid and every edge is contained in a circuit.

We want to show that G can be written as the union of two circuits and then use the matroid decomposition theorem given in the introduction. Let us show first that G contains at least two circuits: A basis B of G contains $2n - 3$ edges, since G is rigid. Then $E - B = \{e_1, e_2\}$, $B + e_1$ contains a circuit C_1, and likewise $B + e_2$ contains a circuit C_2. Since neither C_1 nor C_2 are subsets of B, they are distinct.

Next we show that G can be written as the union of two distinct circuits C_1 and C_2: Assume that this was not the case and that there is an edge e not in $C_1 \cup C_2$. Then, since every edge is contained in a circuit, $e \in C_3$, where C_3 is a circuit distinct from C_1 and C_2. By the sub-modular inequality we have

$$r(C_1 \cup C_2 \cup C_3) \leq |C_1 \cup C_2 \cup C_3| - 3,$$

and consequently, since the addition of any edge increases the rank by at most 1, $r(E) \leq |E| - 3$, a contradiction to $r(E) = |E| - 2$.

So E is the union of two circuits and $r(E) = |E| - 2$ and we can apply the decomposition theorem of the introduction to write $E = \cup E_i$, with E_i disjoint from E_k for distinct i and k, and $E - E_i$ a circuit for all i.

We therefore can write $|E - E_i| = 2|\sigma(E - E_i)| - 3$ which, when subtracted from $|E| = 2|\sigma(E)| - 1$, gives

$$(9) \qquad |E_i| = 2[\sigma(E) - \sigma(E - E_i)] + 1.$$

If E and $E - E_i$ have the same support, $|E_i| = 1$ and E_i is a single edge.

If $\sigma(E) - \sigma(E - E_i) = 1$, then E_i contains all edges of the star of a vertex in G. The equation 9 gives $|E_i| = 3$. Since every vertex in G has valence at least 3, E_i must be a star of a vertex of valence 3, and since $E - E_i$ is a circuit, the two vertices of valence three of G are not joined by an edge.

Note that all edges of the star of a vertex of valence three in G must belong to the same E_i, since the minimum valence of a vertex in $E - E_i$ is 3.

If $\sigma(E) - \sigma(E - E_i) = 2$, then E_i contains all edges of the stars of two vertices of G. The equation 9 gives $|E_i| = 5$. Again, since the minimum valence of a vertex in G is 3, E_i must be the union of two adjacent vertices of valence 3 in G.

If $\sigma(E) - \sigma(E - E_i) > 2$, then E_i contains all edges of the star of 3 vertices of G. One of these must be of valence 4, since G has exactly two vertices of valence 3. But $E - \mathrm{star}(v)$ is rigid, hence independent if $|\mathrm{star}(v)| = 4$, and cannot contain a circuit. The desired decomposition is so established and the proof of the theorem is complete. □

Examples of graphs with a decomposition of type [1] and [2] are given in Figure 1 below.

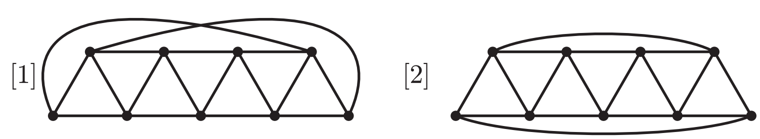


Figure 1. Birigid graphs of excess two

Clearly we can "string together" as many triangles as we wish to obtain birigid graphs of excess two with arbitrarily large size. We can now prove

Corollary 1. *There is no finite field k such that $R(G)$ is representable over k for all G.*

Proof. If $R(G)$ was representable over some finite field k for all G, then consider the union of two circuits whose intersection is rigid. Then the number of the components in the decomposition of the edge set would have to be bounded, a contradiction. □

The simplest birigid graph of excess two, denoted by A, can be obtained from K_5 by deleting an edge. (see Figure 2) or it may be obtained from K_4

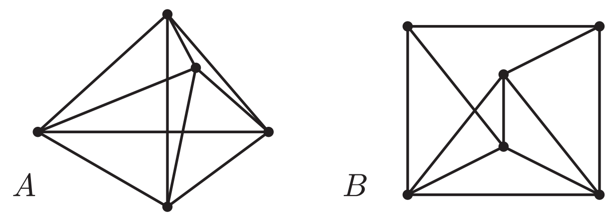


Figure 2.

by attaching a vertex of valence three. The graph A clearly contains other birigid graphs of positive excess, namely the two copies of K_4, (of course every graph might contain a triangle which is a birigid graph of excess zero.)

We may also obtain an edge minimal birigid graph B from K_4 by attaching two adjacent vertices of valence three. See also Figure 2. Note that B also contains K_4 as a subgraph.

The next result shows that A and B are unique in that respect.

Corollary 2. *The birigid graphs of excess two with* $|V| > 6$ *do not contain a birigid subgraph of positive excess.*

Proof. Assume G contains a birigid subgraph H. Proposition 1 implies that $V(H)$ is a proper subset of $V(G)$. If v is of valence 4 in G, $G - v$ is rigid on $n-1$ vertices with $2(n-1)-3$ edges, hence it contains no birigid subgraph of positive excess.

If v is of valence 3 in G, $G - v$ is either the circuit K_4, or $G - v$ is a circuit with a vertex of valence 2 attached. Again, the circuit must be K_4 to be birigid. In either case G has at most six vertices. □

Consider an edge minimal birigid graph $G(V, E)$. For every edge e of G there exists a nonempty set $V(e)$ of vertices of G such that $E - \{e, \text{star}(v)\}$ is nonrigid for all $v \in V(e)$. Elements of $V(e)$ are called *essential vertices* for the edge e.

Given a birigid graph of excess two, how can we construct a larger birigid graph of excess two from it? Can we "attach" a vertex of valence 3 and remove an edge from the given graph to obtain a graph with the desired property? To formalize this idea, we introduce some notation.

Let T be a graph on four vertices and three edges, where one vertex is of valence three and construct a graph $G + T$ by identifying vertices a, b, c of G with the vertices of valence one in T.

To prove the next theorem, we need the following

Lemma 1. *Let G be a birigid graph with two adjacent vertices u and v of valence 3. Then no vertex of G is adjacent to both u and v.*

Proof. Assume that there is a vertex x in G adjacent to both u and v. Then $G - x$ contains two adjacent vertices of valence 2 and can therefore not be rigid. See Figure 3. □

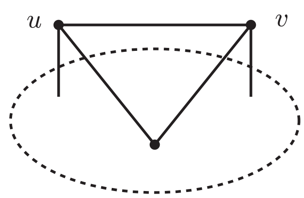


Figure 3. This graph is not birigid.

Theorem 2. *Let G be a birigid graph of excess two, and let T be as described above. Then:*

(1) *$G + T$ is birigid,*

(2) *a necessary and sufficient condition for $G + T$ to be edge minimally birigid is that the set $\{a, b, c\}$ not be contained in $V - V(e)$ for any edge e of G,*

(3) *if $G + T$ is not edge minimally birigid, then there is an edge e such that $G + T - e$ is birigid of excess 2,*
and

(4) *there is always a choice of $\{a, b, c\}$ such that $G - T$ is not edge minimal.*

Proof. (1) The removal of T results in a birigid, and hence rigid graph, and the removal of any vertex $v \in G$ from $G + T$ removes at most one edge from T, and since $G - v$ is rigid, so is $G + T - v$.

(2) Sufficiency: Let e be any edge of G. Since the intersection $V(e) \cap \{a, b, c\}$ is nonempty, the removal of e and any vertex in this intersection, say a, leaves a nonrigid graph, $G - \{e, \mathrm{star}(a)\}$. Since $G + T - \{e, \mathrm{star}(a)\}$ consists of a vertex of valence two attached to $G - \{e, \mathrm{star}(a)\}$, it too is non-rigid.

Necessity: Assume the existence of an edge e of G such that $\{a, b, c\}$ is contained in $V - V(e)$. We conclude that at least one vertex among $\{a, b, c\}$ is an endpoint of e, since at least one of these vertices has valence four and all vertices of valence four in G which are not incident with e are essential for e.

There are two cases.

(a) a and b are endpoints of e and a is of valence three in G. Remove e and $\mathrm{star}(v)$ for some $v \in V(e)$, then $E - \{e, \mathrm{star}(v)\}$ is independent, has one degree of freedom and e is not in the closure of $E - \{e, \mathrm{star}(v)\}$. By (C7),

$$E - \{e, \mathrm{star}(v)\} \cup E(T)$$

is independent and hence rigid.

(b) e has endpoints of valence four, one of them being a, and b and a are of valence three. Remove e and $\mathrm{star}(v)$ for some $v \in V(e)$. Again $E - \{e, \mathrm{star}(v)\}$ is independent with one degree of freedom. Consider the maximal rigid components of $E - \{e, \mathrm{star}(v)\}$, and assume that a, b, and c are contained in the same component. Lemma 1 implies that not all edges of $K\{a, b, c\}$ can belong to G.
Each component is minimally rigid since $E - \{e, \mathrm{star}(v)\}$ is independent. We conclude that there are exactly three edges incident with the component of $E - \{e, \mathrm{star}(v)\}$ containing $\{a, b, c\}$ in $G - e$.
   (i) If none of these three edges belongs to $\mathrm{star}(v)$ and two of them belong to the same rigid component of $E - \{e, \mathrm{star}(v)\}$ and the third represents a single component. then $G - e$ cannot be a circuit, since the removal of the component of $E - \{e, \mathrm{star}(v)\}$ containing $\{a, b, c\}$ and the removal of the edge representing a component of $E - \{e, \mathrm{star}(v)\}$ does not alter the rigidity properties of $G - e$, see Figure 4. This contradicts the circuit structure

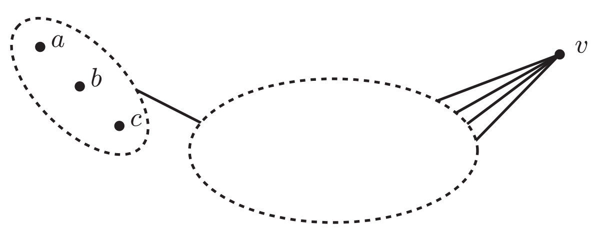


Figure 4. This graph is not birigid.

of G.
Similarly we obtain a contradiction if all three edges leaving the component containing $\{a, b, c\}$ represent different components, see Figure 5.
   (ii) If one edge of $\mathrm{star}(v)$ is incident with the component of $E - \{e, \mathrm{star}(v)\}$ containing $\{a, b, c\}$, the remaining edges

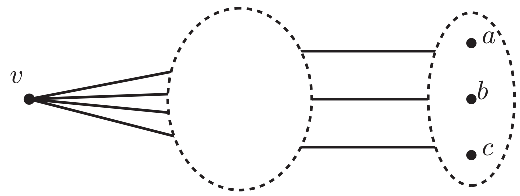


Figure 5. This graph is not birigid.

leaving this component must be incident with (or contained in) the same rigid component since there is only one degree of freedom. In this case three edges of $\mathrm{star}(v)$ are incident with one rigid component of $E-\{e, \mathrm{star}(v)\}$, again contradicting the circuit structure of G. See Figure 6.

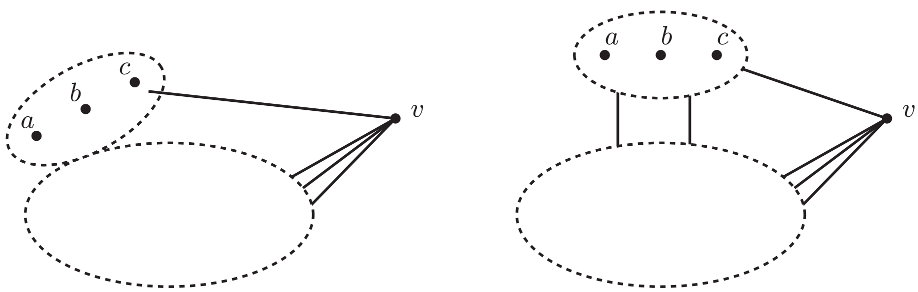


Figure 6. This graph is not birigid.

(iii) If the edges of v are incident with the component of $E-\{e, \mathrm{star}(v)\}$ which contains $\{a, b, c\}$ then$E-\{e, \mathrm{star}(v)\}$ consists of only two rigid components, one of which is a single edge, giving a contradiction to the assumption that v is of valence four. See Figure 7.

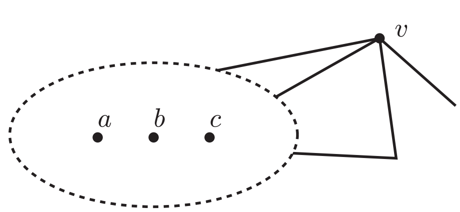


Figure 7. This graph is not birigid.

We conclude that $\{a, b, c\}$ is not contained in one rigid component of $E-\{e, \mathrm{star}(v)\}$, hence there is an edge in $K(\{a, b, c\})$

which is not in the closure of $E - \{e, \text{star}(v)\}$, so

$$E - \{e, \text{star}(v)\} \cup E(T)$$

is independent and therefore rigid for all $v \in V(e)$. so G + T -e is birigid.

(3) If $G + T$ is not edge minimally birigid, then there is an edge e in $G + T$ such that $G + T - e$ is birigid. Since $G + T$ has excess 3, $G + T - e$ has excess 2.

(4) For an edge e with endpoints a and b, both of valence 4, a vertex c of valence 3 is not essential by Theorem 1

The proof of the theorem is now complete. □

Given an edge minimal vertex birigid graph of excess two on n vertices, we can get an edge minimal vertex birigid graph on $n+1$ vertices by choosing an edge e in G with $|V(e)| - 3$ and forming $(G - e) + T$ by identifying three vertices of $V(e)$ with the endpoints of T of valence one. In fact, we obtain all birigid graphs of excess two by this process.

Theorem 3. *Let G be a birigid graph of excess two with $|V| > 5$. Let v be one of its vertices of valence three. $T = star(v)$ and let x, y, and z denote the vertices adjacent to v. Then there is an edge e with endpoints in $\{x, y, z\}$ such that e is not an edge of G and $G - T + e$ is birigid.*

Proof. There are two cases.

(1) The two vertices of valence three in G are adjacent. By Theorem 1the removal of v leaves a circuit, C, with a vertex, x, of valence two attached. Assume x and y are in the same rigid component of $[(C \cup \text{star}(x)) - \text{star}(w)]$, where w is a vertex of valence four in c. We conclude that exactly three edges leave this component. As in the proof of Theorem 2,we derive a contradiction to the circuit structure of G.

Recall that x is not adjacent to y or z, by Lemma 1.

So x and y, y and z are never in the same rigid component of $[C \cup \text{star}(x)] - \text{star}(w)$, therefore $[(C \cup \text{star}(x) - \text{star}(w)] + e$ is rigid if e is one of $(x, y), (x, z)$ respectively.

(2) The two vertices of valence three in G are not adjacent. By Theorem 1, if we remove v, we are left with a circuit, C. We know that for each vertex w of valence four in C there is an edge e with endpoints in $\{x, y, z\}$ such that $C - \text{star}(w) + e$ is rigid. If C contains already two of the possible three edges in $K\{a, b, c\}$, then we are done. If only one of them is contained in C, say (x, y), we have to consider the following: Assume there is a vertex w of valence

four in C such that x and z are in the same rigid component of $C - \text{star}(w)$, and a vertex U of valence four in C such that y and z are in the same rigid component of $C - \text{star}(u)$. These two rigid subgraphs of G intersect in at least one edge, since z is of valence three in C, (x, y) is clearly not in the intersection, and neither are u and v. Since at most four edges leave the rigid component containing x and z and the component containing y and z and one of them is (x, y), we conclude that the intersection of the two rigid components is itself rigid. But since z is of valence three, only five edges can leave a rigid subgraph containing z, which gives a contradiction.

In the remaining case, where no edge of $K\{w, y, z\}$ is contained in C, we derive a similar contradiction. See Figure 8.

Figure 8. This graph is not birigid.

Therefor, there is an edge e in $K\{x, y, z\}$ such that $C - \text{star}(w) + e$ is rigid, i.e., $G - T + e$ is birigid. □

3. An Example

Consider the graph in Figure 9.

We will show that G is edge minimally vertex birigid of excess three and that G does not contain any birigid subgraph of excess two.

To show efficiently that G is birigid, we first show the following

Lemma 2. *Let G_1 and G_2 be rigid graphs with $|V(G_i)| > 3$, $i = 1, 2$, which do not contain any proper rigid subgraph on more than three vertices.*

Let G be the graph obtained from the disjoint union of G_1 and G_2 by connecting four distinct vertices u_i, $i = 1, \ldots 4$ of G_1 to four distinct vertices v_i, $i = 1, \ldots 4$ by edges e_i, $i = 1, \ldots, 4$, such that G is rigid. Then G is a circuit.

Proof. $G - e_i$ is rigid and independent for all i, $i = 1, \ldots 4$. So G contains a circuit. This circuit, being edge birigid, must contain all of the e_i's together with a rigid subgraph of each G_i. Since the v_i's and u_i's are distinct,

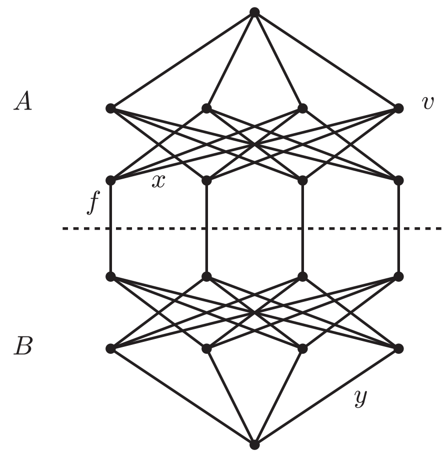


Figure 9. This graph is not birigid.

these subgraphs contain at least four vertices, hence are equal to G_1, G_2 respectively. □

Consider now the subgraph A of G in Figure 9. Remove one of its vertices, v say, to obtain a graph with degree of freedom one, which can be turned into a rigid graph by adding an edge, x. The hypotheses of the lemma are now satisfied, with $G_1 = A - \text{star}(v) + x$, $G_2 = B - y$. So $G - \text{star}(v)$ is rigid, and by symmetry, G is birigid.

G is edge minimally birigid, since the removal of an edge e of A and the star of a vertex in A not incident with e leaves a graph with 17 vertices and 31 edges, which cannot be rigid since B is dependent.

By symmetry, the edges of B are essential. Removal of f and $\text{star}(v)$ obviously destroys rigidity, and so all the edges between A and B are essential.

If G contains a birigid graph H of excess two, then H cannot contain all vertices of G, since G is minimal. But $G - \text{star}(v)$ is rigid and has excess one for every $v \in V$.

Chapter 3

R-Trees

1. Introduction

A tree, a cycle free connected graph, has a natural rigid analogue the *r-tree*, a circuit free rigid graph. There are numerous equivalent definitions for a tree, e.g., G is a tree if and only if every two vertices of G are joined by a unique path. This raises the question: What is an r-path? We will first propose a definition of an r-path and then develop several ways of defining an r-tree.

2. R-Paths

Let $G(V, E)$ be a graph. If x and y are vertices of G, we say that a subset P of E is an *r-path* between x and y if either $p \cup (x, y)$ is a circuit in $R(G)$ or $p = (x, y)$. If $p = (x, y)$ then p is called the *trivial r-path.*

Recall that G is connected if and only if every two vertices of G are joined by a path, and that G is edge biconnected if and only if every two vertices of G are joined by a path of length greater than one. Analogously, we have that G is rigid if and only if every two vertices of G are joined by an r-path, and G is edge birigid if and only if every two vertices of G are joined by a non-trivial r-path.

A maximal rigid subgraph of G is called an *r-component.* We have seen in Chapter 1that two r-components have at most one vertex in common. A non-rigid graph has at least two r-components. In particular, we have

Lemma 1. *A graph with one degree of freedom has an even number of r-components.*

Proof. Let G be a graph with one degree of freedom and consider its r-components $R_1, \ldots, R_n$. In each R_i let V_i be the set of all vertices in R_i which also occur in some R_j, $j \neq i$. To simplify our counting arguments, we want to transform G into a graph G, on an independent edge set as follows. The set of vertices $\bigcup V_i$ remains unaltered.

Each r-component R_i is replaced by a graph on the vertex set $V_i \cup (i_1, i_2)$ with edge set

$$\{(v, i_1), (v, i_2) \mid v \in V_i\} \cup \{(i_1, i_2)\}.$$

An example of this replacement is given in Figure 1. G' has $n + 2\sum |V_i|$

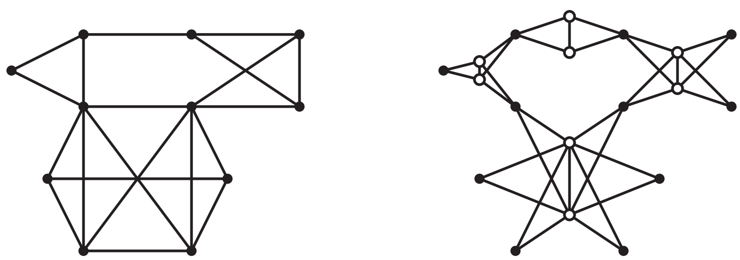


Figure 1.

edges, and, since $E(G')$ is independent, the number of edges is equal to the rank of $E(G')$. We also know that G' has one degree of freedom, which gives $n + 2\sum |V_i| = 2|V(G')| - 4$, and implies that n is even. □

From the proof of Lemma 1, we also have

Lemma 2. *A graph G, of decree of freedom one, with $2k$ r-components, each of which has n_i vertices, has*

$$\sum_{i=1}^{2k} n_i - 3k + 2$$

vertices.

Proof. We use the same G' as in the preceding proof for our counting arguments. By definition of G', we have

$$|V| = |V(G')| - 4k + \sum (n_i - |V_i|).$$

Equation * gives $|V(G')| = \sum |V_i| + k + 2$. and we obtain

$$|V(G)| = \sum |V_i| - 3k + 2 + \sum (n_i - |V_i|) = \sum (n_i) - 3k + 2.$$

□

3. Characterization of R-Trees

An r-tree is a rigid graph which does not contain a circuit. A graph without circuits is called an *r-forest*, its r-components are r-trees. There are thirteen different r-trees on six vertices, as shown in Figure 2.

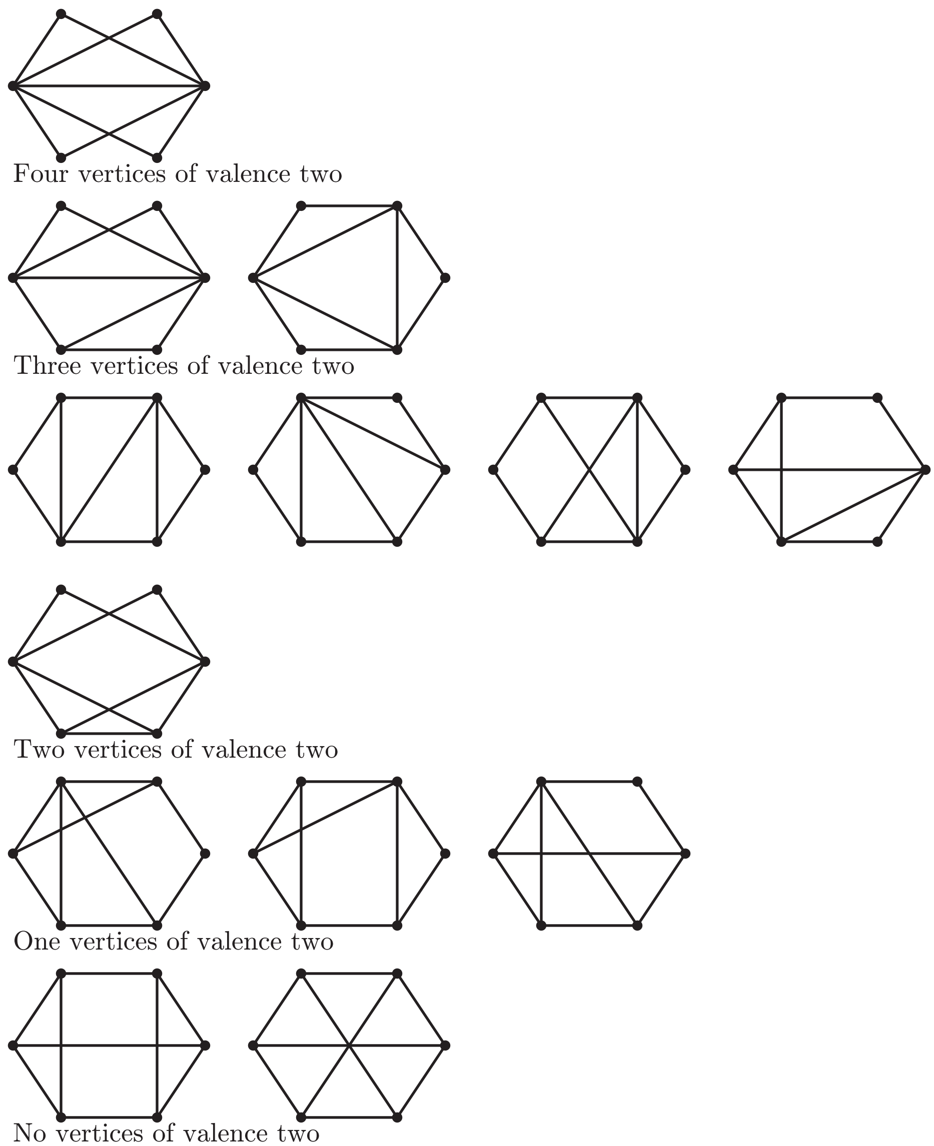


Figure 2. The r trees on six vertices.

An r-tree with more than two vertices of valence two needs more than one additional edge to be birigid, hence is definitely not an r-path. An r-tree with exactly two vertices of valence 2 is an r-path if and only if not both of the vertices of valence 2 are contained in a proper rigid subgraph of G. It is not so easy to decide whether or not an r-tree with fewer than two vertices of valence two is an r-path. In Figure 2, all r-trees with fewer than two vertices of valence two are indeed r-paths. Figure 3 shows an example

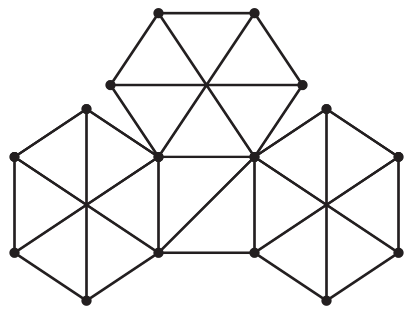


Figure 3. An r-tree hich is not an r-path.

of an r-tree without a vertex of valence two which is not an r-path, since any possible additional edge is contained in a proper rigid subgraph.

Let us call a pair of vertices (x, y) of an r-tree end vertices of T if the union of T and the edge (x, y) is a circuit. We can now distinguish between three kinds of r-trees:

(1) r-trees without a pair of end vertices, i.e. r-trees which are not r-paths.

(2) r-trees with exactly one pair of end vertices, and

(3) r-trees with more than one pair of end vertices.

We now give several characterizations of r-trees.

Theorem 1. *The following statements are equivalent for a graph $G(V, E)$,*

(1) *G is an r-tree,*

(2) *Every two vertices of G are joined by a unique path,*

(3) *G is rigid and $|E| = 2|V| - 3$,*

(4) *$|E| = 2|V| - 3$ and G does not contain a circuit,*

(5) *G does not contain a circuit and if two non-adjacent vertices of G are joined by an edge e not in G then $G + e$ has exactly one circuit,*

(6) *G is rigid, is not $K_{|V|}$ for $|V| > 3$ and if two non-adjacent vertices of G are joined by an edge e, then $G + e$ has exactly one circuit.*

(7) *$|E| = 2|V| - 3$, and if any two non-adjacent vertices of G are joined by an edge e, then $G + e$ has exactly one circuit. Moreover, if K_4 is an r-component of G, then some r-component of G is not complete.*

Proof. 1. implies 2.

Since G is rigid, every two vertices are joined by an r-path. Let P_1 and P_2 be two distinct r-paths joining u and v in G. Then $C_1 = P_1 + (u, v)$ and $C_2 = P_2 + (u, v)$ are distinct circuits in $R(G)$, so $C_1 + C_2 - (u, v)$ contains a circuit of $R(G)$, a contradiction.

2. implies 3.

G is clearly rigid. We prove that $|E| = 2|V| - 3$ by induction. The result is obvious for rigid graphs on two or three vertices. Assume that it is true for graphs on fewer than n vertices. If G has n vertices, the removal of any edge, (u, v), destroys the rigidity of G. Since there was exactly one r-path connecting u and v in G, and, by Lemma 1, the new graph will have $2k$ r-components. By the induction hypothesis, each r-component has $2n_i - 3$ edges if it has n_i vertices and

$$\sum_{i=1}^{2k}(2n_i - 3) = |E| - 1.$$

Moreover, by lemma 2,

$$\sum_{i=1}^{2k} n_i - 3k + 1 = n,$$

which, together with the first equation, gives $|E| = 2|V| - 3$,

3. implies 4.

Assume that G contains a circuit C on n vertices with $2n - 2$ edges. Then there are $|V| - n$ vertices not on this circuit. Choose an edge e of C and extend $C - e$ to a basis B of G. If G is rigid. B contains $2|V| - 3$ edges, and e is not an element of B, so G contains at least $2|V| - 2$ edges.

4. implies 5.

Since G does not contain a circuit, each r-component of G is an r-tree, and E is independent. Therefore, if G has $k > 1$ r-components, then G has degree of freedom at least 1. which means that $|E| < 2|V| - 3$. So $k = 1$ and G is an r-tree. If we add an edge $e = (u, v)$ to G, and P is the unique r-path from u to v in G, then $P + e$ contains a circuit. This circuit is unique because it is the fundamental circuit for e in a base which is an extension of P.

5. implies 6.

Since every K_n for $n \geq 4$ contains a circuit, G cannot be one of them. G must be rigid, since otherwise we could add an edge e between two r-components of G and $G + e$ would not contain a circuit.

6. implies 7.

We will show that any two vertices of G are joined by a unique r-path and thus, because 2. implies 3., $|E| = 2|V| - 3$. Certainly every two vertices of G are joined by some path. If two vertices are joined by two distinct paths, then, by the proof that 1. implies 2., G contains a circuit. This circuit cannot have five or more vertices, because if it did, then we could produce more than one circuit by joining two non-adjacent vertices on the circuit. So the circuit is K_4, which must be a proper subgraph of G, since, by hypothesis, G is not complete with $n \geq 4$. Since G is rigid, we may assume that there is another vertex in G which is joined to this K_4 by at least two edges. Then it is clear that if any edge can be added to G, then one may be added so as to form at least two circuits in $G + e$. If no more edges may be added, so that the second condition is trivially satisfied, then G is K_n with $n \geq 3$, contrary to the hypothesis.

7. implies 1.

If G contains a circuit, then this circuit must be K_4, which is an r-component of G, by an argument in the preceding paragraph. This r-component contains four vertices and six edges. All other r-components of G must be r-trees, and in order to make $|E| = 2|V| - 3$ there can only be one degree of freedom. If one of these other components contains an r-path on more than three vertices, then it will be possible to add an edge e to G and obtain two cycles in $G + e$. Thus all the components must be K_2 or K_3. Therefore G must be a graph with r-components K_4, K_2, or K_3, which are graphs which have been excluded. It follows that G does not contain a circuit, $|E| = 2|V| - 3$ and G is rigid, since 4. implies 5. implies 6. So G is an r-tree and the theorem is proved. □

4. Minimum Cost Spanning R-Trees

Given a graph $G(V, E)$ and a cost function $v : E \to \mathbb{R}$, there are well known algorithms for finding a minimum cost spanning tree of G, and if v is an injection the optimal spanning tree is unique. (See for example Lovasz [1979], [**16**]. For r-trees we have an analogous result.

Theorem 2. *The following algorithm produces a spanning rigid subgraph of the rigid graph $G(V, E)$ with cost function v such that the sum of expenses of the edges is minimal:*

> *At the i'th step choose an edge of minimal cost incident with the closure of the edges already selected, which is not contained in this closure.*

Proof. The resulting graph T is obviously a spanning r-tree. Let H be an optimal spanning r-tree with the maximum number of edges in common with T and let e be an edge of T not in H, selected at the i'th step. Let P

be the r-path in H connecting the endpoints of e. Then there is an edge f of P, which is not in the closure of the previously selected edges, but incident with it, since if all edges of P were in this closure, then the ranks of P and $P + e$ could not be the same. Furthermore, $v(e) \leq v(f)$ since we preferred to choose e rather than f. On the other hand, $H - f + e$ is a spanning r-tree because $P + e$ is the fundamental circuit of e in the base H, and $H - f + e$ has more edges in common with T than H, a contradiction. □

Corollary 1. *If v is an injection, the optimal r-tree is unique.*

Proof. Let S and T be two optimal trees and e be an edge of S not in T. Consider the r-path in T connecting the endpoints of e. Some edge f of it is not in the closure of $S - e$. Now either $S - e + f$ or $T - f + e$ has less expense than S. □

Chapter 4

The Dual

1. Introduction

The concept of dual matroid as introduced by Whitney [1935], [**28**]. There are well known characterizations of the dual of the connectivity matroid of a graph, see Welsh [1976],[**26**] in particular, for a planar graph G, $M^*(G)$ is isomorphic to $M(G^\#)$, where $G^\#$ is the geometric dual of G. Graver [1966][**9**] proved that a matroid is graphic if and only if it is binary and has a 2-complete basis of cocircuits.

We first give a characterization of the dual of $R(G)$ and use it to give necessary and sufficient conditions for a matroid to be the rigidity matroid of a birigid graph, in analogy to Graver's theorem mentioned above.

Finally we show that the geometric dual of a circuit in $R(G)$ is a circuit in $R(G^\#)$.

2. A Characterization of $R^*(G)$

Harary [1969], [**11**], calls a set X of edges of a connected graph G a cutset of G if the removal of X from G results in a disconnected graph, and then defines a cocycle of G to be a minimal cutset of G. We can define an *r-cutset* and a cocircuit analogously for a rigid graph. Welsh [1976], [**26**], extends Harary's definition to disconnected graphs by calling a set X of edges a cutset of G if its removal from G increases the number of connected components. We cannot simply replace connected components by r-components in this definition to obtain a reasonable definition for an r-cutset of a nonrigid graph, since the number of r-components of a graph may actually decrease with the removal of a set of edges, e.g., if G has n r-components, one of which is an edge e, then the removal of e results in a graph with $n-1$ r-components. We know that the rank of $E(G)$ decreases as we remove edges from G, or, equivalently, the degree of freedom of G increases, and we therefore give the following

Definition. A cocircuit of G is a set X of edges of G whose remova1 from G increases its degree of freedom and is minimal with respect to that property.

Lemma 1. *X is a cocircuit of G if and only if X is a minimal subset of $E(G)$ such that X has non-empty intersection with every base of $R(G)$.*

Proof. Assume that there is a base B of $R(G)$ such that $B \cap X = \emptyset$, then $B \subseteq E(G) - X$, so $r(E(G) - X) = r(E)$, a contradiction. □

An immediate consequence of this lemma is

Theorem 1. *If G is a graph and $C^*(G)$ denotes the set of cocycles of G, then $C^*(G)$ is the set of circuits of a matroid $R^*(G)$ on $E(G)$ and*

(1) $R^*(G) = (R(G))^*$,

(2) $R(G) = (R^*(G))^*$.

$R(G)^*$ is called the *cocircuit matroid* of G.

Example: If G is the graph in Figure 1, $R(G)$ has the single circuit $C = \{a, b, c, d, e, f\}$. $C^*(G)$ consists of all two-subsets of C together with $\{g\}$ and $\{h\}$. Note that $R^*(G)$ is not simple.

We can characterize the cocircuits of $R(G)$ as follows:

Theorem 2. *Let $G = (V, E)$ be a rigid graph with $|V| = n$. Let $\{V_1, \dots, V_k\}$ be a system of subsets of V such that $V_1 \cup \dots \cup V_k = V$, V_i is not contained in V_j for all $i \neq j$, and such that*

(1) *V_i induces a rigid subgraph, G_i, of G for all i,*

(2) *for all subsets $\{i_i, \dots, i_l\}$ of the index set holds*

(a) $$|\bigcup_{j=1}^{l} V(G_{i_j})| > \sum_{j=1}^{l} n_{i_j} - 3(l-1)/2 \text{ if } l \text{ is odd, } l \geq 3,$$

(b) $$|\bigcup_{j=1}^{l} V(G_{i_j})| > \sum_{j=1}^{l} n_{i_j} - 3l/2 + 2 \text{ if } l \text{ is even,}$$

where $n_{i_j} = |V(G_{i_j})|$, and

(3) $$\sum_{i=1}^{k} (2n_i - 3) = 2n - 4.$$

Then, the edges of G connecting different V_i form a cocircuit of G, and all edge sets obtained by this process form the collection of cocircuits of $R(G)$.

Proof. Consider a graph G' on the vertex set V which is the union of complete graphs on V_i. Then

$$r(G') = \sum_{i=1}^{k} 2n_i - 3 = r(E) - 1.$$

Therefore $r(E - X) = r(E) - 1$, where $X = E(G) - E(G')$. So X is a cutset. If X is not minimal, then there is an edge e in X such that

$$(E - X + e) = rank(E) - 1 = r(G').$$

The edge e connects different V_i and is contained in a rigid subgraph G'' of G'. We can consider G'' as a union of some G_i's by enlarging G'' by every G_{i_j} which it intersects. Then

$$r(G'') \leq \sum_{j=1}^{l}(2n_{i_j} - 3) < 2|\bigcup_{j=1}^{l} V(G_{i_j}| - 3 = 2|V(G'')| - 3.$$

Therefore G'' cannot be rigid, a contradiction.

Conversely, let C^* be a cocircuit of $R(G)$. Consider the rigid components G_i of $G - C^*$. Then the collection $V(G_i)$ satisfies the hypothesis of the theorem. □

By replacing rigid by connected, and $2n_i - 3$ by $n - 1$ we obtain a simple characterization for the cocycles of $M(G)$ of a connected graph G: Condition (3) of the theorem then implies that $k = 2$ and $V_1 \cap V_2 = \emptyset$, and condition (2) is vacuously satisfied. If we partition V into two sets, V_1 and V_2, each of which induces a connected subgraph, then the edges joining V_1 and V_2 form a cocycle of $M(G)$.

Recall that for a non-rigid graph G, $R(G)$ can be written as a direct sum of its restrictions to the rigid components of G. The set of cocircuits of $R(G)$ is therefore the union over the set of cocircuits of the direct summands.

3. Conditions for a Matroid to be the Rigidity Matroid of a Graph

We shall now be interested in a matroid which satisfies the following conditions:

(1) There is a collection E_i, $i = 1, \ldots, n$ of subsets of E, $E = \cup E_i$, such that $|E_i \cap E_j| \leq 1$ for all $i \neq j$, $|E_i| \geq 3$ for all i and $E_i - e$ is a cocircuit of M for all $e \in E_i$ and each $e \in E$ is contained in exactly two of the E_i's.

(2) For all $F \subseteq E$ we have $r(F) \leq 2n|F| - 3$, where $n(F)$ is the number of E_i's with non-empty intersection with F.

Given now a matroid satisfying conditions 1 and 2, we associate with M a graph G in the following manner:

- The vertices of G are the sets E_i
- The edges of G are the elements in E
- An edge e has endpoint E_i if $e \in E_i$.

Observe that, because of condition 1, each edge has exactly two distinct endpoints and no two edges have the same pair of endvertices, so G has no loops or parallel edges. Moreover, since $|E_i| \geq 3$, every vertex of G has valence at least three. We will show that M and $R(G)$ are isomorphic, but first we need to prove two lemmas.

Lemma 2. *Any independent set I in $R(G)$ determines uniquely an independent set in M.*

Proof. Obviously a one edge set determines an independent set in M, since every element is contained in a cocircuit of cardinality at least two. We proceed by induction on the cardinality of I. Let $|I| = m+1$ and assume the lemma holds for independent sets in $R(G)$ of smaller cardinality. I induces a subgraph G_I of G which contains at least one vertex of valence less than or equal to three. Remove from G_I the star I_1 of a vertex v_1 of minimum valence, and all possible isolated vertices, to obtain another independent edge set $I - I_1$ of $R(G)$, and proceed in this manner to get a partition of I, $I = I_1 \cup \cdots \cup I_m$, and a sequence of vertices of G_I, $v_1, v_2, \ldots, v_k$, such that $|I_i| \leq 3$ and no element of I_i has v_j as an endpoint if $i > j$ and every element of I_i has v_i as an endpoint. Assume the elements $e_1, \ldots, e_{n+1}$ of I are ordered such that $e_i < e_j$ if $e_i \in I_{i'}$, $e_j \in I_{j'}$ and $i' < j'$. Observe that I_m contains exactly one element, namely $e_(n+1)$.

Recall that the vertices of G correspond to unions of cocircuits of M. For subsets of elements of M the above therefore means that I_i is contained in $E - v_j$ if $i < j$ and $e_i \in v_j$ means that e_i is in I_j. Note that $E - v_i$ is a closed set, namely the intersection of hyperplanes which are the complements of the cocircuits in v_i. By induction we assume that $e_1, \ldots, e_n$ is an independent set of elements in M. Now if e_{n+1} is dependent on $e_1, \ldots, e_n$, then e_{n+1} is contained in

$$S = c(I_{m-1} \cup \cdots \cup I_1) \cap (E - v_1) \cap \cdots \cap (E - v_k).$$

If, for example, $I_1 = \{e_1, e_2, e_3\}$ then by condition (1) there are three hyperplanes, h_1, h_2, and h_3 such that $(E - v_1)$ is equal to $h_1 \cap h_2 \cap h_3$ and e_i

is not contained in h_i, $1 \leq i \leq 3$. We thereby get

$$\begin{aligned} c(I_{m-1} \cup \cdots \cup I_1) \cap (E - v_1) &= c(e_n + ... + e_1) \cap h_1 \cap h_2 \cap h_3 \\ &\subseteq c(e_n + ... + e_2) \cap h_2 \cap h(3) \subseteq \cdots \\ &\subseteq c(e_n + \cdots + e_4). \end{aligned}$$

Similarly, if I_1 is of smaller cardinality. Rewriting $(E - v_i)$ for all $1 \leq i \leq k$ in this manner, we see that S is empty, a contradiction. □

Lemma 3. *Any set of edges in G which forms a circuit in $R(G)$ determines a circuit in M.*

Proof. Let $C = \{e_i e_{2k-2}\}$ be a circuit on the vertices $v_1, \ldots, v_k$. By Lemma 2,it is clear that any subset of C is independent in M. Moreover, condition 2 implies that C is dependent. □

Theorem 3. *A matroid M satisfying conditions 1 and 2 is isomorphic to the rigidity matroid of a graph G.*

Proof. By Lemma 3, a circuit in $R(G)$ determines a circuit in M. Let C be a circuit in M which is not a circuit in $R(G)$. If C is independent in $R(G)$ then Lemma 2gives a contradiction. If in $R(G)$, C properly contains a circuit C', then, by Lemma 3, C' is a circuit in M which is properly contained in C, again a contradiction. □

Theorem 4. *If G is birigid, then $R(G)$ satiefies conditions 1 and 2.*

Proof. If G is birigid, the removal of star(v) decreases the rank of $E(G)$ by 2, hence star(v) $- e$ is a cocircuit for every $e \in E$, moreover $|\text{star}(v)|$ 3 for every v. We conclude that the stars of the vertices of G determine the desired partition of the edge set and condition 1 is satisfied. Condition 2 is Laman's condition. □

Theorem 5. *A matroid n is isomorphic to the rigidity matroid $R(G)$ of a birigid graph G if and only if n is connected and satisfies conditions 1 and 2.*

Proof. Since n is connected and isomorphic to the rigidity matroid of a graph G. G is rigid by proposition 2 in chapter 1. So $R(G)$ has rank $2n - 3$ if G has n vertices.

Furthermore, for every vertex v in G and any edge e in star(v). $E -$ star(v) $+ e$ is a hyperplane, hence has rank $2n - 4$, so $E -$ star(v) has rank $2(n-1) - 3$ which is to say that G is birigid.

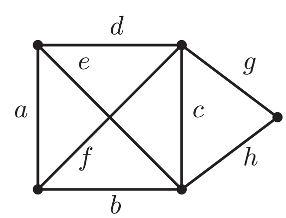
Conversely, if G is birigid, then $R(G)$ is connected by Theorem 3, in chapter 1, and Theorem 4 completes the proof. □

4. The Geometric Dual of a Circuit

Theorem 6. *Let C be a planar graph and let $C^{\#}$ be its geometric dual. Then $R(C)$ consists of a circuit if and only if $R(C^{\#})$ consists of a circuit.*

Proof. Let C have n vertices and $2n-2$ edges. As an immediate consequence of the Euler characteristic $C^{\#}$ has the same number of vertices and edges as C. If $C^{\#}$ is not a circuit, then it must properly contain a circuit C' on k vertices and $2k-2$ edges. The geometric dual of C' is a contraction of C. This contraction is obtained by contracting $2(n-k)$ edges of C. Each connected component of the subgraph induced by these $2(n-k)$ edges is contracted to a single vertex. Since these edge sets are free in $R(C)$, their supports satisfy Laman's inequality. Let $x_1, \ldots, x(r)$ be the cardinalities of the connected components. Then the contracted graph has at most $n + r - \sum(1/2)(x_i + 3) \leq k - 1$ vertices, contradicting the fact that C' has k vertices. $\square$

Examples: Wheels are self dual.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Chapter 5

Complete Graphs

1. The Connectivity Matroid of K_n

Theorem 1. *A matroid M of rank $n-1$ on a set E with $|E| = \binom{n}{2}$ is $M(K_n)$, the connectivity matroid of the complete graph K_n on n vertices, if and only if*

(1) *For every closed subset F of E there is a unique decomposition*

$$F = F_1 \cup \cdots \cup F_k$$

so that $r(F) = \bigcup_{i=1}^{k} r(F_i)$ *and* $|F_i| = \binom{r(F_i)+1}{2}$.

(2) *For every edge e in a closed subset F of E with $|F| = \binom{r(F)+1}{2}$, there are $r(F)-1$ pairs $\{a,b\}$ of elements of $F - \{e\}$ such that $e \in c(a,b)$.*

Observe that condition (1) implies that M is simple. We will call the two elements $e, e' \in E$ incident if e and e' are contained in a circuit of cardinality three. A circuit of cardinality three will be called a triangle. Elements of E will be called edges.

The proof will be preceded by five lemmas.

Lemma 1. *Two triangles intersect in at most one edge.*

Proof. Assume that $\{a,b,c\}$ and $\{a,b,d\}$ are triangles. Then $\{a,b,c,d\}$ is contained in $c(a,b)$, a contradiction. □

Lemma 2. *Let $k \leq r(M)-2$. An edge e of M together with any k triangles containing e spans a subset F of E with $r(F) = k+1$ and $|F| = \binom{k+2}{2}$.*

Proof. Condition (2) implies that every edge e is in k triangles $T_i = \{e, a_i, b_i\}$, $1 \leq i \leq k$, for every $1 \leq k \leq r(M) - 2$. Any two of these triangles intersect in exactly the edge e by Lemma 1, hence the union of two such triangles has five edges and its closure has trivial decomposition, rank three, and therefore, by condition (1), has six elements. Let the additional element in the closure of $T_i \cup T_j$ be denoted by $e_{i,j}$. We want to show that

(A) $\{e_{i,k}, e\}$ is closed, which implies that $e_{i,k} \in \cup\{a_i, b_i\}$, and

(B) $e_{i,j} \neq e_{r,s}$ for $(i,j) \neq (r,s)$.

To show (A), assume w.l.o.g. that $c(e_{i,j}, e) = \{e_{i,j}, e, a_1\}$. Then

$$\{e, a_1, b_1, e_{i,j}\} \subseteq c(e, a_1)$$

contradicting [1].

To show [B], assume first that $e_{i,j} = e_{i,k}$, $j \neq k$. Then

$$\{e, a_i, b_i, a_j, b_j, a_k, b_k, e_{i,j}\} \subseteq c(e_{i,j}, a_i e),$$

contradicting [1]. Assume next that $e_{i,j} = e_{k,l}$, $i \neq k$. $j \neq l$. Let

$$F = \{e_{i,j}, e, a_i, b_i, a_k, b_k, a_j, b_j, a_l, b_l\}$$

then $F \subseteq c(e_{i,j}, e, a_i, a_k)$. The set F is closed by [1]. Therefore $e_{i,j} \in F$. By the previous observation $e_{i,k} \neq e_{i,j}$. Furthermore, $e_{i,j}$ is not an element of $\cup\{a_j, b_j\}$ by [A]. So F is a proper subset of $c(a_i, b_i, a_k, b_k)$, a contradiction. We conclude that $|F| = \binom{k+2}{2}$, and $r(F) = k + 1$. □

Corollary 1. *A set S of mutually adjacent edges is independent if $|S| > 3$.*

Lemma 3. *If e_0 is incident to e_1 and e_2, and either $\{e_0, e_1, e_2\}$ is a triangle or e_1 and e_2 are non-incident, then an edge f which is incident to e_0 is incident to exactly one of e_1 and e_2.*

Proof. Assume f was incident to both e_1, e_2, i.e., we have the triangles $\{e_0, f, x\}$, $\{e_i, f, y\}$ and $\{e_2, f, z\}$. By Lemma 1, x, y, and z are all distinct. Then

$$\{f, e_0, e_1, e_2, x, y, z\} \subseteq c(f, e_i, e_2),$$

contradicting [1].

Assume next that f is not incident to e_1 or e_2. By Lemma 2, the triangles $\{e_0, e_1, e_2\}$ and $\{e_0, f, x\}$ span a set F of rank three. The edge f has to be contained in at least two triangles of F to satisfy condition [2]. Lemma 1 forces this second triangle to contain either e_1 or e_2. In the case that e_1 and e_2 are non-incident we have a similar contradiction. □

Lemma 4. *If S is a set of mutually incident edges with $|S| > 3$. If e is not in the closure of S but is incident with an element of S, then either e is incident with all members of S or with exactly one member of S.*

Proof. Let $a, b, c \in S$. By the Corollary to Lemma 2, the closure of $\{a, b, c\}$ has six elements, which we will denote by $\{a, b, c, \alpha, \beta, \gamma\}$, and contains the four triangles $\{a, b, \alpha\}$. $\{a, c, \beta\}$, $\{b, c, \gamma\}$, and $\{\alpha, \beta, \gamma\}$. Assume that e is incident with a and b but not c. Since e is not in the closure of S by assumption, $c\{a, b\}$ has six elements, which we will denote by $\{a, b, e, \alpha, x, y\}$, and contains the triangles $\{a, b, \alpha\}$, $\{a, e, x\}$, $\{a, e, y\}$, and $\{\alpha, x, y\}$. Similarly, in the closure of $\{a, c, e\}$ we have the elements $\{a, c, e, \alpha, x, z\}$ and the triangles $\{a, c, \beta\}$, $\{a, e, x\}$, $\{e, \beta, z\}$, and $\{e, x, z\}$. In the closure of $\{b, c, e\}$ we find the elements $\{b, c, e, \gamma, y, o\}$ and the triangles $\{b, c, \gamma\}$, $\{b, e, y\}$, $\{e\gamma, o\}$, and $\{c, y, o\}$. Since the closure of $\{a, b, c, e\}$ contains ten elements, o must be contained in the set $\{a, b, c, e, x, y, z, \alpha, \beta, \gamma\}$, which leads to a contradiction with lemma 1. □

Lemma 3 and Lemma 4 give

Lemma 5. *The union of k triangles containing e is the union of two sets of mutually incident edges of cardinality $k + 1$ whose intersection is e.*

Proof. Proof of the Theorem: We procede by induction on n. If $n = 2$, M consists of exactly one edge, and is thus isomorphic to K_2.

Consider any element e of M and the triangles containing e. By Lemma 2, all but one of them span a subset W of rank $r(M) - 1$ so that, by the induction hypothesis, there is an isomorphism $f : W \to K_{n-1}$. Let x be a vertex in K_{n-1} and $S(x) = f^{-1}(\text{star}(x))$. $S(x)$ is a set of mutually incident edges. For every element e in $S(x)$, there are two elements, a and b, such that $\{e, a, b\}$ is a triangle and $\{a, b\}$ is not contained in $c(S(x))$, which follows from Lemma 2 and condition 2. By Lemmas 3 and 4, exactly one of a and b, a say, is incident with all elements of $S(x)$. We define $F(a)$ to be the edge connecting x to the vertex n. We have to show that F is a matroid isomorphism. F maps M onto K_n and the preimage of each triangle in K_n is a triangle in M. Hence M contains all cycles of K_n.

Assume that there was a circuit C in M such that $F(C)$ was not a cycle of K_n. The circuit C has n edges and rank $n - 1$, otherwise $F(C)$ would be a cycle of K_n by the induction hypothesis. So $F(C)$ is dependent in $M(K_n)$. Since $F(C)$ is not a cycle of K_n, it properly contains a cycle of K_n which, by the induction hypothesis, corresponds to a circuit of M. Therefore C properly contains another circuit, a contradiction. □

Conditions (1) and (2) of Theorem 1 are independent in the sense that there do exist matroids of rank $n - 1$ on a set of $\binom{n}{2}$ elements satisfying either (1) or (2) which are not isomorphic to K_n: Consider the uniform matroid U of rank $n - 1$ on S, with $S = \binom{n}{2}$. U satisfies (1) since every set of elements whose cardinality is strictly smaller than $n - 1$ is free and closed, and therefore has a unique decomposition into single edges, whereas $c(E) = S$ for all E with $|E| \geq n-1$. For $n > 3$, U contains no triangles, hence U cannot be $M(K_n)$.

On the other hand, consider a graph G with $n-1$ components, n even, each of which contains $n/2$ parallel edges. $M(G)$ satisfies (2) as soon as $n \geq 12$.

2. The Rigidity Matroid of K_n

Theorem 2. *A matroid M on E with $r(M) = 2n-3$ and $|E| = \binom{n}{2}$ is the rigidity matroid of the complete graph K_n on n vertices if and only if*

(1) *For every closed subset F of E there is a unique decomposition $F = F_1 \cup F_2 \cup \cdots \cup F_k$ such that $r(F) = \sum r(F_i)$ and $|F| = \binom{(r(F_i)+3)/2}{2}$ with k minimal (the F_i's are called cliques) and*

(2) *For every pair $e, f \in F$, where F is a clique, there are elements $a, b, c, d, \in F - \{e, f\}$ such that $e \in c(a, b, c, d, f)$ and $f \in c(a, b, c, d, e)$.*

Elements of E will be called edges. Condition (1) implies that edge sets of cardinality less than six are independent. A circuit of cardinality six will be called a tetrahedron.

As above, the proof of the theorem will be preceded by several lemmas.

Lemma 1. *Two tetrahedra with non-empty intersection intersect in one, three, or six edges.*

Proof. Let S and T be tetrahedra, $|S \cap T| \neq 0$. The closure of $S \cup T$ is a clique, therefore, by condition (1), it has odd rank not less than five. If $r(S \cup T) = 5$ then $S = T$. $S \neq T$ implies that $r(S \cup T) \geq 7$, since S and T are closed by (1). The sub modular inequality

$$r(S \cup T) + r(S \cap T) \leq r(S) + r(T) = 10. \tag{10}$$

gives $|S \cup T| \leq 3$. Assume $|S \cap T| = 2$. Then $|S \cup T| = 10$ and $r(S \cup T) \geq 8$ by (10). In $S \cup T$, however, not every pair of elements is contained in a tetrahedron, contradicting (2). □

If two tetrahedra meet in three edges, we will call their intersection a *triangle.*

Lemma 2. *Every edge in a clique K, of rank exceeding five, is contained in a triangle in K.*

Proof. Condition (2) implies that every edge e of K is contained in at least two tetrahedra, say S and T. Assume $S \cap T = \{e\}$. We have $|S \cup T| = 11$, which implies $r(S \cup T) = 9$. Thus $c(S \cup T) - (S \cup T)$ contains four edges. The edge e together with anyone of these four edges is contained in a tetrahedron which must intersect S or T in an edge distinct from e, hence in a triangle containing e. □

Lemma 3. *k (distinct) tetrahedra intersecting in a triangle T span a clique of rank $2k+3$.*

Proof. By Lemma 2, we know that if $n > 4$ then every edge in K is in a triangle T, hence there are at least two tetrahedra intersecting in T. Let $T_1, \ldots, T_k$ be tetrahedra containing T. $r(T_i \cup T_j) > 5$, since $T_i \neq T_j$, and $c(T_i \cup T_j) \leq 7$, by the sub modular inequality, so $r(T_i \cup T_j) = 7$ and

$$c(T_i \cup T_j) - (T_i \cup T_j) = e_{i,j}.$$

We want to show that $e_{i.j} \neq e_{k,l}$ whenever $(i,j) \neq (k,l)$. Assume $e_{i,j} = e_{i,k}$, $j \neq k$. Extend $e_{i,j} + T$ to a basis $B_{i,j}$ of $T_i \cup T_j$ such that $B_{i,j}$ contains four edges of T_i. (Every five-subset of edges is free and can be extended to a basis). Extend the same four-subset of T_i to a basis $B_{i,k}$ of $T_i \cup T_j$ such that $B_{i,k}$ contains $e_{i,k} = e_{i,j}$. Then the set $B_{i,j} \cup B_{i,k}$ spans $T_i \cup T_j \cup T_k$, hence $r(T_i) \cup T_j \cup T_k) \leq 8$. But $|T_i \cup T_j \cup T_k| = 12$, violating condition (1).

Assume next that $e_{i,j} = e_{k,l}$, $i \neq k$, $j \neq l$,

$$c(T_i \cup T(j) > -(T_i \cup T_j >= e(i.j),$$

$$c(T_k \cup T(l - (T_k \cup T(l = e(k.l).$$

Extend $T \cup e_{i,j}$ to a basis $B_{i,j}$ of $T_i \cup T_j$ and also to a basis $B_{k,l}$ of $T_k \cup T_1$. $|B_{i,j} \cup B_{k,l}| = 10$, therefore $r(T_i \cup T_j \cup T_k \cup T_l) \leq 10$. We have $|T_i \cup T_j \cup T_k \cup T_l| = 15$, which implies that $e_{i,j}$ is contained in the union of these four tetrahedra, contradicting Lemma 1.

We conclude that the closure of k tetrahedra with base T contains $B_{k+3,2}$ elements and has rank $2(k+3) - 3 = 2k+3$. □

Lemma 4. *Every triangle T in a clique K of rank $2k+3$, with $k > 1$, is contained in k tetrahedra.*

Proof. By Lemma 2, every edge is in a triangle and by Lemma 3, every triangle is in at most k tetrahedra.

Assume that there is a triangle which is contained in only t tetrahedra, $T_1, \ldots, T(t)$, t maximal and $t < k$, (t is at least 2 by lemma 2). The union of the T_i's spans a clique C of rank $2t+3$. Every edge f in $K - C$ and every edge e of T is contained in a tetrahedron, T_{t+1}. If T_{t+1} intersects one T_i in a triangle for some i, $1 \leq i \leq t$, then the closure of $T_i \cup T_{t+1}$ contains a tetrahedron containing T which is not contained in the closure of the union of the T_i's, contrary to the assumption that t is maximal.

Let us therefore assume that T_{t+1} intersects T_i in e only:

$$T_i = \{e, f, g, 1, 2, 3\}, T_{t+1} = \{e, 4, 5, 6, 7, 8\},$$
$$c(T_i \cup T_{t+1}) - (T_i \cup T_{t+1}) = \{\alpha, \beta, \gamma, \delta\},$$

$T = \{e, f, g\}$. Consider the tetrahedron in $c(T_i \cup T_{t+1})$ containing e and a. Since this tetrahedron must not intersect T_i in a triangle, otherwise it would violate the maximality condition on t, it must therefore intersect T_{t+1} in a triangle and contain three edges from $\{\alpha, \beta, \gamma, \delta\}$. W.l.o.g., let $\{e, \alpha, 4, 5, \beta, \gamma\}$ be a circuit, and again w.l.o.g. $\{e, \delta, \alpha, \beta, 6, 7\}$ be a circuit. Then the tetrahedron containing e and 8 must intersect T_i in a triangle. This contradiction proves the lemma. □

With these results and some straightforward calculations analogous to the corresponding calculations in the previous section, we easily get

Lemma 5. *If e_0 is incident to e_1 and e_2, and either $\{e_0, e_1, e_2\}$ is a triangle or e_1 and e_2 are non-incident, then an edge f which is incident to e_0 is incident to exactly one of e_1 and e_2.*

Lemma 6. *A set S of mutually incident edges is independent.*

Lemma 7. *If S is a set of mutually incident edges with $|S| > 3$. If e is not in the closure of S but is incident with an element of S, then either e is incident with all members of S or with exactly one member of S.*

Lemma 8. *The union of k tetrahedra containing the triangle $T = \{a, b, c\}$ is the union of three sets A, B and C of mutually incident edges of cardinality $k + 2$, and $A \cap B = \{a\}$, $B \cap C = \{b\}$, and $C \cap A = \{c\}$. Furthermore, the union of an two of A, B or C is a basis of M.*

We are now ready for the

Proof. (Proof of Theorem 2) We proceed by induction on n. The theorem is trivially true for $n = 1, 2, 34$.

Let T be any triangle in M and consider the tetrahedra containing T. All but one of them span a subset, W, of rank $r(M) - 2$, so that, by the induction hypothesis, there is an isomorphism $f : W \to K_{n-1}$. Let $e = (u, v)$ be an edge in K_{n-1} and $A = f^{-1}(\text{star}(u)$. $B = f^{-1}(\text{star}(v))$. A and B are sets of mutually incident edges, and by Lemma 8, $A \cup B$ is a basis of W. For every triangle T of W which is contained in $A \cup B$, there are three elements a, b, $c \in M - W$ such that $T \cup \{a, b, c\}$ is a tetrahedron. By Lemmas 5 and 7, exactly one of a, b, or c, say a, is incident with all elements of A. We define $F(a)$ to be the edge connecting u to the vertex n. We have to show that F is a matroid isomorphism. Clearly all tetrahedra, and consequently all circuits of K_n. are contained in M.

Assume there is a circuit C in M which is not a circuit in K_n. C has rank $2n-3$ and $2n-2$ edges, otherwise C would be a circuit of K_n by the induction hypothesis. So C is dependent in $R(K_n)$. Since C is not a circuit of K_n, it properly contains a circuit of K_n which, by the induction hypothesis, is a circuit of M. Therefore C properly contains another circuit, a contradiction. □

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