PATH PARTITIONS OF RIGID GRAPHS

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Laman [4] proved that a simple graph $G = (V, E)$ is (generically) *rigid* in the plane, if and only if there is a subset F of E such that

$$|F| = 2|V| - 3, \text{ and} \tag{1}$$

$$|F'| \leq 2|\sigma(F')| - 3, \text{ for all nonempty } F' \subseteq F, \tag{2}$$

where $\sigma(X)$ denotes the set of endpoints of the edge set X. Condition (1) ensures that G has enough edges to be rigid, while condition (2) ensures that no subset of vertices is overbraced by the edges satisfying (1).

Let $G = (V, E)$ be a graph. An edge set $F \subseteq E$ is called *independent* if condition (2) is satisfied. The independent subsets of E are the collection of independent sets of a matroid on E which we denoted by $M(2, E)$. If the graph is K_n, the complete graph on n vertices, then the matroid is denoted by $M(2, n)$ and is called the *2–dimensional generic rigidity matroid*. Since the independence of $F \subseteq E$ is not affected by any of the edges in G not in F, we may regard G as a subgraph of K_n for some $n > |V|$ and $M(2, E)$ will then be the restriction of $M(2, n)$ to E.

A set F of edges is said to be *isostatic* if it is both independent and rigid, and if $E = F$, then we sa y the graph $G = (V, E)$ is isostatic. Lovasz and Yemini [5] observed that Laman's condition for a graph G to be isostatic is related to a theorem by Nash–Williams [6, 7]: $E(G)$ is isostatic if and only if adding any edge to G yields the edge disjoint union of two spanning trees. Equivalently, $E(G)$ is independent in $M(2, n)$, if, after adding any edge to G, the resulting graph can be decomposed into two spanning forests. Recski [8, 9] proved that in the above statements, "adding any edge to G" may be replaced "by doubling any edge of G", and it has been shown in [1] that

Theorem 1. *A graph G on n vertices is a cycle in $M(2, n)$ if and only if it is the edge disjoint union of two trees no two of whose subtrees have the same span.*

For a direct proof see [11]. Two such decompositions are illustrated in Figure 1.

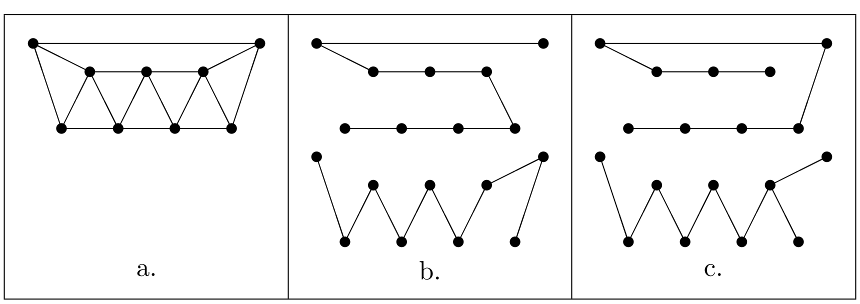

Figure 1. A graph partitioned into two trees

A graph G is called *edge birigid* if its edge set is rigid in $M(2,n)$ and the removal of any edge of G leaves a rigid graph. It is easy to show, see [10], that cycles in $M(2,n)$ are edge birigid. Examples of cycles and an account of their properties can be found in [12]. We note that the average valence of a vertex in such a cycle is $4-4/n$ and that the minimum valence is 3. It follows that every cycle has at least 4 vertices of valence 3. We would like to consider cycles that have exactly 4 vertices of valence 3 and all other vertices of valence 4. An example is given in Figure 1a. The graph depicted there has a decomposition into the two Hamiltonian paths of Figure 1b. This motivates the following

Problem 1. *If a graph is the edge disjoint union of two spanning paths such that no two subpaths have the same span, then its edges form a cycle in $M(2,n)$ with exactly 4 vertices of valence 3 and all other vertices of valence 4. Is the convers true?*

In [10], birigidity in the plane is examined. A graph G is called *(vertex) birigid* if its edge set is rigid in $M(2,n)$ and remains rigid even after the removal of any vertex of G, together with the edges incident with it. A simple infinite family of birigid graphs in 3–space, which are in some sense the 3–dimensional analogue to the graphs under consideration in the previous problem, may be constructed as follows. Take m disjoint tetrahedra $\{T_1,\ldots,T_m\}$ and for each $1<i<m$ glue two distinct faces of T_i to T_{i-1} and T_{i+1} respectively. Notice that this graph is planar and also that it may be constructed such that it has no vertex of valence greater than 6, and exactly 6 vertices of valence less than 6. These last six vertices may be paired off with three additional edges such that each has valence 5. Figure 2 has an example. In [11] such a graph is shown to be birigid in 3–space. Figure 2 shows a

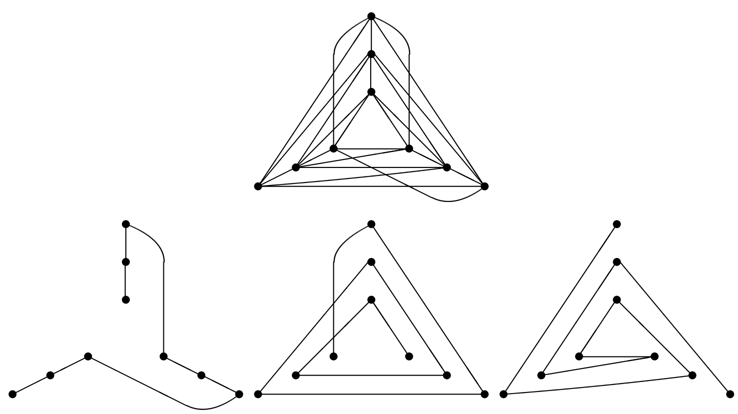

Figure 2. A decomposition of a graph birigid in 3–space.

decomposition of such a graph into the edge disjoint union of three spanning paths, motivating the following ambitious

Problem 2. *Characterize graphs with k-path decompositions in terms of rigidity in dimension k.*

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