A note on Gaussian graphs

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Abstract

A graph is Gaussian if it is the graph of arcs and self-intersections of a closed C^∞ curve in the plane. In this note we describe a recursive characterization of 4-regular Gaussian graphs and give conditions under which the Gaussian property is a graph invariant.

1 Introduction

Let G be an Eulerian planar graph, with planar embedding $\mathcal{G}$. We allow G to have loops and multiple edges. Let N be the set of edges incident with a vertex v. If we label the edges of N clockwise in $\mathcal{G}$ from $\{1, \ldots, \deg(v)\}$, then pairs of vertices with the same label modulo $\deg(v)/2$ are said to be *parallel*. A trail in G is said to be *transverse in* $\mathcal{G}$ if its successive edges are parallel. A transverse circuit is said to be *Gaussian* if it contains all the edges of G. The graph G is said to be Gaussian if there is an embedding $\mathcal{G}$ which contains a Gaussian circuit. In Figure 1 there is an example of a Gaussian graph and its transverse circuit. This example is the smallest simple 4-regular Gaussian graph [2].

Gaussian graphs were introduced in [2], motivated by work of Gauss, [3] on the theory of knots. Gauss observed that in the sequence of multiple

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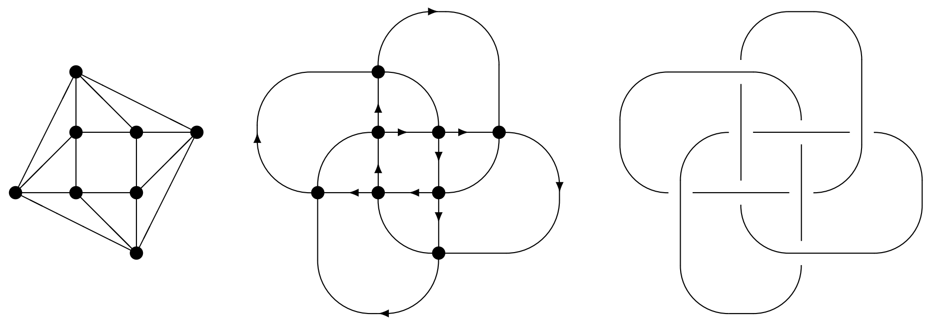


a) Gaussian Graph b) Transverse Circuit c) Alternating Knot

Figure 1:

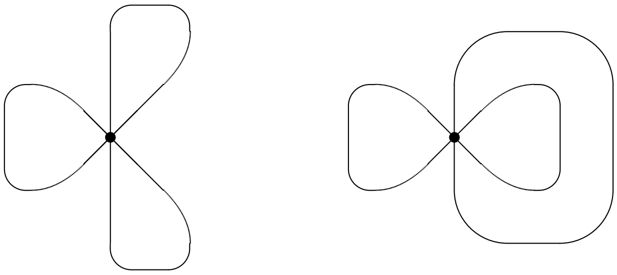
points encountered while traversing a closed curve in the plane with only double points, each multiple point occurs twice, once in an even position, and once in an odd position. Gauss's observation was proved in [4]. See [5] for an elementary treatment. A graph theoretic reformulation and proof of this theorem is given in [2].

THEOREM 1 (THEOREM OF GAUSS) *Let G be a 4-regular planar connected Gaussian graph with planar embedding $\mathcal{G}$ and Gaussian circuit C. Then every proper subcircuit of C is of odd length.*

An easy consequence of this result is that every 4-regular Gaussian graph corresponds to the diagram of an alternating knot, see Figure 1.

2 Embedding invariance

If $\mathcal{G}$ is the planar embedding of an Eularian graph G, then the edges of G are partitioned into transverse circuits. The number of transverse circuits will in general depend on the embedding $\mathcal{G}$, see Figure 2. Note that the example in Figure 2 is only 1-connected. There are also 2-connected examples of this type, however, if G is 3-connected, then the embedding $\mathcal{G}$ is essentially unique, and the number of transverse circuits is a graph invariant. The next theorem shows that for 4-regular graphs this invariance does not depend on connectivity.



a) 1 transverse circuit b) 2 transverse circuits

Figure 2:

THEOREM 2 *Let G be a 4-regular planar graph. Then the number of transverse circuits of G is independent of the planar embedding.*

PROOF: To eliminate the consideration of superfluous special cases, we will assume that the graph is embedded on the sphere.

If G is 3-connected, then combinatorially there is only one embedding, by Whitney's theorem, see [7], so the result follows in this case.

Suppose that G is 2-connected. Then, [8], any two embeddings of G are connected by a sequence of Whitney twists, see Figure 3.

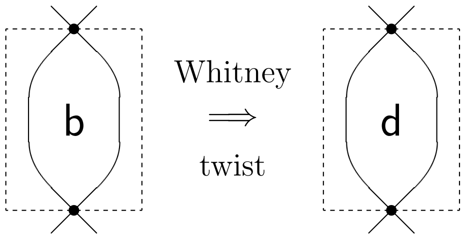


Figure 3:

Let us consider the possible cases for a single Whitney twist. Let $\{v, w\}$ be the cutset. Then either the portion of the graph to be twisted contains an odd number of edges incident with v, and an odd number incident with w, in which case twisting does not change the parallel classes at v and w, and so does not alter the partition into transverse circuits, or both these numbers are even, in which case the possibilities are illustrated schematically in Figure 4.

Note that there may be other transverse circuits completely contained in the dotted sections. In the first case, the twist does not affect the partition

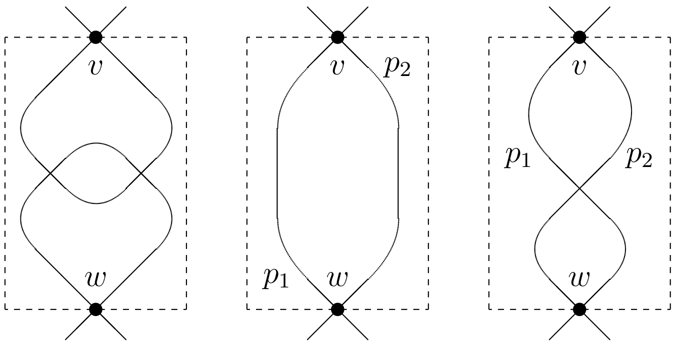


Figure 4:

into transverse circuits, while in the other two, a twist does not alter the number transverse circuits, since twisting simply exchanges the edges in paths p_1 and p_2. Thus the result is true if G is 2-connected.

Finally, we must consider the case that G is 1-connected. We use induction on the number of blocks. If G has a single block, then G is 2-connected, and the result follows from above. Otherwise, G has a cutvertex, v, and edges reading clockwise e_1, e_2, e_3, e_4. Since v is a cutvertex, all four edges must belong to the same transverse circuit. Let us cut G at v into two graphs, G_1 and G_2, with e_1 and e_2 in G_1 and e_3 and e_4 in G_2, and let us also replace e_1 and e_2 with a single edge E_1 and e_3 and e_4 with a single edge E_2. G_1 and G_2 are both 4-regular and planar, and by induction the number of their transverse circuits is independent of the embedding, hence it follows that the number of transverse circuits of G is independent of the embedding. $\square$

Consequently, for a 4-regular planar graph the property of being Gaussian is a graph invariant. Thus to determine whether or not a given 4-regular graph is Gaussian is no more difficult than to decide planarity.

3 Recursive construction of Gaussian graphs

We would like to describe how to recursively generate all 4-regular Gaussian graphs together with their Gaussian circuits. First we define an operation we call vertex splitting.

Definition 1 *Let G be a 4-regular Gaussian graph with embedding $\mathcal{G}$ and Gaussian circuit C. Let v be a vertex of G, and let the edges of G incident* *with v occur in C in the order $\{e_1, e_2, e_3, e_4\}$, with orientations as shown in Figure 5. We modify G to form a new graph $G\rightleftarrows v$ by removing v, replacing e_1 and e_3 with a single edge e_{13} as well as replacing e_2 and e_4 with a single edge e_{24}. This process is called* vertex splitting at v.

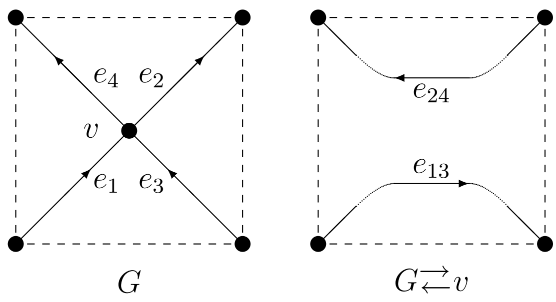


Figure 5:

It is easy to see that $G\rightleftarrows v$ is Gaussian. Note that, in the definition of the split, if we were to pair the edges the other way, replacing e_1 and e_4 with a single edge, as well as e_2 and e_3, then the result would *not* be Gaussian, but have precisely two transverse circuits. Finally, note that performing vertex splits on a simple graph may yield graphs with loops and parallel edges.

Since G may be split at any vertex to form a smaller 4-regular Gaussian graph, we have the following result.

THEOREM 3 *Every 4-regular Gaussian graph may be transformed into the figure eight graph (one vertex and two loops) by a sequence of vertex splits.*

The reverse operation to vertex splitting is edge splicing.

DEFINITION 2 *Suppose G is a 4-regular graph with embedding $\mathcal{G}$ on the sphere. The sequence of edges in the Gaussian circuit induces an orientation of the edges of G. Let F be a face of $\mathcal{G}$. We say that two edges e_1 and e_2 bounding F are* co-oriented *if they both have the same orientation with respect to the orientation of F, and* disoriented *otherwise. (Note that a vertex split always results in a pair of co-oriented edges.)*

DEFINITION 3 *Let G be a 4-regular Gaussian graph and e_1 and e_2 a pair of co-oriented edges. Modify G to form $G\times\{e_1, e_2\}$ by replacing the edges e_1 and e_2 with a new degree 4 vertex adjacent to the endpoints of e_1 and e_2. $G\times\{e_1, e_2\}$ is called the splice of G with respect to e_1 and e_2.*

We may now restate Theorem 3 in terms of splices.

THEOREM 4 *Every 4-regular Gaussian graph may be obtained from the figure eight graph by a sequence of splices.*

As an example, Figure 6 shows how to construct the smallest simple 4-regular Gaussian graph from the figure eight by edge splicings. In each figure, the edges to be spliced are indicated by arrows.

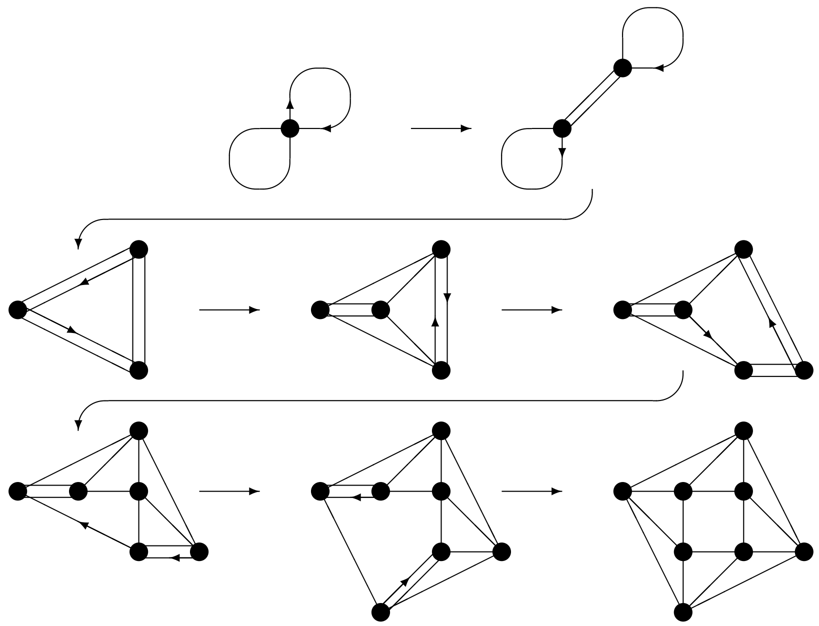


Figure 6:

This recursive construction may be used to give an easy inductive proof of the Theorem of Gauss, as well as the result that every regular projection of a knot is also the regular projection of an alternating knot, as well as the projection of a trivial knot.

To explicitly convert any Gaussian graph into a 4-valent Gaussian graph, replace every vertex of degree $2n$, $n > 2$, with a complete n-line, see [1]. This process is clearly reversible and shows that our recursive construction is sufficient to generate all Gaussian graphs.

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