THE 2-DIMENSIONAL GENERIC RIGIDITY MATROID AND ITS DUAL

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1. Abstract

We consider the 2-dimensional generic rigidity matroid $R(G)$ of a graph G and give a characterization of the dual of $R(G)$. We show that the connectivity of $R(G)$ implies the birigidity of G but not conversely. Finally we give necessary and sufficient conditions for a connected matroid to be the rigidity matroid of a birigid graph.

2. Introduction and Basic Definitions

Let $G = (V, E)$ be a graph on the edge set E, vertex set V. We define the *support* $\sigma(F)$ of a subset F of E to be the set of endpoints of edges in F.

We define a subset F of E to be *independent* if $|F'| \leq 2|\sigma(F')| - 3$ holds for all subsets F' of F. It is well known, see [8] and [14], that these independent edge sets are the independent sets of a matroid, the so-called *2-dimensional generic rigidity matroid*, $R(G)$, of the graph G. The closure operator and rank function of this matroid will be denoted by c and r respectively. The term *circuit* will always refer to a circuit in $R(G)$.

$G = (V, E)$ is called *rigid* if $r(E) = 2n - 3$, where $|V| = n$. G is called *edge birigid*, if $r(E - e) = 2n - 3$ for every $e \in E$. G is called *birigid* if G is rigid and $r(E - \text{star(v)}) = 2(\text{n} - 1) - 3 = 2\text{n} - 5$ for every $v \in V$, where star(v) denotes the set of edges adjacent to v. We will henceforth abbreviate $E - \text{star(v)}$ with $E - v$. To simplify notation and language we will not distinguish between sets of edges and the subgraphs they induce.

The following observations are immediate consequences of the definitions. The union of two graphs G_1 and G_2 having at most one vertex in common is not rigid, and $c(G_1 \cup G_2) = c(G_1) \cup c(G_2)$. If two rigid graphs intersect in two or more vertices, their union is rigid. Rigidity induces an equivalence relation on the edge set of G. The equivalence classes are called *r-components*. It follows that r-components have at most one vertex in common and that birigid graphs are at least 3-connected. Moreover, $R(G)$ can be written as the direct sum over the r-components of G.

3. A Characterization of $R^*(G)$

Harary [1969] calls a set X of edges of a connected graph G a *cutset* of G if the removal of X from G results in a disconnected graph, and then defines a *cocycle* of G to be a minimal cutset of G. We can define an *r-cutset* and a *cocircuit* analogously for a rigid graph. Welsh [1976] extends Harary's definition to disconnected graphs by calling a set X of edges a *cutset* of G if its removal from G increases the number

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of connected components. We cannot simply replace connected components by r-components in this definition to obtain a reasonable definition for an r-cutset of a non-rigid graph, since the number of r-components of a graph may actually decrease with the removal of a set of edges, e.g., if G has n r-components, one of which is an edge e, then the removal of e results in a graph with n-1 r-components. We know that the rank of E(G) decreases as we remove edges from G, or, equivalently, the degree of freedom of G increases, and we therefore define a cocircuit of G to be a set X of edges of G whose removal from G increases its degree of freedom and is minimal with respect to that property. An immediate consequence of this definition is

Lemma 1. *The set X is a cocircuit of G if and only if X is a minimal subset of $E(G)$ such that X has non-empty intersection with every base of $R(G)$.*

and

Theorem 1. *If G is a graph and $C^*(G)$ denotes the set of cocycles of G, then $C^*(G)$ is the set of circuits of a matroid $R^*(G)$ on $E(G)$ and*
(1) $R^*(G) = (R(G))^*$
(2) $R(G) = (R^*(G))^*$.
$R(G)^*$ *is called the* cocircuit matroid *of G.*

We can characterize the cocircuits of $R(G)$ as follows:

It will in general not be possible to construct a G such that every vertex of G corresponds to a cocycle of $R(G)$, because the next theorem shows that birigidity of G implies the connectivity of $R(G)$ and that this implication is not an equivalence.

Theorem 2. *If $G = (V, E)$ is birigid and $|V| > 3$, then $R(G)$ is connected but not conversely.*

Proof. Assume that G is birigid and that $R(G)$ is not connected. Consider the connected components R_i of $R(G)$. Then there is a partition of E,

$$E = E_1 \cup E_2 \cup \cdots \cup E_k,$$

such that

$$R(G) = R_1 + R_2 + ... + R_k,$$

where $R_i = R(G_i)$, with $G_i = (\sigma(E_i), E_i)$. since every G_i is rigid, we have

$$2|V| - 3 = r(G) = \Sigma_{i=1}^k r(G_i) = \Sigma_{i=1}^k (2|\sigma(E_i)| - 3)$$

. Let us define N_i, n_i, N, and n by the following equations:
(1) $N_i = |\sigma(E_i) - \cup_{i\neq j}\sigma(E_j)|$
(2) $|\sigma(E_i)| = n_i + N_i$
(3) $N = \Sigma_{i=1}^k N_i$
(4) $|V| = n + N$.

So

$$n \leq \frac{1}{2}\Sigma_{i=1}^k n_i$$

Rewriting [1] in this new notation we obtain

$$2n + 2N - 3 = \Sigma_{i=1}^k (2(n_i + N_i) - 3)$$

or

$$2n = 3(1 - k) + \Sigma_{i=1}^k 2n_i$$

so that [2] and [3] give

$$(\Sigma_{i=1}^{k} 2n_i) - 3(k-1) \leq \Sigma_{i=1}^{k} n_i,$$

or

$$\Sigma_{i=1}^{k} n_i \leq 3(k-1).$$

Furthermore, since every cutset in a birigid graph has cardinality at least 3, we have that

$$|\sigma(E_i) \cap (\cup_{i \neq j} \sigma(E_j))| \geq 3,$$

which implies that $n_i \geq 3$ for all i. This combined with [4] gives

$$3k \leq \Sigma_{i=1}^{k} n_i \leq 3(k-1),$$

a contradiction.

If $R(G)$ is connected, G need not be birigid: If G is a wheel, $R(G)$ consists of a single circuit and hence is connected. But the removal of the center vertex leaves a non-rigid graph if the number of spokes is larger than 3. □

References

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