Blanuša Double

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Abstract

A snark is a non-trivial cubic graph admitting no Tait coloring. We examine the structure of the two known snarks on 18 vertices, the Blanuša graph and the Blanuša double. By showing that one is of genus 1, the other of genus 2, we obtain maps on the torus and double torus which are not 4-colorable. The Blanuša graphs appear also to be a counter example for the conjecture that $\gamma(P^n) = n - 1$ (P^n is a dot product of n Petersen graphs) given by Tinsley and Watkins in [10]. We also prove that the 6 known snarks of order 20 are all of genus 2.

1 Introduction

Ever since Tait [9] proved that the four-color theorem is equivalent to the statement that every planar bridgeless cubic graph is edge 3-colorable, snarks, i.e. non-trivial cubic graphs possessing no proper edge-3-coloring, have been investigated. For an explanation of the term non-trivial in the definition of snark, see [7]. The smallest snark is a graph on 10 vertices, namely the Peterson graph. It was discovered in 1898, [8].

The second snark appearing in the literature is a graph on 18 vertices, discovered by the Croatian mathematician Danilo Blanuša, [1]. It is constructed from two copies of the Petersen graph by a construction generalized in [3] to obtain infinite families of snarks.

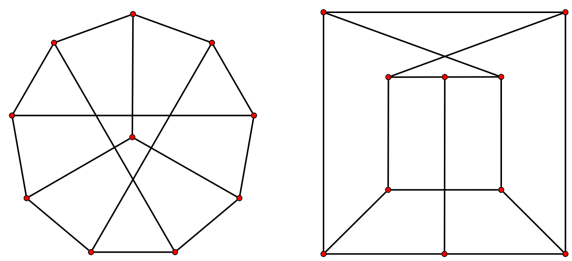


Figure 1: The Petersen graph, the smallest and oldest snark. The drawing is a variant of the one presented by A. B. Kempe already in 1886. The second drawing is due to Danilo Blanuša.

Tutte conjectured that, in fact, every snark contains the Petersen graph as a minor. The conjecture was recently proved by Robertson, Sanders, Seymour and Thomas and it is currently in the process of publishing in series of articles.

Later it was discovered that this operation can be applied in two different ways and another snark on 18 vertices was constructed. We call the first one *the Blanuša snark* and the second one *the Blanuša double.*

Also an interesting and still open conjecture on snarks and their embeddings into orientable surfaces is also a Grünbaum's conjecture [2] which is a generalization of the 4-color theorem.

2 The dot product

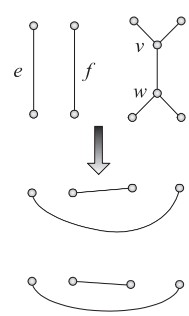


Figure 2: The dot product.

The dot product, $S_1 \cdot S_2$, of two snarks S_1 and S_2 is defined in [3] by removing two non-incident edges e and f from S_1 and two adjacent vertices v and w from S_2 and joining the endpoints of e and f to neighbors of u and v like in the Figure 2. Performing this operation using the Petersen graph for both S_1 and S_2, yields, depending on the choice of the deleted edges, the Blanuša snark and the Blanuša double.

Both graphs are cubic graphs on 18 vertices and are difficult to identify. Actually, most drawings of the two graphs make hard to distinguish between them, see for instance [12], where the figure of the same graph appears twice. In this note we present a result that will help distinguish the two graphs in the future.

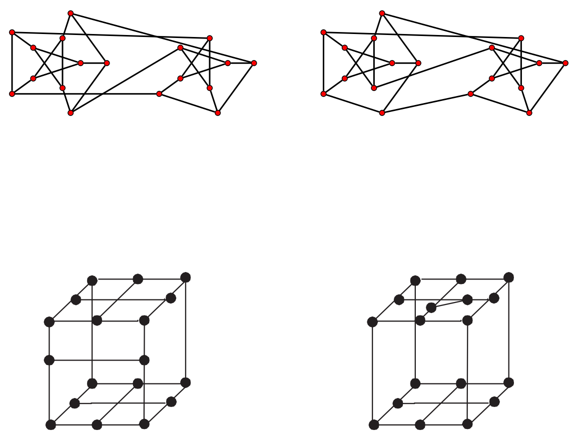


Figure 3: The Blanuša snark and its double. They can be distinguished in several ways. While the genus of the Blanuša snark is 1, the genus of its double is 2. The computer search for the number of 1-factors (Kekule structures) shows that there are $K = 19$ Kekule structures in the Blanuša snark and $K = 20$ in its double.

3 Genus embeddings

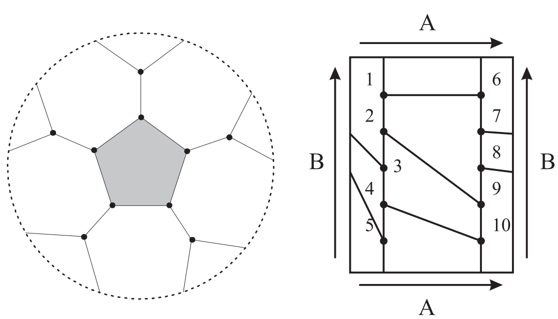


Figure 4: The Petersen graph is not planar. However, it can be embedded in the projective plane and in the torus.

Theorem 3.1. The genus of the Blanuša snark is 1 while the genus of the Blanuša double is 2.

However, we first look at the Petersen graph. It is clearly non-planar. The simplest non-orientable surface, in which we can embed the Petersen graph is the projective plane (see the Figure 4). The embedding in this case is pentagonal and highly symmetric. In order to exhibit a toroidal embedding, one has to provide a suitable collection of facial walks. We obtain 3 pentagonal faces, one hexagonal face and one nonagonal. We can represent this symbolically as : $5^3 6^1 9^1$.

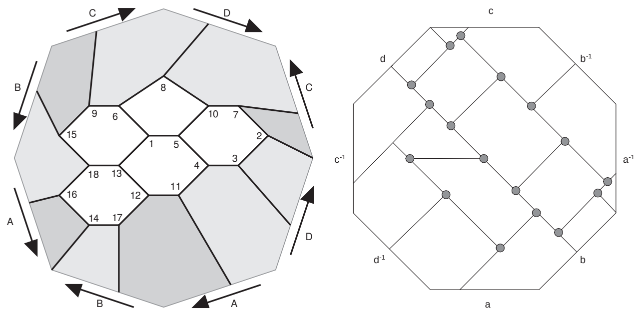


Figure 5: The Blanuša double on the double torus. The drawing on the left was produced by Herman Servatius.

Petersen snark on the torus:

1	6	10	4	5				
2	3	4	10	9				
7	8	5	4	3				
6	7	3	2	1	5	8	9	10
1	2	9	8	7	6			

Similarly, we can define the embeddings of the other two snarks by giving the collection of facial walks (i.e. combinatorial faces).

Blanuša snark on the torus:

14	16	18	15	17					
11	12	13	18	16					
4	5	10	7	9					
2	3	11	16	14					
2	7	10	8	3					
1	6	8	10	5					
1	5	4	15	18	13				
1	13	12	17	15	4	9	6		
2	14	17	12	11	3	8	6	9	7

Blanuša double on the double torus:

1	5	10	8	6											
12	13	18	16	14	17										
2	7	10	5	4	3										
1	6	9	15	18	13										
1	13	12	11	4	5										
2	14	16	11	12	17	15	9	7							
2	3	8	10	7	9	6	8	3	4	11	16	18	15	17	14

A proof for an embeddability of a graph is simply a combinatorial embedding or a list of faces. These can be produced by some torus embedding program like the program EMBED written by one of authors according to [4].

To prove a non-toroidality of a graph G one would usually find a minor of the graph G for which it is known that it is not toroidal. There is a finite list of minimal forbidden minors for each surface (see [6]). Unfortunately, a complete list of minimal forbidden minors for a torus is currently not known.

To prove non-toroidality of a graph G we used the program EMBED to find a candidates for minimal forbidden minors. Starting with a graph G one sequentially

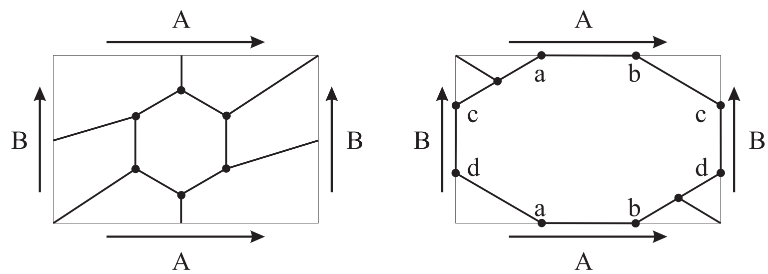


Figure 6: Two essentially different unlabelled embeddings of $K_{3,3}$ into torus

removes and contracts the edges. If a removal or a contraction of an edge produces a toroidal graph then the operation is ignored. Otherwise the process is repeated on the obtained graph. When no removal or contraction of any edge is possible without obtaining a toroidal graph, the candidate for a minimal forbidden graph is obtained. It only remains to prove that the candidate is indeed non-toroidal.

For the snarks in the sequel it appears that the obtained candidates for the forbidden minors have similar properties so we were able to prove their non-toroidality by a general proposition.

The Kuratowski graph $K_{3,3}$ admits exactly two essentially different unlabelled embeddings into torus. The two embeddings are shown on fundamental polygons for the torus in the Figure 6.

The first embedding in the Figure 6 does not have a singular face. A face is called *singular* if the corresponding face walk is not a cycle. This also means, that the closure of the face is not homeomorphic to the closed disk. An embedding is *cellular* if all the faces are discs. In a cellular embedding the non-singular faces are homeomorphic to the closed disk. In a singular face some vertex or a sub-path occurs more than once along the facial walk. The repeated parts are called *singular parts*. Each face walk of a graph embedded into a torus is oriented according to the positive orientation of the surface (in our case the torus). This orientation induces the orientation of the singular parts (if they are sub-paths).

The second embedding in the Figure 6 has one singular face, with the singular parts ba, cd, ab and dc. Transversal of the face has been done in direction of positive orientation of the fundamental polygon.

Let G be a graph and K subgraph of G. All vertices of K having degree at least 3 are called *main vertices*. A path in K between two main vertices that does not contain any main vertex in its interior is called a *branch*. A connected component C of $G - K$ together with all edges from G having one vertex in K and the other in C is called a *bridge*. An edge of G not contained in K but having both vertices in K is also considered as a bridge. The edges of a bridge that have one vertex in K are called *legs* of the bridge and the vertices of the legs contained in K are called *attachment vertices*. All other vertices of the bridge are called *interior vertices*.

A *non-contractible curve* on a torus is a simple closed curve such that if we cut the torus along the curve, then the obtained surface is still connected. It is commonly known, that if one cuts the torus along any of non-contractible curves, than the obtained surface can be embedded into plane.

Let us prove the following proposition:

Proposition 3.2. Let graph G consist of K, which is a subdivision of $K_{3,3}$, and some bridges. Let one of bridges, say a bridge B, satisfy one of the following properties:

1. Bridge B has at most 3 legs and if one joins the attachment vertices with a path P, then $P \cup B$ contains a subdivision of $K_{3,3}$.

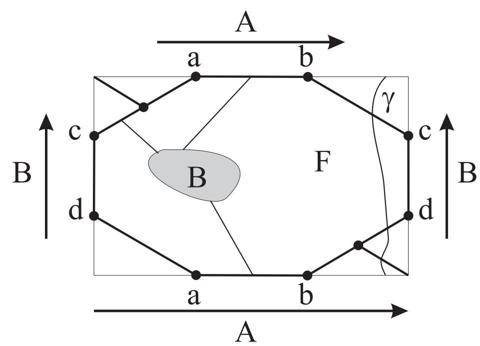


Figure 7: Cutting torus with non-contractible curve γ near the unused singular part.

2. Bridge B has at most 5 different attachment vertices, two of them are in the interior of two different incident branches b_1 and b_2 and the others are attached to main vertices of b_1 and b_2. No two legs of B are attached to the same attachment vertex. A union $b_1 \cup b_2 \cup B$ contains a subdivision of Kuratowski graph $K_{3,3}$.

Then graph G is not toroidal.

Proof. If the graph G was toroidal than K should be embedded in one of ways shown in the Figure 6. Obviously any bridge B must be embedded into a single face F of the embedding of K. The face cannot be non-singular, since in both cases the face walk and bridge B together with a subdivision contain $K_{3,3}$ embedded into the closed face F and hence into the plane. This is a contradiction to the Kuratowski theorem. Hence K is embedded into a torus with 2-singular face F with B embedded into F. In the case 1 of the proposition B has at most 3 attachment vertices, hence it can be attached on at most 3 singular parts. In the case 2 of the proposition the branches b_1 and b_2 must be consecutive along the facial walk. This means that only one branch of b_1 or b_2 can be contained in some singular part s_1 and the other branch has at most one main vertex contained in some other singular part, say s_2. According to the properties of the bridge B in the case 2, only one leg can be attached to the singular part s_2. This means again that at most 3 singular parts are used in any embedding of B into the singular face F.

In each of the cases one can cut the torus along a non-contractible curve γ in a manner shown in the Figure 7. Removing the edges which intersect with the curve γ one obtains a planar embedding of the bridge B together with the connected rest of the facial walk of F containing a subdivision of $K_{3,3}$. Again a contradiction to the Kuratowski theorem.

□

For the labelled Blanuša double graph in the Figure 8 and its forbidden minor returned by the program EMBED a description how to obtain the minor has to be given.

The following notation:

```
n : (a b c) : i j k l
```

means: vertex labelled n in minor was obtained by the contraction of all edges in a connected component on the vertices labelled a, b, c in the original graph. The vertex n is adjacent to the vertices i, j, k, l in the minor. If the vertex n in the minor has no adjacent edges in the description then vertex may be removed.

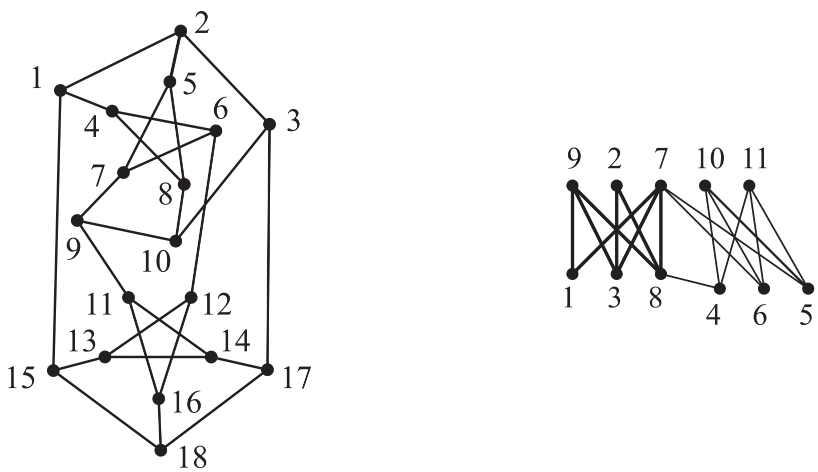


Figure 8: Labelled Blanuša double and its minimal forbidden minor for a torus.

The forbidden minor from the Figure 8 was obtained in the following way:

```
Sn2:
Vertices: 11, Edges: 5
1: (5): 7 9 2
2: (8): 3 1 8
3: (10): 9 7 2
4: (12): 10 11 8
5: (14): 7 11 10
6: (18): 7 10 11
7: (1 2 3 17): 3 8 1 5 6
8: (4 6): 2 7 9 4
9: (7 9): 8 1 3
10: (11 16): 5 4 6
11: (13 15): 4 5 6
```

Applying the proposition 3.2 one can easily see that the minor is not toroidal. Corresponding subdivison of $K_{3,3}$ (K in the proposition) is marked with bold edges in the F igure 8.

Combining all together the theorem 3.1 follows.

The consequence of the theorem is a counter example for the conjecture of F. C. Tinsley and J. J. Watkins. Their conjecture was that if P is a Petersen graph and P^n stands for a dot product of n Petersen graphs, then $\gamma(P^n) = n - 1$. The Blanuša snarks are both obtained as a dot product of two Petersen snarks, but the genus is different. Another counter example is Szekeres's snark, that is obtained as a dot product of 5 Petersen graphs. From the Figure 9 one can easily find 5 disjoint subdivisions of $K_{3,3}$ which cannot be embedded into a surface of the genus 4.

4 The 6 snarks on 20 vertices

The labelled 6 snarks on 20 vertices are shown in Figure 10. We name the 6 snarks Sn4,Sn5,. . ., Sn9. Snark Sn4 is also known as the smallest *Flower snark* from the infinite family of the flower snarks (see [3]).

There are exactly 6 snarks of order 20, [12].

Theorem 4.1. All snarks on 20 vertices have genus 2.

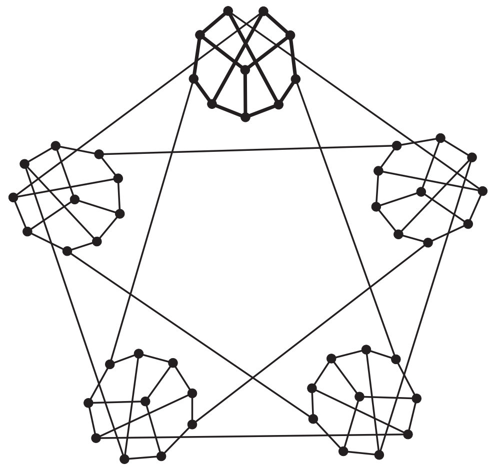


Figure 9: The Szekeres snark with one of the 5 disjoint subgraphs $K_{3,3}$ marked bold.

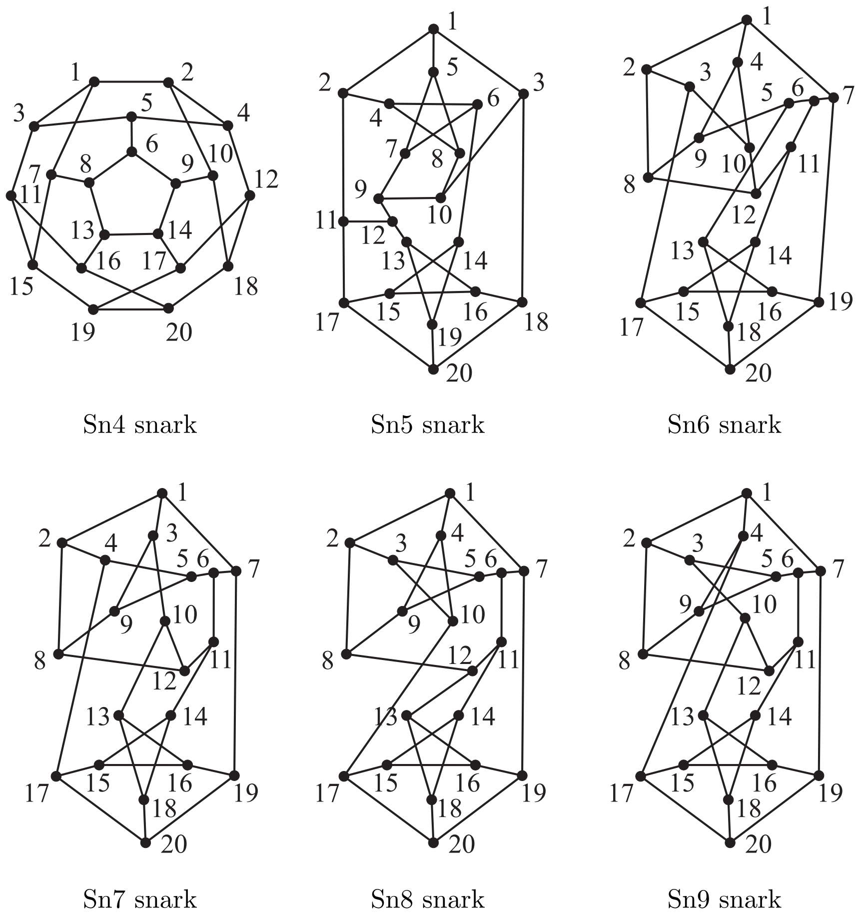
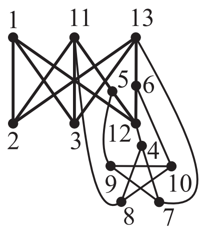
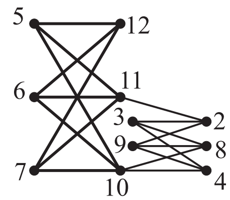


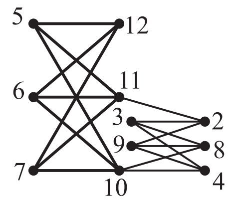
Figure 10: The 6 snarks on 20 vertices.



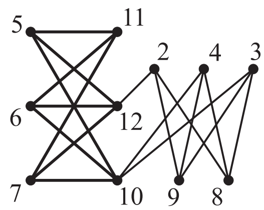
Sn4's forbidden minor



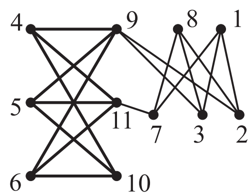
Sn5's forbidden minor



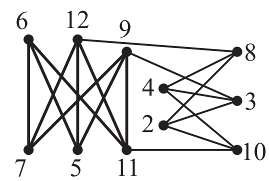
Sn6's forbidden minor



Sn7's forbidden minor



Sn8's forbidden minor



Sn9's forbidden minor

The forbidden minors for a torus of the 6 labelled snarks are shown in the Figure 4. Using the proposition 3.2 one can easily see that all 6 given minors are non-toroidal.

The minors were obtained the following way:

```
Sn4:
Vertices: 13, Edges: 6
1: (6): 3 12 2
2: (8): 13 11 1
3: (9): 11 13 1
4: (12): 7 12 8
5: (15): 11 12 9
6: (16): 10 13 12
7: (17): 9 4 13
8: (18): 11 10 4
9: (19): 10 5 7
10: (20): 6 8 9
11: (1 2 7 10): 5 8 3 2
12: (3 4 11 5): 6 1 4 5
13: (13 14 ): 6 7 2 3
```

```
Sn5:
Vertices: 12, Edges: 6
2: (6): 3 9 11
3: (7): 8 2 4
4: (9): 9 3 10
5: (13): 10 12 11
6: (15): 10 11 12
7: (20): 11 10 12
8: (1 2 5): 3 10 9
9: (4 8 10): 8 4 2
10: (11 12 17): 7 8 6 5 4
11: (14 18): 5 6 7 2
12: (16 19): 7 6 5
Vertex 3 from original
graph removed
```

```
Sn6:
Vertices: 12, Edges: 6
2: (3): 12 8 9
3: (9): 8 10 9
4: (12): 8 9 10
5: (14): 12 11 10
6: (16): 11 12 10
7: (20): 10 11 12
8: (2 8): 3 2 4
9: (4 10): 4 3 2
10: (5 6 11 7 19): 5 6 7 3 4
11: (13 18): 5 6 7
12: (15 17): 7 6 5 2
Vertex 1 from original
graph removed
```

```
Sn7:
Vertices: 11, Edges: 5
1: (8): 7 2 3
2: (9): 1 9 8
3: (12): 9 1 8
4: (14): 9 11 10
5: (16): 9 10 11
6: (20): 11 10 9
7: (2 4): 1 8 11
8: (1 3 10): 3 7 2
9: (5 6 11 7 19): 4 6 5 3 2
10: (13 18): 4 6 5
11: (15 17): 5 6 4 7
```

```
Sn8:
Vertices: 12, Edges: 6
2: (3): 3 10 8
3: (5): 4 9 2
4: (9): 3 8 10
5: (14): 6 9 12
6: (18): 7 5 11
7: (20): 9 6 12
8: (4 10): 12 4 2
9: (6 11 7 19): 3 11 5 7
10: (2 8 12): 11 2 4
11: (13 16): 6 12 9 10
12: (15 17): 5 11 8 7
Vertex 1 from original
graph removed
```

```
Sn9:
Vertices: 11, Edges: 5
1: (5): 7 8 2
2: (9): 9 1 6
3: (14): 4 8 11
4: (18): 5 3 10
5: (20): 11 8 4
6: (1 2 4): 11 7 2
7: (3 10): 1 6 9
8: (6 11 7 19): 10 3 5 1 9
9: (8 12): 7 2 8
10: (13 16): 8 11 4
11: (15 17): 5 10 3 6
```

To prove that the genus of all 6 snarks is 2 one have to produce embeddings of those graphs into double torus. The embeddings are shown in Figure 4. The embeddings were found by program VEGA [11] using an algorithm that checks all possible combinations of local rotations. Having local rotations for embeddings the figures were produced. The lists of facial walks for the embeddings are also shown below.

Sn4 faces in double torus

13	16	20	19	17	14												
7	8	13	14	9	10	18	20	16	11	15	19	20	18	12	17	19	15
4	6	9	14	17	12	5											
3	11	16	13	8	6	4											
2	5	12	18	10													
1	2	10	9	6	8	7											
1	7	15	11	3													
1	3	4	5	2													

Sn5 faces in double torus

14	15	16	19	20	18							
11	12	13	16	15	17							
4	6	14	18	13	12	9	7	5	8			
3	19	16	13	18	20	17	15	14	6	7	9	10
2	4	8	10	9	12	11						
1	2	11	17	20	19	3						
1	3	10	8	5								
1	5	7	6	4	2							

Sn6 faces in double torus

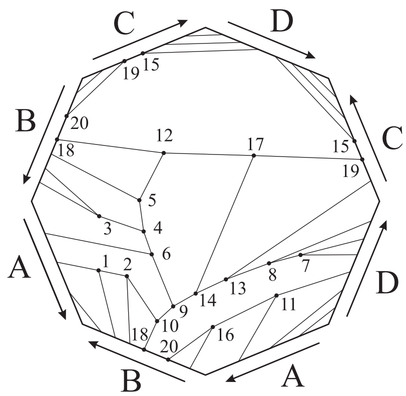
13	16	15	14	18								
6	7	19	20	18	14	11						
4	9	8	12	10								
3	10	12	11	14	15	17						
2	3	17	20	19	16	13	5	6	11	12	8	
1	2	8	9	5	13	18	20	17	15	16	19	7
1	7	6	5	9	4							
1	4	10	3	2								

Sn7 faces in double torus

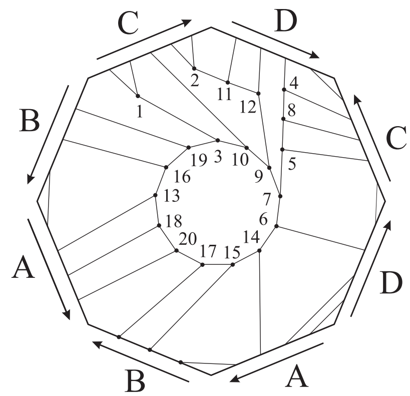
13	16	15	14	18						
6	7	19	20	18	14	11				
4	5	9	8	12	11	14	15	17		
3	9	5	6	11	12	10				
2	4	17	20	19	16	13	10	12	8	
1	2	8	9	3						
1	3	10	13	18	20	17	15	16	19	7
1	7	6	5	4	2					

Sn8 faces in double torus

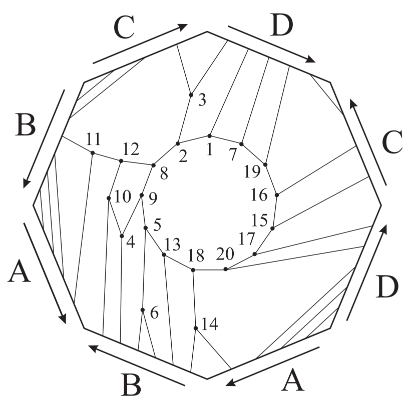
13	18	20	17	15	16							
11	14	18	13	12								
5	6	11	12	8	9							
3	5	9	4	10								
2	8	12	13	16	19	7	6	5	3			
1	2	3	10	17	20	19	16	15	14	11	6	7
1	7	19	20	8	14	15	17	10	4			
1	4	9	8	2								



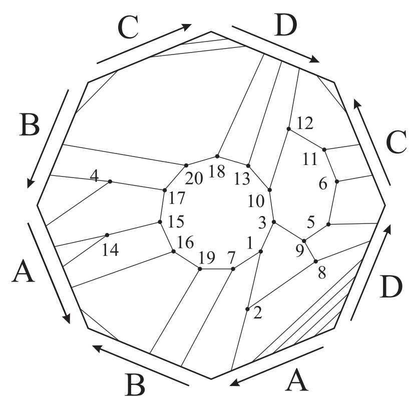
Sn4 (flower snark) embedded into double torus



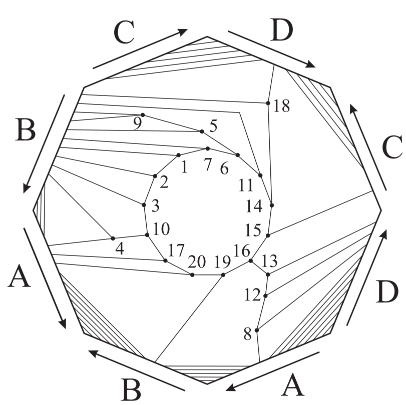
Sn5 embedded into double torus



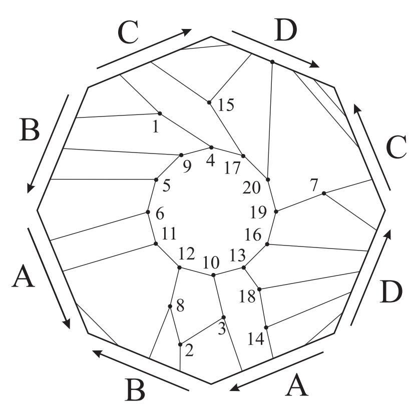
Sn6 embedded into double torus



Sn7 embedded into double torus



Sn8 embedded into double torus



Sn9 embedded into double torus

Sn9 faces in double torus

14	15	17	20	18							
6	7	19	16	15	14	11					
4	9	5	6	11	12	10	13	16	19	20	17
3	5	9	8	12	11	14	18	13	10		
2	3	10	12	8							
1	2	8	9	4							
1	4	17	15	16	13	18	20	19	7		
1	7	6	5	3	2						

Using the VEGA program the automorphisms of the 6 snarks were calculated.

Snark Sn4 (Flower snark) has 10 automorphisms and 3 vertex orbits: $5^2 \cdot 10 = \{\{4,7,10,16,17\},\{6,8,9,13,14\},\{1,2,11,12,19,20,18,15,5,3\}\}$

Snark Sn5 has 4 automorphisms and 8 vertex orbits $2^6 \cdot 4^2 = \{\{2,9\}, \{3,6\}, \{5,8\}, \{11,12\}, \{13,17\}, \{14,19\}, \{15,16,18,20\}, \{1,4,10,7\}\}$

Snark Sn6 has 4 automorphisms and $1^2 \cdot 2^3 \cdot 4^3 = \{\{5\},\{6\},\{7,11\}, \{3,17\}, \{9,13\}, \{1,12,19,14\},\{2,10,20,15\},\{4,8,16,18\}\}$

Snark Sn7 has trivial automorphism group.

Snark Sn8 has trivial automorphism group.

Snark Sn9 has 2 automorphisms and 11 vertex orbits: $1^2 \cdot 2^9 = \{\{5\},\{6\}, \{1,12\}, \{2,8\}, \{3,9\}, \{4,10\},\{7,11\},\{13,17\},\{14,19\},\{15,16\},\{18,20\}\}$

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