BIRIGIDITY IN THE PLANE

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Abstract. We consider the 2-dimensional generic rigidity matroid $R(G)$ of a graph G. The notions of vertex and edge birigidity are introduced. We prove that vertex birigidity of G implies the connectivity of R(G) and that the connectivity of $R(G)$ implies the edge birigidity of G. These implications are not equivalences.

A class of minimal vertex birigid graphs is exhibited and used to show that $R(G)$ is not representable over any finite field.

1. Introduction and Basic Definitions

Let $G = (V, E)$ be a simple graph on the edge set E, vertex set V. We define the *support* $\sigma(F)$ of a subset F of E to be the set of endpoints of edges in F.

We define a subset F of E to be *independent* if $|F'| \leq 2|\sigma(F')| - 3$ holds for all subsets F' of F. It is well known, see [1] and [3], that these independent edge sets are the independent sets of a matroid, the so-called *2-dimensional generic rigidity matroid*, $R(G)$, of the graph G. The closure operator and rank function of this matroid will be denoted by c and r respectively. The term *circuit* will always refer to a circuit in $R(G)$. Some properties of circuits are discussed in [4]. Note that $R(G)$ may be considered as a restriction of the rigidity matroid of a sufficiently large complete graph.

$G = (V, E)$ is called *rigid* if $r(E) = 2n - 3$, where $|V| = n$. G is called *edge birigid*, if $r(E - e) = 2n - 3$ for every $e \in E$. G is called *vertex birigid*, if G is rigid and $r(E - star(v)) = 2(n - 1) - 3 = 2n - 5$ for every $v \in V$, where $star(v)$ denotes the set of edges adjacent to v. We will henceforth abbreviate $E - star(v)$ with $E - v$.

To simplify notation and language we will not distinguish between sets of edges and the subgraphs they induce. Some simple examples of graphs with specified rigidity properties are given in figure 1.

The following observations are immediate consequences of the definitions. The union of two graphs G_1 and G_2 having at most one vertex in common is not rigid, and $c(G_1 \bigcap G_2) = c(G_1) \bigcap c(G_2)$. If two rigid graphs intersect in two or more vertices, their union is rigid.

Let us call two edges of G *related* if they are both contained in a rigid subgraph of G. Clearly the so defined relation is symmetric. An edge constitutes a rigid subgraph, which shows reflexivity. For transitivity; if edges e and f are contained in a rigid subgraph H_1 of G and f and h are contained in a rigid subgraph H_2 of G, then H_1 and H_2 intersect in at least two vertices, namely the endpoints of f, so their union is a rigid graph containing e and h. Thus rigidity induces an equivalence relation the edge set of G. The equivalence classes are called r-components. It follows that r-components have at most one vertex in common and that birigid graphs are at least 3-connected. Moreover, $R(G)$ can be written as the direct sum over the r-components of G. This follows from the observation that circuits are rigid, in fact edge birigid, see [4].

We shall often use the following property of $R(G)$: Assume the edge set F induces a subgraph of G containing a vertex, v, of valence three. Then F is independent if and only if there is an edge e connecting neighbors of v such that e is not contained in F and $F-v+e$ is independent. We say R(G) satisfies the *1-extendability property*, see [1]. Note that e need not be contained in G.

If the vertices of G are "generically" embedded in the plane, see [3], and the edges of G are replaced by rigid bars, which are pin-jointed at their endpoints, the resulting structure will be rigid if and only if G is rigid in the sense defined above. See [2].

If the vertices of G are restricted to a line, and the edges of G are again replaced by rigid bars, the resulting structure will be rigid if and only if G is connected, and we may characterize the 1-dimensional generic rigidity matroid $M(G)$, of the graph G as follows: a subset F of E is independent if and only if $|F'| \leq |\sigma(F')| - 1$, holds for all subsets F' of F, i.e., the independent sets in this matroid are simply the edge sets of subforests of G. $M(G)$, is called the cycle matroid $M(G)$, of G, see [6].

Observe that $M(G)$, and $R(G)$ are matroids defined on the edge set of G, and that the vertex set of G is used only via the support function, to define independent sets. Consequently, there is no property of $M(G)$, or $R(G)$ that corresponds directly to the connectivity of G. Whitney, [7], calls a matroid M on S *connected* if $r(A) + r(S-A) > r(S)$ holds for every non-empty proper subset a of S. With this definition $M(G)$, is connected if and only if G is biconnected. It is natural to ask for relations between the connectivity of $R(G)$ and the rigidity of G. This will be done in section II.

Every pair of edges in a biconnected graph is contained in a cycle. A cycle is an edge-minimal vertex-biconnected graph. Note that any cycle has exactly one edge more than it needs to be connected. A biconnected graph can simply be thought of as a union of sufficiently intersecting cycles.

It is natural to look for a rigid analogue: Given a birigid graph, can we write it as a union of birigid graphs of minimal excess, where the *excess* of a rigid graph $G = (V, E)$ is defined to be $|E| - r(E)$. Observe that the only birigid graph of excess one is the complete graph on four vertices, since the average valence in a birigid graph of excess one on n vertices is greater than or equal to $4 - (4/n)$. Therefore, a birigid graph on more than four vertices contains a vertex of valence at least four. The removal of a vertex of valence four decreases the excess by two, therefore a birigid graph on more than four vertices has to have excess at least two. In section 3 we show that there are infinitely many birigid graphs of excess two. We give an inductive procedure to construct them all. We also show that they do not, unfortunately, fullfill the role of universal building blocks of birigid graphs.

2. Birigidity of G and Connectivity of $R(G)$

Theorem 1. *If G has no isolated vertices and more than one edge, and $R(G)$ is connected, then G is edge birigid, but not conversely.*

Proof. $G = (V, E)$ is rigid, otherwise $R(G)$ could be written as the direct sum over the rigid components of G. Hence $r(E) = 2|V| - 3$.

Assume that there is an edge, e, such that $G-e$ is not rigid. Then $r(E-e)=2|V|-4$ and $r(E-e)+r(e)=r(E)$. The last equation contradicts the connectivity of $R(G)$.

The converse is not true: Let G_o be minimally rigid, having n_o vertices and $2n_o-3$ edges. We attach to each edge e_i a circuit C_i, $1\le i\le(2n_o-3)$, C_i having n_i vertices, by identifying one edge of each C_i with one edge of G_o. Then the resulting graph is clearly rigid and hence has rank $2n-3$, where

$$n=n_0+\sum_{i=1}^{2n_o-3}(n_i-2).$$

So $\sum_{i=1}^{n} n_i=n+3n_0-6$. The rank if each C_i is $2n_i-3$.

If we sum over the ranks, we get

$$\begin{aligned}\sum_{i=1}^{2n_o-3} r(C_i) &= \sum_{i=1}^{2n_o-3}(2n_i-3)=-3(2n_o-3)+2\sum_{i=1}^{2n_o-3} n_i\\ &= -6n_o+9+2n+6n_o-12=2n-3=r(G).\end{aligned}$$

So M(G)is not connected. On the other hand, G is clearly edge birigid. □

An example with $n_0=3$ is drawn in figure 1(vi).

Theorem 2. *If $G=(V,E)$ is birigid and $|V|>3$, then $R(G)$ is connected but not conversely.*

Proof. Assume that G is birigid and that $R(G)$ is not connected.

Consider the connected components R_i of $R(G)$. Then there is a partition of E, $E=E_1\bigcup E_2\bigcup\cdots\bigcup E_k$, such that

$$R(G)=R_1+R_2+...+R_k,$$

where $R_i=R(G_i)$, with $G_i=(\sigma(E_i),E_i)$. Every G_i is rigid, so it follows that

$$2|V|-3=r(G)=\sum_{i=1}^{k} r(G_i)=\sum_{i=1}^{k}(2|\sigma(E_i)|-3). \tag{1}$$

Let n_i be the number of vertices in the support of E_i which are also contained in the support of some $E_j, i\neq j$ and let N_i be the number of vertices contained only in the support of E_i. Denote by N the number of vertices of G which are contained in exactly one of the $\sigma(E_i)$'s, and by n the number of vertices which occur in more than one of these supports. N_i, N, n_i, and n satisfy the following equations:

$$i.\ N_i=|\sigma(E_i)-\bigcup_{j\neq i}\sigma(E_j)| \qquad ii.\ |\sigma(E_i)|=n_i+N_i$$

$$iii.\ N=\sum_{i=1}^{k} N_i \qquad iv.\ |V|=n+N.$$

So

$$n\le\frac{1}{2}\sum_{i=1}^{k} n_i. \tag{2}$$

Rewriting 1 in this new notation we obtain

$$2n + 2N - 3 = \sum_{i=1}^{k}(2(n_i + N_i) - 3)$$

or

$$2n = 3(1-k) + \sum_{i=1}^{k} 2n_i \tag{3}$$

so that 2 and 3 give

$$[\sum_{i=1}^{k} 2n_i] - 3(k-1) \le \sum_{i=1}^{k} n_i,$$

or

$$\sum_{i=1}^{k} n_i \le 3(k-1). \tag{4}$$

Furthermore, since every cutset in a birigid graph has cardinality at least 3, we have that

$$|\sigma(E_i) \bigcap_{i \neq j} \sigma(E_j)| \ge 3,$$

which implies that $n_i \ge 3$ for all i. This combined with 4 gives

$$3k \le \sum_{i=1}^{k} n_i \le 3(k-1),$$

a contradiction.

If $R(G)$ is connected, G need not be birigid: If G is a wheel, $R(G)$ consists of a single circuit and hence is connected. □

But the removal of the center vertex leaves a non-rigid graph if the number of spokes is larger than 3.

3. Birigid Graphs of Excess Two

G is called *edge minimally birigid* if G is birigid but $G - e$ is not birigid for all $e \in E(V)$.

In this section we will restrict our attention to an edge minimal vertex birigid graph $G = (V, E)$, which has exactly two edges more than it needs to be rigid, i.e.

$$|E| = 2|V| - 1,$$

$$r(E) = 2|V| - 3.$$

We first list some elementary properties of G.

Proposition 1. *Let G be a birigid graph of excess two. Then*

1. *G contains at least five vertices,*
2. *If $e \in E(G)$, then $G - e$ is not birigid, and*
3. *G has exactly two vertices of valence three and the remaining vertices each have valence four.*

Proof. (1) Simple graphs on less than five vertices do not contain enough edges to satisfy $|E| = 2|V| - 1$.

(2) $G - e$ is not a complete graph. $G - e$ has excess one. Since the only birigid graph of excess one is K_4, $G - e$ is not birigid.
(3) Since G is rigid, it contains no vertex of valence less than two. Suppose that G had a vertex v of valence two. Let w be adjacent to v. Then $G - w$ contains a vertex of valence one and is not rigid. Now suppose G has a vertex v of valence k. Then $G - v$ has $n - 1$ vertices and $2(n-1) - (k-1)$ edges. Since $G - v$ is rigid, $k - 1 < 4$, which implies that $k < 5$. Finally, if there are m vertices of valence three, we have $3m + 4(n - m) = 2(2n - 1)$, which gives $m = 2$.

□

The simplest birigid graph of excess two can be obtained from K_5 by deleting an edge. This graph contains two copies of K_4 as subgraphs. By "attaching" to K_4 two adjacent vertices of valence three, we obtain a birigid graph on six vertices. We remark that birigid graphs on more than 6 vertices do not contain a birigid subgraph of positive excess.

Next, we examine the circuit structure of R(G):

Theorem 3. *A graph on n vertices with* $2n - 1$ *edges is birigid if and only if there is a partition of the edge set E of G,* $E = E_1 \bigcap E_2 \bigcap \cdots \bigcap E_k$ *such that* $E - E_i$ *is a circuit in* $R(G)$ *for all* i*, and either*

(1) E_i *is an edge for* $3 \le i \le k$ *and* E_1 *and* E_2 *are stars of two vertices of valence three, or,*
(2) E_i *is an edge for* $2 \le i \le k$ *and* E_1 *is the union of stars of two adjacent vertices of valence three.*

Proof. Assume that there exists such a partition. Consider a class containing exactly one element e. Then $E - e$ is a circuit of $R(G)$, so $G - e$ is a graph with minimum valence at least three, and e has endpoints of valence at least four in G. Condition i) or ii) imply that G has 2 vertices of valence three and we conclude by a simple counting argument that all other vertices are of valence four.

Depending on whether or not the two vertices of valence three are adjacent in G, conditions i) or ii) imply that the removal of a vertex of valence three of G results in a circuit or in a circuit with a vertex of valence two attached, a rigid graph in both cases.

Consider a vertex v of valence four in G. Remove an edge e of star(v) with endpoints of valence four. $E - e$ is a circuit by assumption, and v has valence three in this circuit. Recall that a circuit is edge birigid. By deleting an edge in star v, we therefore obtain a rigid graph in which v has valence two. The removal of a vertex of valence two does not destroy the rigidity of a graph, so $E - v$ is rigid.

Conversely, assume that G is edge birigid on n vertices and $2n - 1$ edges. Since $r(E) = |E| - 2$, and every edge is contained in a circuit, E is the union of two distinct circuits, and can be partitioned into a collection of sets E_i such that $E - E_i$ is a circuit for each i, and $|E - E_i| = 2|\sigma(E - E_i)| - 2$, see for example [5] or [6]. Subtracting this equation from $|E| = 2|\sigma(E)| - 1$ gives

$$* \; |E_i| = 2|\sigma(E) - \sigma(E - E_i)| + 1.$$

If E and $E - E_i$ have the same support then E_i is a single edge. If $\sigma(E) - \sigma(E - E_i) = 1$, then E_i contains all edges of the star of a vertex in G. The equation * gives $|E_i| = 3$. Since every vertex in G has valence at least three, E_i must be a star of a vertex of valence three, and the two vertices of valence three in G are not adjacent because $E - E_i$ is a circuit.

If $|\sigma(E)-\sigma(E-E_i)| = 2$, then E_i contains all edges of the stars of two vertices of G. The equation * gives $|E_i| = 5$, so E_i must be the union of two adjacent vertices of valence three in G.

If $|\sigma(E)-\sigma(E-E_i)| > 2$, then E_i contains all edges of the star of 3 vertices of G. One of these must be of valence four. But the removal of a vertex of valence four leaves an independent set since G is birigid. The desired partition is so established and the proof of the theorem is complete. □

Examples of graphs with a partition of type i) and ii) are given in the figure below.

Clearly we can "string together" as many triangles as we wish to obtain birigid graphs of excess two of arbitrarily large size. Also the number of classes in the partition described in theorem 1 is unbounded. From a theorem of Tutte [5], we know that, if M is a matroid representable over a finite field k of order n and S is the union of two cycles of M with $r(S) = |S| - 2$ and $S_1, \cdots, S_m$ is a partition of S such that $S - S_i$ is a cycle of M, then m is bounded by $n + 1$. Hence we have proved

Corollary 1. *There is no finite field k such that $R(G)$ is representable over k for all G.*

Consider an edge minimal birigid graph $G = (V, E)$. For every edge e in E there exists a nonempty set V_e of vertices of G such that $E - e - v$ is nonrigid for all $v \in V_e$. Elements of V_e are called *essential vertices* for the edge e.

From a given birigid graph of excess two, we want to construct a larger birigid graph of excess two, by attaching a vertex of valence three and removing one edge from the given graph.

To formalize this idea, we introduce some notation:

Let T be a graph on four vertices and three edges, where one vertex is of valence three and construct a graph $G + T$ by identifying vertices a, b, c of G with the vertices of valence one in T.

We can now prove

Theorem 4. *Let G be a birigid graph of excess two, and let T and $\{a, b, c\}$ be as described above. Then:*

(1) *$G + T$ is birigid,*
(2) *a necessary and sufficient condition for $G + T$ to be edge minimally birigid is that the set $\{a, b, c\}$ not be contained in $V - V_e$ for any edge e of G;*
(3) *if $G + T$ is not edge minimally birigid, then there is an edge e such that $G + T - e$ is birigid of excess two; and*
(4) *there is always a choice of $\{a, b, c\}$ such that $G - T$ is not edge minimal.*

Proof. (1) The removal of T results in a birigid, and hence rigid graph, and the removal of any vertex $v \in G$ from $G + T$ removes at most one edge from T, and since $G - v$ is rigid, so is $G + T - v$.

(2) **Sufficiency::** Let e be any edge of G. Since the intersection of V_e with $\{a, b, c\}$ is nonempty, the removal of e and any vertex v in this intersection leaves a nonrigid graph, $G - e - v$, which has the same rigidity properties as $G + T - e - v$.

Necessity:: Assume the existence of an edge e of G such that $\{a, b, c\}$ is contained in $V - V_e$. Observe that all vertices of valence four of G which are not endpoints of e are elements of V_e. Therefore least one vertex in the set $\{a, b, c\}$ is an endpoint of e.

There are two cases.

(a) a and b are endpoints of e and c is of valence three in G. Theorem 3.1 implies that a vertex v of valence three is essential for a nonempty set of edges only if the two vertices of valence three in G are adjacent. In this case v is essential for the two edges not contained in a circuit in $G - v$. It follows that c is not adjacent to a possible endpoint of e of valence three and all essential vertices for e are of valence four. This means that $E - v$ is rigid of zero excess, i.e. independent for all $v \in V_e$. Therefore $E - v - e$ is independent and e is not in its closure. By the 1-extendability property $E + T - e - v$ is independent and hence rigid.

(b) e has endpoints of valence four in G, one of them being a, and b and c are of valence three. Remove e and star(v) for some $v \in V_e$. Repeating the argument in (a) we show that $E - e - v$ is independent and non-rigid. Consider the $r-$*components* of $E - e - v$, and assume that a, b, and c are contained in the same r-component. This component is independent, and we count that exactly 3 edges of $G - e$ are incident with it, contradicting the fact that $G - e$ is a circuit by theorem 1. So, $\{a, b, c\}$ is not contained in one r- component of $E - e - v$ and the 1-extendability property implies that $E + T - e - v$ is independent and hence rigid for all v in V_e, so $G + T - e$ is birigid.

(3) If $G + T$ is not edge minimally birigid, then there is an edge e in $G + T$ such that $G + T - e$ is birigid. $G + T - e$ has excess two.

(4) For an edge e with endpoints a and b, both of valence four, a vertex c of valence three is not essential by theorem 1.

The proof of the theorem is now complete. □

Given an edge minimal vertex birigid graph of excess two on n vertices, we can get an edge minimal vertex birigid graph on $n + 1$ vertices by choosing an edge e in G with $|V - V_e| \geq 3$ and forming $(G - e) + T$ by identifying three vertices of $V - V_e$ with the endpoints of T of valence one. In fact, we obtain all birigid graphs of excess two by this process.

THEOREM 5. *Let G be a birigid graph of excess two with $|V| > 5$.*

Let v be one of its vertices of valence three, $T = star(v)$ and let x, y, and z denote the vertices adjacent to v. Then there is an edge e with endpoints in $\{x, y, z\}$ such that e is not an edge of G and $G - T + e$ is birigid.

Proof. : $|V| > 5$ insures that $G - T$ is not complete. There are two cases.

(1) The two vertices of valence three in G are adjacent. By Theorem 1, the removal of v leaves a circuit, C, with a vertex, x, of valence two attached. Assume x and y are in the same rigid component of $C + x - w$, where w is a vertex of valence four in C. We count that exactly three edges leave this component, contradicting the fact that a cutset of C has cardinality greater

than three. Observe that x is not adjacent to y or z, this would contradict the birigidity of G. So x and y are never in the same rigid component of $C + x - w$, and neither are y and z, therefore $C + x - w + e$ is rigid if e is one of (x, y), (x, z) respectively.

(2) the two vertices of valence three in G are not adjacent.

By Theorem 1, if we remove v, we are left with a circuit, C. Let w be a vertex of valence four in C. $C - w + T$ is rigid of zero excess, hence independent and consequently $C - w$ is independent and nonrigid. By the 1-extendability property there exists an edge e with endpoints in $\{x, y, z\}$ such that $C - w + e$ is rigid. However, the choice of e depends on w, and we have to find an e that achieves rigidity independently from the choice of the removed vertex w.

If C contains already two of the possible three edges with endpoints in x, y, z we are done. Assume now that C does not contain $e = (x, z)$ and $f = (y, z)$ and there is a vertex w of valence four in C such that x and z are in the same rigid component A of $C - w$, but $C - w + f$ is rigid and that there is a vertex u of C such that y and z are in the same rigid component B of $C - u$ and $C - u + e$ is rigid. A and B intersect in at least one edge, since z is of valence three in C, and their union is not equal to C. a contains at least two vertices which are not in B, so there are at least three edges of $A - B$ incident with vertices of B, and by symmetry, three edges of $B - A$ are incident with vertices of A. We count that exactly four edges leave each of A and B. So $|C - (A \bigcup B)| \leq 2$, contradicting the fact that $C - (A \bigcup B)$ contains a vertex.

Therefore, we can always find an edge e with endpoints in $\{x, y, z\}$ such that $C - w + e$ is rigid for all vertices w in C, i.e., $G - T + e$ is birigid.

□

We have now found all birigid graphs of excess two, and we have seen that they are not only edge minimally birigid, but also minimal in the sense that they do not, with the exception of the ones on five and six vertices, contain any birigid subgraph of positive excess. Now we ask if every birigid graph on more than six vertices contains a birigid graph of excess two. The answer is no: The graph in figure 3 is birigid, has excess three and is minimal. The question if there are minimally birigid graphs of arbitrary excess is still open.

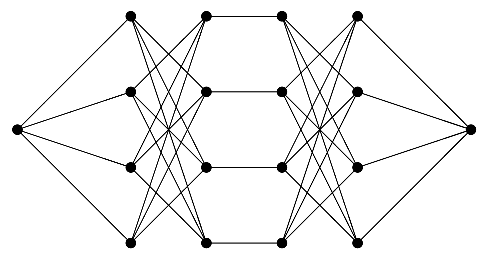

FIGURE 1. This is our favorite figure

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