GROUPS ASSEMBLED FROM FREE AND DIRECT PRODUCTS

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Abstract. Let $\mathcal{A}$ be the collection of groups which can be assembled from infinite cyclic groups using the binary operations free and direct product. These groups can be described in several ways by graphs. The group $(Z * Z) \times (Z * Z)$ has been shown by [1] to have a rich subgroup structure. In this article we examine subgroups of $\mathcal{A}$–groups.

Define $\mathcal{A}$ to be the smallest class of groups which contains the infinite cyclic group Z, and which contains both the free product and the direct product of any two of its members. So $\mathcal{A}$ contains f.g. free groups, f.g. free abelian groups, direct products of f.g. free groups, free products of f.g. free abelian groups, etc., see Figure 1. In particular, groups in $\mathcal{A}$ are finitely generated. Groups in $\mathcal{A}$ are examples of

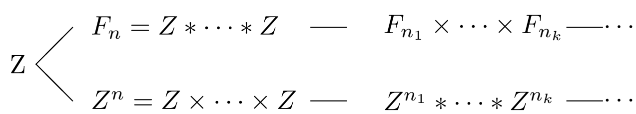


Figure 1. Generating groups in $\mathcal{A}$

graph groups [4]. Given a graph $\Gamma = (V, E)$, the graph group F_Γ is generated by the vertex set V, with a defining relation $vw = wv$ whenever the pair of vertices (v, w) is connected by an edge. The monoid with this presentation, S_Γ, was introduced in [2] in the study of derangements of sets. Given a graph $\Gamma = (V, E)$, the *complement* of Γ is the graph $\Gamma^c = (V, E^c)$, where $e \in E \iff e \notin E^c$. Given graphs Γ and Σ, $\Gamma \sqcup \Sigma$ denotes their disjoint union, and $\Gamma \bowtie \Sigma$ their *join*: $\Gamma \bowtie \Sigma = (\Gamma^c \sqcup \Sigma^c)^c$. That is, $\Gamma \bowtie \Sigma$ is obtained from $\Gamma \sqcup \Sigma$ by setting every vertex of Γ adjacent to each vertex of Σ. It is clear from the presentations that $F_{\Gamma \sqcup \Sigma} = F_\Gamma * F_\Sigma$ and $F_{\Gamma \bowtie \Sigma} = F_\Gamma \times F_\Sigma$.

We call a finite graph an *assembly* if it belongs to the smallest collection of graphs containing the one-point graph K_1, and which is closed under the operations $\sqcup$ and $\bowtie$. It is straightforward to verify that a finite graph Γ is an assembly if and only if no four vertices of Γ induce the subgraph $L =$ •—•—•—•, and that a group G belongs to $\mathcal{A}$ if and only if $G \cong F_A$ for some assembly A.

For any element $u \in F_\Gamma$, the *length* of u, $|u|$, is the length of any shortest word in the letters $V^{\pm 1} = V \cup V^{-1}$ representing u, and the *support* of u, $\operatorname{supp}(u)$, is the collection of letters occurring in any one such word. We call a shortest word *reduced*. Any two reduced words representing the same element are composed of the same letters, perhaps rearranged as allowed by the adjacencies, hence the support of an element in F_Γ is well-defined [9]. Let $\operatorname{link}(u)$ denote the set of vertices which are adjacent to every vertex of $\operatorname{supp}(u)$. If $u, v \in F_\Gamma$ and $|uv| = |u| + |v|$, we say that the product uv is *reduced as written*.

A group is said to be *freely indecomposable* if it cannot be written as the free product of two non–trivial subgroups.

PROPOSITION 1. *Let Γ be a connected graph. Then F_Γ is freely indecomposable.*

Proof. This is clear if $\Gamma = K_1$. Otherwise, suppose $F_\Gamma = G*H$. Choose two vertices v and w which are adjacent in Γ. Then, since v and w commute and generate a non-cyclic subgroup of F_Γ, they must lie in the same conjugate of either G or of H [8, Corollary 4.1.6]. So, since Γ is connected, it follows that all the vertices of Γ are conjugate to elements in, say, G. Since such elements cannot generate $G * H$, we have a contradiction. □

Let G be any group, and suppose that

$$G \cong G_1 * G_2 * \cdots * G_n \cong H_1 * H_2 * \cdots * H_m$$

where all the groups G_i and H_j are freely indecomposable. Then [8, remarks after Corollary 4.9.2] $n = m$, and the G_i can be renumbered so that for each i, $G_i \cong H_i$. In particular, if G is finitely generated, and $G \cong A * B$, then the G_i can be renumbered so that $A \cong G_1 * \cdots * G_k$ and $B \cong G_{k+1} * \cdots * G_n$. Thus, by Proposition 1 we can conclude that any free factor of an assembly group is itself an assembly group.

We shall need the following classical result from combinatorial group theory (see [8], Corollary 4.9.1):

THEOREM 1 (Kurosh Subgroup Theorem). *If $G \cong G_1 * G_2 * \cdots G_n$, then any subgroup of G is itself a free product of groups, each of which is either infinite cyclic or isomorphic to a subgroup of one of the G_i.*

In [7] it was shown that any finitely generated graph monoid S_Γ embeds in the direct product of the submonoids generated by pairs of vertices of Γ and that this product monoid is isomorphic to S_Σ, where Σ is an assembly. By contrast, we will show that a graph group which embeds in any assembly group must itself be an assembly group.

LEMMA 1. *Let Γ be a graph with more than one vertex, and suppose that Γ^c is connected. If x and y are nontrivial elements of F_Γ, then some conjugate of y does not commute with x.*

Proof. Let V be the vertex set of Γ and let $[x, y] = 1$. Suppose that $\mathrm{supp}(x) = V$, then the centralizer of x is cyclic [9]. Let r be a generator of the centralizer of x. Then $\mathrm{supp}(r) = V$, $x = r^n$ for some n, and $y = r^m$ for some m. If v is any vertex of Γ, then $v^{-1}rv \neq r^k$ for any k, since $\mathrm{supp}(r)$ contains vertices not adjacent to v. Consequently, $v^{-1}yv$ does not commute with x.

On the other hand, suppose that $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are proper subsets of V. Since Γ^c is connected, $\mathrm{supp}(x) \sqcup \mathrm{link}(x)$ and $\mathrm{supp}(y) \sqcup \mathrm{link}(y)$ are proper subsets of V. Let $a \in (\mathrm{supp}(x) \sqcup \mathrm{link}(x))^c$ and $b \in (\mathrm{supp}(y) \sqcup \mathrm{link}(y))^c$, and choose a path $a = v_0, v_1, \cdots, v_n = b$ in Γ^c. Define $g = v_0 \cdots v_n$. Then the product gyg^{-1} is reduced as written, and $\mathrm{supp}(gyg^{-1}) \not\subseteq \mathrm{supp}(x) \sqcup \mathrm{link}(x)$, and consequently [9], gyg^{-1} does not commute with x. □

LEMMA 2. *Let G_1 and G_2 be groups and let Γ have connected complement. If $F_\Gamma \leq G_1 \times G_2$, then one of the projections $p_i : G_1 \times G_2 \to G_i$ is injective when restricted to F_Γ.*

Proof. Let h_1 and h_2 be non-trivial elements of F_Γ which belong to $ker(p_1)$ and $ker(p_2)$ respectively. Then every conjugate of h_1 belongs to $ker(p_1)$, and hence commutes with h_2, a contradiction of Lemma 1. □

Recall that L denotes the graph $\bullet\!-\!\!\!-\!\bullet\!-\!\!\!-\!\bullet\!-\!\!\!-\!\bullet$.

Proposition 2. *Let A be an assembly. Then no subgroup of F_A is isomorphic to F_L.*

Proof. This is clear if $A = K_1$. Otherwise, suppose $F_L \cong H \leq F_A$. If $A = A' \sqcup A''$, then H must be a subgroup of one of $F_{A'}$ or $F_{A''}$, since it is non-cyclic and indecomposable. If $A = A' \bowtie A''$, then Lemma 2 implies that one of $F_{A'}$ or $F_{A''}$ has a subgroup isomorphic to H. In either case, the result follows by induction. □

Theorem 2. *Let A be an assembly, and let Γ be a finite graph. If there is a monomorphism $f : F_\Gamma \to F_A$, then Γ is an assembly.*

Proof. This follows immediately from Proposition 2. □

As an application of this theorem we note that the 4–string pure braid group P_4 contains F_L as a subgroup, see [5]. In [6] it was proved that the quotients of the lower central series of P_4 are isomorphic to those of the direct product $G = F_1 \times F_2 \times F_3$ (F_i free of rank i). The authors prove, however, that P_4 is not isomorphic to G. We may in fact conclude the following stronger statement:

Corollary 1. *If $n > 3$, then neither B_n (the full n–string braid group) nor P_n embeds in any direct product of free groups.*

Let A be a fixed assembly. We will now characterize the assemblies A' such that F_A has a subgroup isomorphic to $F_{A'}$. We begin with a relation $\preceq$, defined inductively on the collection of finite assemblies, by the rules:

(1) $\Gamma \preceq \Sigma$ if Γ is an induced subgraph of Σ.
(2) $\Gamma_1 \bowtie \cdots \bowtie \Gamma_m \preceq \Sigma_1 \bowtie \cdots \bowtie \Sigma_n$ if $0 \leq m \leq n$ and the Σ_i's can be ordered so that $\Gamma_i \preceq \Sigma_i$.
(3) $\Gamma_1 \sqcup \cdots \sqcup \Gamma_m \preceq \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ if $2 \leq m, n$ and for each Γ_i there is some Σ_j so that $\Gamma_i \preceq \Sigma_j$.

For example, if Σ is the star $\Sigma = K_1 \bowtie (K_1 \sqcup K_1 \sqcup K_1)$ and $\Gamma \preceq \Sigma$, then either Γ is discrete or $\Gamma = K_1 \bowtie (K_1 \sqcup \cdots \sqcup K_1)$. So, in general, if $\Gamma \preceq \Sigma$, it may indeed happen that that the graph Γ has more vertices than Σ, but nevertheless, the degree of nestedness of the expression for Γ in terms of $\bowtie$, $\sqcup$ and K_1 is less than that of Σ, and it follows that to decide if $\Gamma \preceq \Sigma$ is a finite decision procedure.

Proposition 3. *If $A' \preceq A$, then F_A has a subgroup isomorphic to $F_{A'}$.*

Proof. If A is either K_1 or $A_1 \bowtie \cdots \bowtie A_n$, the result is clear. Suppose $A = A_1 \sqcup \cdots \sqcup A_n$ and that $A' \preceq A$, say,

$$A' = \big(A_{11} \sqcup \cdots \sqcup A_{1,k(1)}\big) \sqcup \cdots \sqcup \big(A_{n1} \sqcup \cdots \sqcup A_{n,k(n)}\big)$$

where for each i and j, $A_{ij} \preceq A_i$. By induction, for each i and $j \leq k(i)$, F_{A_i} has a subgroup H_{ij} isomorphic to $F_{A_{ij}}$. For each i, let $g_{i1}, \cdots, g_{i,k(i)}$ be distinct elements of some F_{A_k} with $k \neq i$. Then the subgroups $g_{ij}^{-1} H_{ij} g_{ij}$ $(1 \leq i \leq n,\ 1 \leq j \leq n(i))$ generate their free product. Clearly this subgroup is isomorphic to $F_{A'}$. □

We now prove the converse of this statement.

THEOREM 3. *Let A and A' be finite assemblies and let $f : F_{A'} \to F_A$ be a monomorphism. Then $A' \preceq A$.*

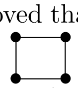
Proof. This is clear if $A = K_1$.

If $A = A_1 \sqcup \cdots \sqcup A_n$, then, by the Kurosh subgroup theorem, any subgroup of F_A can be written as a free product of groups, each of which is either infinite cyclic or isomorphic to a subgroup of one of the free factors F_{A_i}. Each infinite cyclic factor is isomorphic to F_{K_1}, and $K_1 \preceq A_i$ for each i. Each of the other factors has the form F_Σ for some assembly Σ by the remarks following Proposition 1, and is isomorphic to a subgroup of F_{A_i} for some i. By induction, $\Sigma \preceq A_i$.

Now suppose that $F_A = F_{A_1} \times \cdots \times F_{A_n}$, where each $F_{A_i} \in \mathcal{A}$ is either infinite cyclic or a non-trivial free product of $\mathcal{A}$-groups. If A' is not connected, the result follows from Lemma 2, so assume that $F_{A'} = F_{A'_1} \times \cdots \times F_{A'_m}$, with each factor either cyclic or a non-trivial free product.

If all direct factors of $F_{A'}$ are cyclic (so that A' is an m-clique), then the cohomological dimension of $F_{A'}$ is m, [3]. Thus, the cohomological dimension of F_A is at least m, which implies that A has a clique of size m, so $A' \preceq A$.

Otherwise, we may assume that $F_{A'_1}$ is a non-trivial free product. Let p_1 and p_2 denote the projections of F_A onto F_{A_1} and $F_{A_2} \times \cdots \times F_{A_n}$ respectively. By Lemma 2 we may suppose that $p_1 f : F_{A'_1} \longrightarrow F_{A_1}$ is an injection. Thus, F_{A_1} can't be cyclic, and so it must also be a non–trivial free product, say $F_{A_1} = F_P * F_Q$ for assemblies P and Q. If $p_2 f : F_{A'_2} \times \cdots \times F_{A'_m} \longrightarrow F_{A_2} \times \cdots \times F_{A_n}$ is injective as well, we are done, so assume there is an element $\alpha \in F_{A'_2} \times \cdots \times F_{A'_m}$ with $f(\alpha) = \beta \in F_{A_1}$. Since α commutes with each element of $F_{A'_1}$, β must commute with all elements of $p_1 f(F_{A'_1})$. Thus, since F_{A_1} is a non–trivial free product, β and $p_1 f(F_{A'_1})$ lie in the same conjugate either of F_P or of F_Q [8, Corollary 4.1.6]; we may assume without loss of generality that they lie in F_P. But the subgroups $F_{A'_1}$ and $F_{A'_2} \times \cdots \times F_{A'_m}$ commute, so $p_1 f\left(F_{A'_2} \times \cdots \times F_{A'_m}\right)$ lies in F_P also. That is, $f(F_{A'}) \leq F_P \times F_{A_2} \times \cdots \times F_{A_n}$. The latter is an assembly group whose graph is a proper full subgraph of A, so by induction, $A' \preceq A$. □

A finitely generated free abelian group has finitely many isomorphism classes of subgroups, and a finitely generated free group countably many. Baumslag and Roseblade, [1], proved that $(Z * Z) \times (Z * Z)$, the graph group corresponding to the square graph $S =$  , has uncountably many nonisomorphic subgroups. It follows from Theorem 3 that only countably many of these non–isomorphic subgroups belong to $\mathcal{A}$, since any graph which precedes the square under $\preceq$ is either discrete or the join of two discrete graphs.

PROPOSITION 4. *Let $P =$. Then F_P has uncountably many non-isomorphic subgroups.*

Proof. Note that $F_P \cong Z * (Z \times (Z * Z))$. For each integer $i \geq 0$, the group $Z \times (Z * Z)$ has a subgroup isomorphic to $Z \times F_i$, where F_i is free of rank i. Thus, for every increasing sequence $i_1, i_2, \cdots$ of non-negative integers (of which there are uncountably many), F_P has a subgroup isomorphic to the free product $\mathop{\ast}\limits_{j=1}^{\infty} (\mathrm{Z} \times \mathrm{F}_{\mathrm{i_j}})$, and these groups are clearly mutually nonisomorphic. □

Thus, if an assembly Γ contains either P or S as a full subgraph, then F_Γ has uncountably many nonisomorphic subgroups. The converse of this statement is also true:

THEOREM 4. *If Γ is an assembly which contains neither S nor P as a full subgraph, then F_Γ has only countably many nonisomorphic subgroups.*

Proof. Suppose Γ satisfies the hypothesis. Since S is not a subgraph of Γ, Γ has the form $C \bowtie \Sigma$, where C is a (possibly empty) complete graph, and Σ is either disconnected or empty [4]. If Σ is empty, the result is clearly true. If Σ is nonempty, then each component of Σ must be complete, since P is not a subgraph of Σ. Thus, F_Σ is the free product of a finite number of free abelian groups, and so has only countably many non-isomorphic subgroups. F_Γ is the direct product of F_Σ with a f.g. free abelian group, so any subgroup of F_Γ is isomorphic to $G \times H$, where G is f.g. free abelian, and H is a subgroup of F_Σ, [4]. Thus F_Γ also has only countably many nonisomorphic subgroups. $\square$

Finally, we remark that if Γ is not an assembly, then F_Γ contains a copy of F_T for every countable forest T [5], and so F_Γ has uncountably many nonisomorphic subgroups.

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