THE GEOMETRY OF FRAMEWORKS: RIGIDITY, MECHANISMS AND CAD

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ABSTRACT. This paper is an introduction to the the study of constrained geometric structures. The motion and rigidity of frameworks of rods and joints is examined, and connections are drawn to other constrained structures that are useful in many applications, in particular to Computer Aided Design.

1. THE GEOMETRY OF FRAMEWORKS

Mathematical applications, beyond increasing our understanding of the world, often refocus our attention on the underlying mathematics, shifting our point of view and deepening our understanding of a familiar abstract relationship. The most mundane theorems may be transformed by this process of redirection and reinterpretation.

THEOREM 1 (SSS). *If the lengths of corresponding sides of two triangles are equal, then the triangles are congruent.*

Let us consider this theorem in the context of two physical triangles constructed of straight rods joined at their endpoints, where the second triangle represents the result of forces and stresses acting on the first. We may interpret the theorem as stating that a triangular framework retains its structural integrity as long as the individual rods do, and as long as the joints don't break apart. This means that a triangular framework made of rigid members is rigid even if each individual joint allows twisting, as does a pin joint, see Figure 1. There is no SSSS theorem for

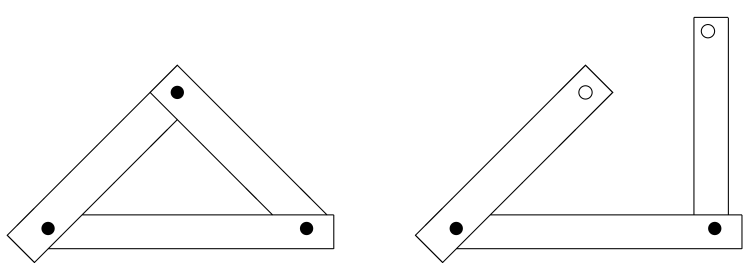

FIGURE 1.

quadrilaterals, as we see in Figure 2. Even if the joints of a simple rod structure do fix angles, for analysis it is often preferable to idealize them as pin joints (or ball joints in three dimensions.) This allows for the fact that any angle fixing joint is

in Geometry at Work, C. Gorini, Ed., MAA Notes Number 53, pp. 81-88.

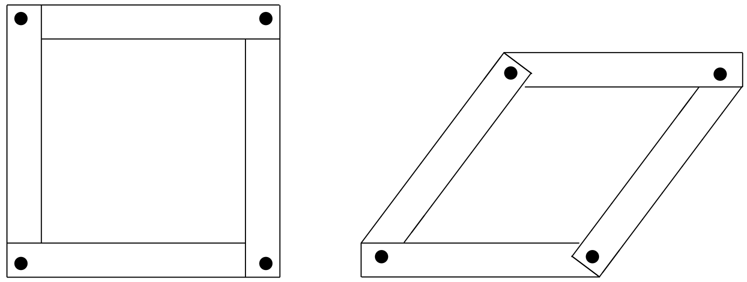

FIGURE 2.

prone to failure due to twisting, since each of the rods it connects acts as a lever with respect to it.

With a mathematical model for a framework we would like to be able to determine which frameworks are, like the triangle, rigid, so that they would be candidates for the supporting structure of a building or a bridge. On the other hand, if the framework is flexible like the square, or like the frameworks of mechanical machines or DNA molecules, we would like to know how it moves.

A nice example of a plane framework is an $n \times m$ grid of squares. Since, as we

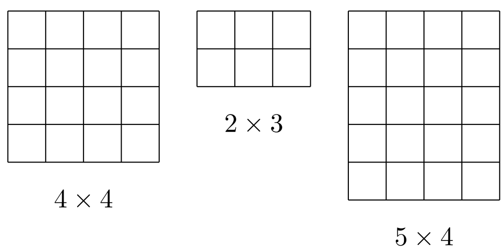

FIGURE 3. Grids

have seen, a single square made of rigid rods is not rigid in the plane, we expect a grid of squares to allow even more complicated deformations. An easy measure of the deformability of a framework is its *degree of freedom*. A point in the plane has two degrees of freedom, say its x and y coordinate, while a more complicated geometric object requires three values to specify it among its congruence class to allow for the possibility of rotation.

An $n \times m$ grid has $(n+1)(m+1)$ vertices, each of which has two degrees of freedom, but the grid as a whole has three degrees of freedom (two translations and one rotation) which are regarded as trivial since they correspond rigid motions of the entire grid. This gives $(n+1)(m+1) - 3 = 2nm + 2n + 2m - 1$ degrees of freedom which must be constrained in order for the grid to be rigid. The edges, each fixing the distance between two vertices, give $n(m+1)+m(n+1)$ constraints, so we expect that at least $n+m-1$ additional constraints are required to rigidify the grid.

A rigid bracing with exactly $n+m-1$ braces is achieved by adding diagonal braces in all the squares of the first row and column of the grid, since the braced squares then are triangulated, hence rigid, and it is easy to see that the whole braced grid is then rigid.

As might be expected, it is not sufficient to match the degree of freedom with the number of constraints and, indeed, not all choices of $n + m - 1$ braces will rigidify an $n \times m$ grid, see Figure 4. Analyzing which braces are required to rigidify a grid

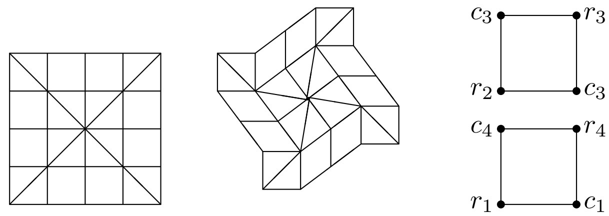

FIGURE 4. A braced grid with a motion, and its brace graph.

may be reduced to a simple combinatorial exercise. Since a square deforms to a rhombus, any deformation of a grid leaves all the vertical edges in row i parallel, and likewise with the horizontal edges in column j. Thus, a brace in square (i, j) forces the vertical rods in row i to remain perpendicular to the horizontal rods in column j under any deformation. Define a graph, called the *brace graph*, on the set of rows and columns of the $n \times m$ grid, and declare row i adjacent to column j if there is a brace in square (i, j).

THEOREM 2. *An $n \times m$ braced grid is rigid if and only if its brace graph is connected.*

Proof. If the brace graph is connected then there is a path from every row or column to column 1. A simple induction on the length of the path shows that, under any deformation, the rods in every column are constrained to be parallel to those in column 1, and the rods in every row are constrained to be perpendicular to the rods in column 1, hence the grid is rigid.

If, on the other hand, the brace graph is disconnected, then make the rods in the rows and columns connected in the brace graph to row 1 vertical and horizontal respectively, and assign to those not connected to row 1 two other perpendicular directions. This gives a deformation of the grid, see Figure 4. □

If we brace a square with cables instead of rods, then, since cables can buckle but not stretch, the constraints become inequalities, namely that the distance between the endpoints of the cable is at most the length of the diagonal. In other words, the constraint is that the angles where the cables are attached are not acute, and the rigidity of the grid may be analyzed by a directed graph. For an elementary treatment of grids see [16]. Also, an analogous development may be attempted for grids of cubes in space, and other natural generalizations, see [15].

A framework which deforms, while not desirable for the structures of civil engineering, is fundamental in mechanical engineering, robotics and some branches of organic chemistry. From the standpoint of machinery, the simplest and most useful type of framework has its deformations governed by a single parameter which is associated with time by a driving force, so that the behavior of the framework is predictable. A 1-parameter framework is called a *mechanism.* Even more complicated programmable machines may be analyzed as an assembly of mechanisms. The deformations of a mechanism are called its *motions.*

One of the classical problems in this area was to find a mechanism one point of which traces out a straight line, analogous to the compass which traces out a circle.

The mechanism of James Watt is pictured in Figure 5, in which two vertices are "grounded" to indicate that their positions are to be held fixed. As the mechanism

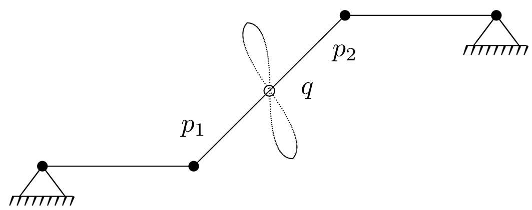

FIGURE 5. The Watt mechanism

moves, points p_1 and p_2 are constrained to move on circles centered at the grounds, and the point q traces out a curve which is straight to third order at its inflection point. This line approximating engine is so mechanically simple that one finds it in almost universal use even though a mechanism drawing perfect lines requires just twice as many bars. Since the requirement is to transform circular motion about a "ground" vertex into straight line motion, we will make use of the fact that inversion in a circle transforms the collection of circles and lines into itself. Recall that inversion in a the circle $|z| = R$ in the complex plane is given by the linear fractional transformation $w = -R^2/z$, see [1]. We may calculate the altitude of the triangle in Figure 6 in two different ways to get

$$b^2 - (\frac{y+x}{2})^2 = a^2 - (\frac{y-x}{2})^2$$

which gives $xy = b^2 - a^2$, so the points P and Q are inversions of each other in the circle of radius $\sqrt{b^2 - a^2}$ and center O. Inversion in this circle sends every circle passing through O into a circle passing through infinity, that is, a straight line. If, therefore, we fix O, force P, Q and O to lie on a straight line, and force Q to move in a circle passing through O, then P will move along a straight line, which is accomplished by the mechanism of Peaucellier, see Figure 7.

A more general idea than looking at curves traced out by mechanisms is to look at the *configuration space* of a framework. We may regard a framework with n vertices in $\mathbb{R}^d$ as a single point in $\mathbb{R}^{dn}$ by simply listing all the coordinates of all the vertices. The configuration space of a framework is the subset of $\mathbb{R}^{dn}$ consisting of the framework together with all its deformations. Since there are only distance constraints between the vertices, Pythagoras' Theorem implies that the configuration space is the solution of a system of quadratic equations.

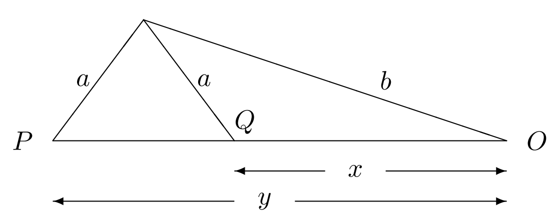

FIGURE 6.

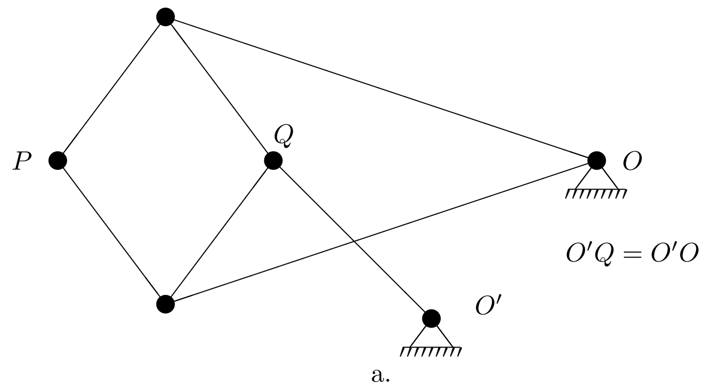

FIGURE 7.

A true mechanism will have a 1-dimensional configuration space, for instance, the configuration space of the Watt mechanism is homeomorphic to a circle, with the homeomorphism given by $(\alpha, \beta)/\sqrt{\alpha^2 + \beta^2}$, where α is the counterclockwise angle the leftmost bar makes with the horizontal, and β the clockwise angle of the rightmost bar, $-\pi < \alpha, \beta < \pi$. The configuration space of a planar two-bar robot arm is a torus, of a pentagon in the plane is a 5-holed torus. For some topological aspects see [20]. The configuration spaces of frameworks have long been used as examples of manifolds, however Figure 8 shows a recently discovered curiosity: a

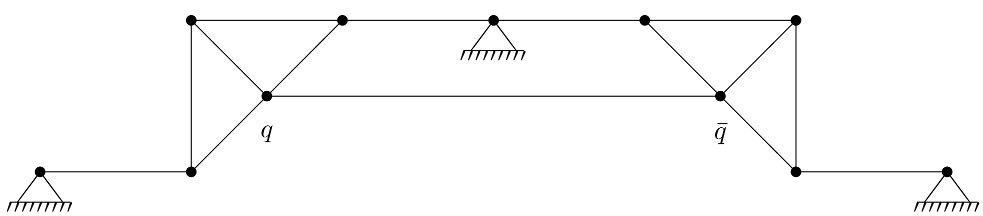

FIGURE 8. A plane mechanism in a cusp configuration.

mechanism in the plane whose configuration space contains a cusp, see [4]. A consequence of this is that any differentiable motion of this mechanism must have zero velocity in the cusp position.

2. RIGIDITY AND STRONG RIGIDITY

Up until now we have been content with an intuitive definition of framework, but it is time for a precise definition. A *framework* $\mathcal{F} = (G, \mathbf{p})$ is a graph $G = (V, E)$ together with an embedding $\mathbf{p}$ of V into Euclidean space, in practice usually two or three dimensional. By this definition, a framework is a mathematical model of a physical framework in a fixed position in space. A *motion* of the framework $\mathcal{F} = (G, \mathbf{p})$ is a continuous one-parameter family $\mathbf{p}(t)$ of embeddings of V so that $\mathbf{p}(0) = \mathbf{p}$ and so that for all t the distance between points corresponding to adjacent vertices is constant. We say a framework is *rigid* if the only motions which it admits arise from Euclidean congruences.

Consider a quadrilateral with one diagonal in the plane. If we regard this frame-

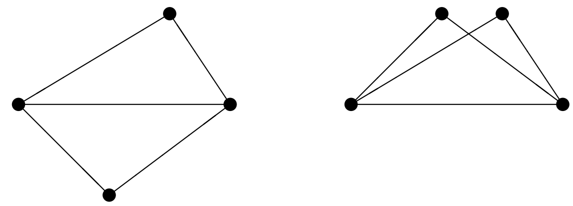

FIGURE 9.

work as being embedded in 3-space, then it is not rigid since the diagonal rod acts as a hinge between the rigid triangles, (illustrating the fact that the rigidity of a framework depends upon the dimension of the space in which it is embedded.) If we hold one triangle fixed and turn the other on the hinge through 180°, the framework again lies in a plane, but now in a non-congruent form, see Figure 9, and each of these two forms lies in a different connected component of the configuration space of the plane framework. Nevertheless, both forms of the quadrilateral with one diagonal are rigid in the plane, since there are no non-trivial motions in the plane which deform them. We say that the rectangle with one diagonal is not strongly rigid. A framework is *strongly rigid* if the underlying graph, together with specified edge lengths, determines the congruence class of the framework. Obviously any strongly rigid framework is rigid, however, testing for strong rigidity is very delicate, and the complexity is daunting even for moderate numbers of vertices, see [3].

For a comprehensive history and background into rigidity see [9, 21].

3. Infinitesimal rigidity and parallel drawings

Suppose there is a differentiable motion, $\mathbf{p}(t)$, of a framework. Let $\{\mathbf{p}_i', \mathbf{p}_j'\}$ denote the initial velocities of the endpoints of edge (i, j). Since the distance between $\mathbf{p}_i$ and $\mathbf{p}_j$ is held fixed during the motion, the components of $\mathbf{p}_i'$ and $\mathbf{p}_j'$ in the direction parallel to the edge must be equal, i.e.

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot \mathbf{p}_i' = (\mathbf{p}_i - \mathbf{p}_j) \cdot \mathbf{p}_j' \text{ for } (i, j) \in E \tag{1}$$

This condition gives us a system of linear equations whose variables are the coordinates of the vectors $\mathbf{p}_i'$, where i ranges over V, with one equation for each edge in E. A solution to this linear system of equations 1 is called an *infinitesimal motion,* or a *flex.* If a framework has a differentiable motion, then the instantaneous velocities of the vertices form a flex. We denote a flex in a diagram by drawing the initial velocity vectors at each vertex, see Figure 10. If the only infinitesimal motions are

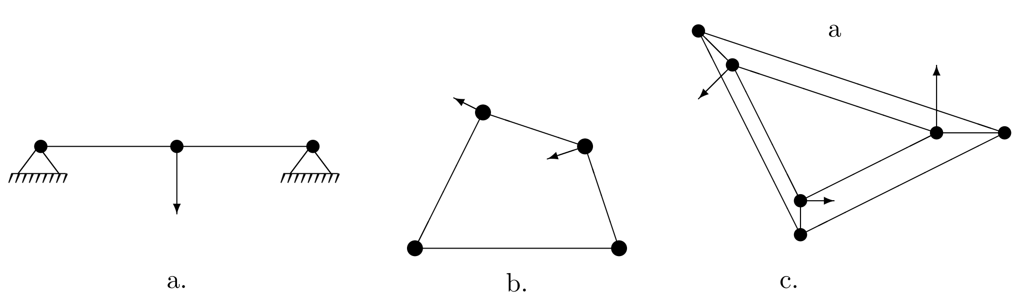

FIGURE 10. Infinitesimal motions.

trivial, that is, they arise from infinitesimal translations or rotations of $\mathbb{R}^d$, then we say that the framework is *infinitesimally rigid*.

If a framework is infinitesimally rigid, then it is rigid, see for example [2]. On the other hand, since not every flex need be realized as the initial velocities of an actual motion of the framework, a rigid graph is not necessarily infinitesimally rigid. For instance, the framework of Figure 10a is clearly rigid, yet it has a non-trivial infinitesimal motion. We shall see later that the rigid framework of Figure 10c is also not infinitesimally rigid.

One computational advantage of infinitesimal rigidity is that the set of length constraints, a system of quadratic equations, is replaced by a system of linear equations. There is also a greater engineering utility to infinitesimally rigid structures. The vulnerability of Figure 10a to sagging is typical of rigid, but non-infinitesimally rigid frameworks.

Infinitesimal motions may not correspond to real motions, and so it is often difficult to make use of our geometric intuition in detecting them. An old engineering trick is to transform the rigidity problem into one of parallel redrawing, see [21]. Two frameworks on the same graph are *parallel drawings* if corresponding edges are parallel. Translations and dilations trivially result in parallel drawings, however any infinitesimal motion can be transformed into a parallel drawing by rotating each of the velocity vectors 90° and adding them to the position vectors, see Figure 11. Conversely, any parallel redrawing corresponds to an infinitesimal motion. In the

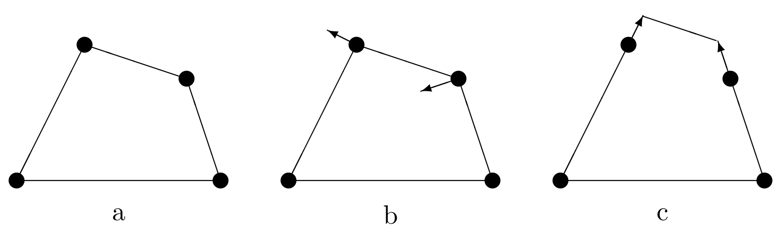

FIGURE 11. Infinitesimal motions and parallel redrawings

plane frameworks of Figure 12a, the segments which join the two triangles may be extended to intersect at a single point, so the inner triangle may be dilated with respect to the point of intersection producing the parallel drawing Figure 12b, which

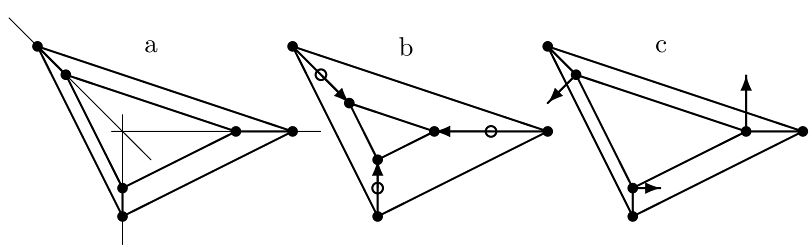

FIGURE 12.

corresponds to the infinitesimal motion of Figure 12c. This infinitesimal motion does not correspond to a real motion, and in fact this framework can be shown to be rigid.

4. Generic rigidity

If a small perturbation in the embedding of a framework alters its rigidity properties then such a framework is in some sense a singularity and, consequently, often a poor design choice. We call a framework *generic* if the vertices are embedded so that we can can "wiggle" the vertices a little bit without altering any of the framework's rigidity properties. Most embeddings are generic, and, in fact, generic embeddings form an open dense subset in the space of all embeddings.

Since the rigidity of a generic framework does not depend on the particular embedding, we may speak of the rigidity of the graph itself. A graph G is called *generically rigid* (in dimension d) if there is a generic embedding of G in $\mathbb{R}^d$ which is rigid. A non-generic framework may have more or less rigidity than a generically embedded one. In Figure 13a and b we see two frameworks on the same graph, with

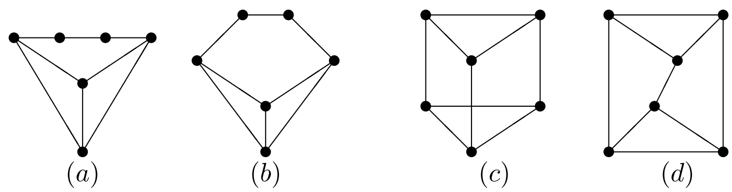

FIGURE 13. Generic and non-generic frameworks

the first being rigid since the path with three edges is pulled taut. The second embedding is generic and flexible. In Figure 13c and d we again have two frameworks on the same graph, but now the "singular" framework is flexible, since we can hold the lower triangle fixed and simultaneously rotate the corresponding vertices of the upper triangle about them. On the other hand, the generic embedding, Figure 13d, is rigid.

If G is generically rigid, then almost all embeddings of G will yield a rigid framework. While generic rigidity is purely a combinatorial concept, no combinatorial characterization is known for dimension 3 and up. The characterization question in dimension 3, in particular, is one of the most compelling open problems in geometry today. For dimension 1, rigidity on a line, generic rigidity is equivalent to ordinary connectivity. For two dimensional frameworks, we have the following theorem.

THEOREM 3. **Laman, 1970.** *A graph $G = (V, E)$ is generically rigid if and only if there is a subset F of edges so that $|F| = 2|V| - 3$ and $|F'| \leq 2|V(F')| - 3$ for all subsets $F' \subseteq F$, where $V(F')$ denotes the set of vertices which are endpoints of F'.*

The conditions of Laman's Theorem justify the intuitive ideas of constraint and degrees of freedom with which we began. The condition that $|F| = 2|V| - 3$ insures that G has enough edges to constrain the two degrees of freedom of each vertex leaving aside the three which arise from isometries. The condition that $|F'| \leq 2|V(F')| - 3$ insures that no subset of these constraints is used wastefully to overbrace some subset of vertices. This combinatorial form of rigidity is also of interest just as a concept of graph theory – the one dimensional version of this theorem states that every connected graph has a spanning tree.

As was observed in [12], the edge sets F which satisfy the conditions of Laman's Theorem form the bases of a matroid, so many of the questions of generic rigidity in two and higher dimensions are stated in terms of the rigidity matroid, see [8, 10]. For instance, matroid algorithms can be used to test for rigidity in dimension 2 in polynomial time even though checking the condition of Laman's Theorem directly requires testing all subsets of edges, see [7, 5]. See [14] for background on matroids.

Laman's theorem may also be reworded to apply to parallel drawings in the plane. Given a generic drawing of a graph G in the plane, if G is satisfies Laman's condition, then all parallel redrawings are related by similarity. If G is not generically rigid, then there definitely are non-similar parallel redrawings.

5. Applications to CAD

In computer aided design, (CAD), the object is to specify and represent a complicated design using a standard collection of geometric objects such as points, circular arcs, line segments, etc. The design is specified by requiring the objects to satisfy certain constraints. For example, certain point-line incidences, segment lengths, or angles may be prescribed. The basic design problems are: realizability - (does there exist a design satisfying the constraint?), (local) uniqueness - (do the constraints determine the congruence class of the design?), and constraint independence - (are all the constraints necessary?).

Theoretically, just as with rigidity of frameworks, the constraints may be written as a system of algebraic equations whose variables are the coordinates or other parameters of the geometric objects, see [13, 18]. The rank of the Jacobian of the system of constraints is used to answer some of the basic design questions. Even these linear computations may be slow or unstable because of degeneracies, for example the simple cusp framework in Figure 8 crashed our CAD software. Since typical CAD designs use hundreds of objects, avoiding degeneracies to achieve computational stability is useful, and may be accomplished by using generic parameters.

For plane length designs, where the objects are points and the constraints specify distances between certain pairs of points, rigidity theory provides a purely combinatorial solution to generic design questions. Parallel drawings serve the same role for direction designs. For designs with both direction and length constraints, see [17], an analogue of Laman's theorem is also valid.

Purely combinatorial characterizations are available in this setting as well: A set of direction and length constraints is independent and achieves local uniqueness if and only if there is a decomposition of the edges of the underlying graph into two spanning trees, such that no two proper subtrees of the same constraint type have the same span. Figure 14 shows a two tree decomposition of a direction length design in which the edges which correspond to direction constraints are indicated by arrows in their interior.

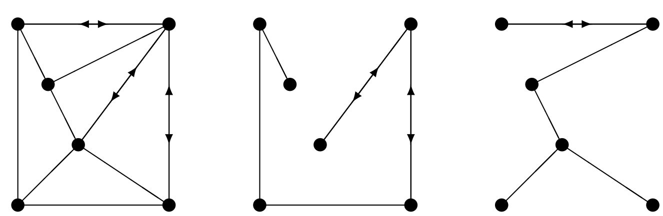

Figure 14. Decomposing a direction length design.

Other combinations of CAD motivated constraint systems have not yet been satisfactorily analyzed. Observe that a direction constraint corresponds to an equation among the coordinates of two points, and the same is true for a length constraint. An angle constraint, by contrast, involves three points. A theory of designs mixing lengths and angles would be very interesting for map making however, so far, none has been formulated.

References

[1] L. Alfors, (1966). *Complex Analysis*, McGraw-Hill.

[2] R. Connelly, (1980). *The rigidity of certain cabled frameworks and the second-order rigidity of arbitrary triangulated convex surfaces,* Advances in Math. **37**, 272–298.

[3] R. Connelly, (1990). *On generic global rigidity,* in "Applied Geometry and Discrete Mathematics", DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **4**, 147–155.

[4] R. Connelly and H. Servatius, (1994). *Higher-order rigidity, what is the proper definition?* J. Discrete and Computational Geometry **11**, 2, 193-199.

[5] H. Crapo, (1993). *On the generic rigidity of structures in the plane,* Advances in Applied Math., to appear.

[6] H. Crapo and W. Whiteley, (1993). *The Geometry of Rigid Structures,* Encyclopedia of Math., Cambridge University Press.

[7] J. Edmonds, (1965). *Minimum partition of a matroid into independent subsets,* J. Res. Nat. Bur. Standards B **69**, 67–72.

[8] Jack E. Graver, (1991). *Rigidity Matroids,* SIAM J. Discrete Math. **4**, 355-368.

[9] J. Graver, B. Servatius and H. Servatius (1993). *Combinatorial Rigidity,* Graduate Studies in Math., AMS.

[10] J. Graver, B. Servatius and H. Servatius (1993). *Abstract rigidity matroids,* Jerusalem Combinatorics '93, Contemporary Math. **178**, 145–152.

[11] G. Laman, (1970). *On Graphs and rigidity of plane skeletal structures,* J. Engrg. Math. **4**, 331–340.

[12] L. Lovász and Y. Yemini, (1982). *On Generic rigidity in the plane,* SIAM J. Alg. Disc. Methods **3**, 91–98.

[13] J. C. Owen, (1991). *Algebraic solutions for geometry from dimensional constraints,* Symposium on Solid Modeling Foundations and CAD/CAM applications, ACM press.

[14] J. Oxley (1966). *Matriod Theory*, Oxford University Press.

[15] A. Recski, (1988). *Bracing cubic grids - a necessary condition,* Discrete Math. **73**, 199–206.

[16] B. Servatius, (1995). *Graphs, digraphs, and the rigidity of grids,* The UMAP Journal, **16**, No. 1, 43–70.

[17] B. Servatius and W. Whiteley, (1995). *Constraining plane configurations in CAD: Combinatorics of directions and lengths,* Preprint.

[18] K. Sugihara, (1985). *Detection of structural inconsistency in systems of equations with degrees of freedom,* J. Disc Appl. Math. **10**, 297–312.

[19] T. S. Tay, (1993). *A new proof of Laman's theorem,* Graphs and combinatorics. **9**, No. 4, 365–348.

[20] W. P. Thurston and J. R. Weeks, (1984). *The mathematics of three–dimensional manifolds,* Scientific American **251** 1, 108–120.

[21] W. Whiteley, (1996). *Some Matroids from Discrete Applied Geometry,* in Matroid Theory, J. Bonin, J. Oxley and B. Servatius eds., Contemporary Math., AMS.

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