Discrete Time		Continuous Time
z - transform	⟷	Laplace transform

⟵ Fourier Transform ⟶

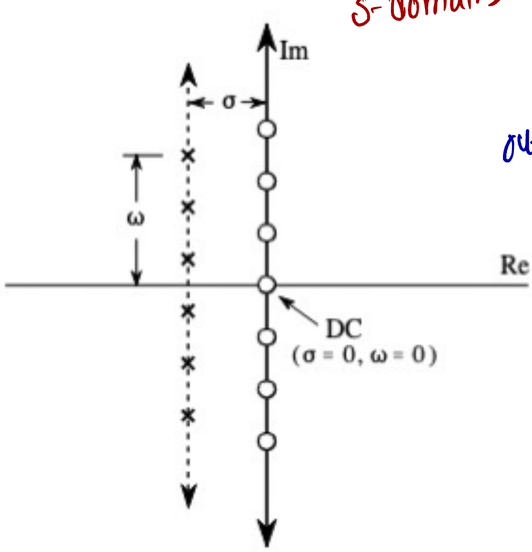
Z - Transform

Z-Transform converts a discrete-time signal, which is a sequence of real or complex numbers, into complex frequency-domain (z-domain or z-plane) representation.

In Laplace Transformation → s - domain's imaginary line

In Z-Transformation → unit circle in z domain.

s - Plane



rectangular

right

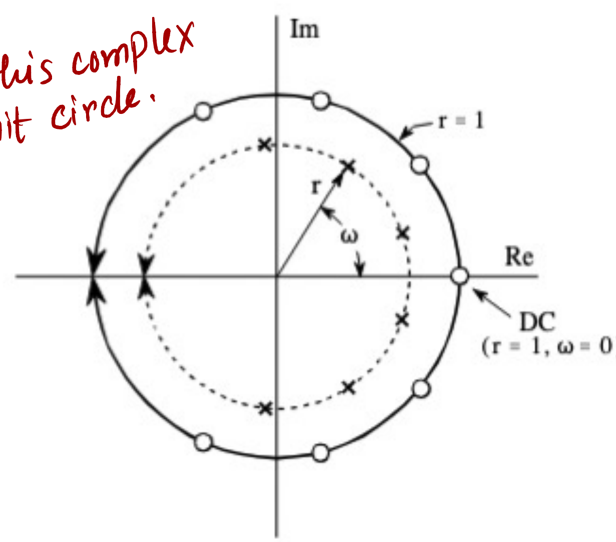
s-domain's left half plane

↓

inside this complex

outside unit circle.

z - Plane



polar.

- z-Transform is defined by-

Bilateral z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

for $z = r e^{j\omega}$

r → magnitude

ω → angle phase.

$= r(\cos\omega + j\sin\omega)$

$$X(z) = Z\{x[n]\}$$

$$x[n] \Longleftrightarrow X(z)$$

Time domain — Frequency Domain.

With z-Transform, the s-plane is represented as a set of complex exponential signals.

For any given LTI system, some of these signals may cause the output of the system to converge, while others may cause the output to diverge.

The set of signals that cause the system's output to Converge lies in the Region of Convergence (ROC)

$X(z)$ is a power series → must specify the ROC

values of z s.t. $|X(z)| < \infty$

$$\boxed{\text{power series. } \sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots}$$

z-Transform of Impulse function. $\delta[n]$

If $x[n] = \delta[n]$

$$X(z) = \mathcal{Z}\{x[n]\} = \mathcal{Z}\{\delta[n]\}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n]\, z^{-n}$$

$$\downarrow$$

For $n<0 \rightarrow \delta[n] = 0$,

$$\sum_{n=0}^{\infty} \delta[n]\, z^{-n}$$

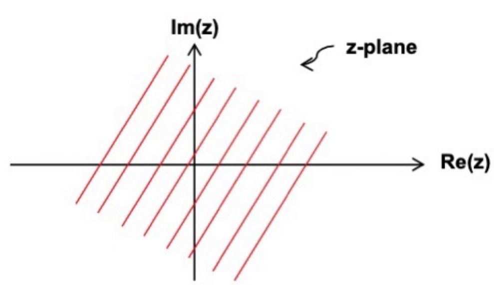
$$\downarrow$$

For $n>0$, $\delta[n] = 0$

$$\delta[0]\, z^{-0} = 1 \cdot \frac{1}{z^0} = 1.$$

$$\boxed{X(z) = 1, \ \forall z}$$

ROC



z-Transform of a unit step

$$x[n] = \mu[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \mu[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \mu[n]\, z^{-n}$$

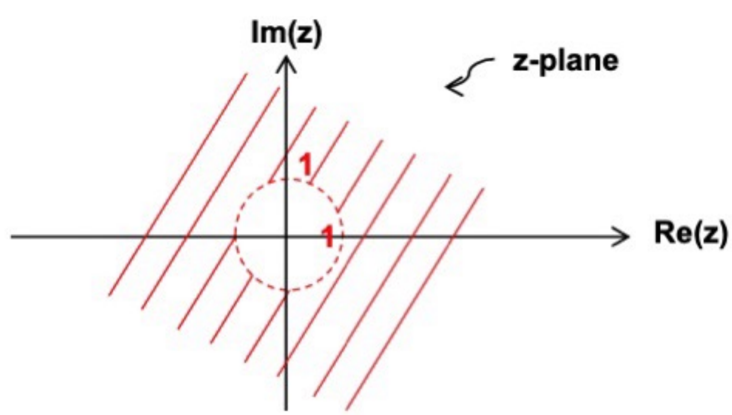
$$= \sum_{n=0}^{\infty} (1) \cdot z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$\frac{z}{z-1} \rightarrow$ if $z = 1$ the $\frac{1}{0} = \infty$ not converging

ROC $|z| > 1$



z-Transform of a Power Series.

$x[n] = a^n \mu[n]$, $a > 0$ & real

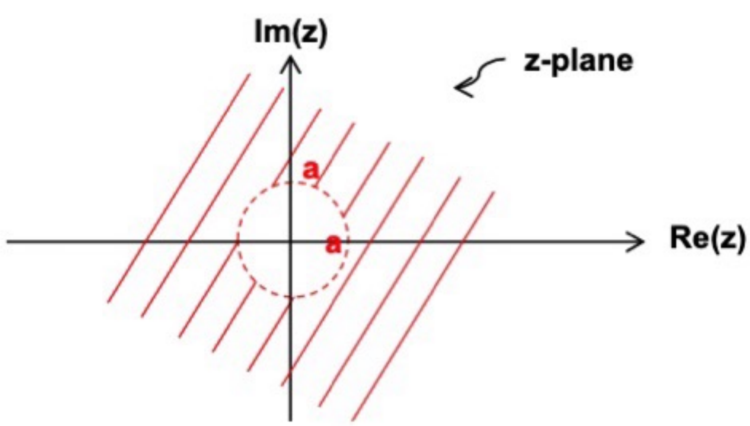
$$X(z) = \sum_{n=-\infty}^{\infty} a^n \mu[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n (1) z^{-n}$$

$$= a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots$$

$$= \frac{1}{1 - az^{-1}} = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

ROC $|z| > a$



z Transform of Anti Causal Power Series.

$$x[n] = -a^n \mu[-n-1] \quad , a > 0 \text{ \& real}$$

$$= \sum_{n=-\infty}^{\infty} -a^n \mu[-n-1]\, z^{-n}.$$

let $m = -n$

$$= \sum_{n=-\infty}^{-1} -a^n (1) z^{-n} \rightarrow -\sum_{n=1}^{\infty} a^{-m} z^{m}$$

Time reversed so

$-\infty$ to 0 have value 1

for -1 in $\mu[-n-1]$

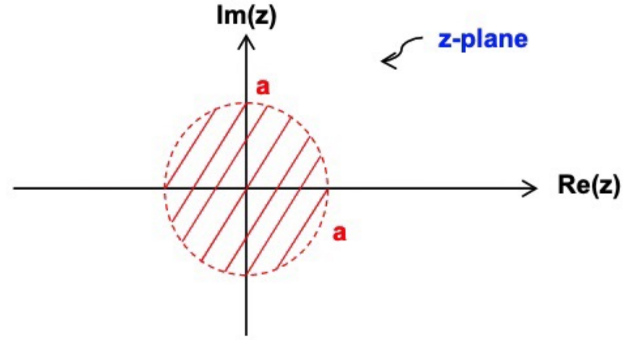
the bound is -1 instead of 0

$$= \frac{-1}{1 - az^{-1}}$$

$$= \frac{-z}{z-a}$$

$$= \frac{z}{a-z}$$

ROC $|z| < a$



z Transform of a finite duration sequence.

If $x[n] = \{ \underset{\substack{\uparrow \\ 0}}{4} , 1 , -7 \}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= 4 \cdot z^{-0} + 1 \cdot z^{1} + (-7)\, z^{-2}$$

$$= 4 \cdot 1 + 1\, z^{-1} - 7\, z^{-2}$$

$$= 4 + z^{-1} - 7\, z^{-2}$$

$$= 4 + \frac{1}{z} - \frac{7}{z^{2}}.$$

ROC $z \neq 0$

If $x[n] = \{8, 3, \underset{\uparrow}{2}, 1\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= 8z^{2} + 3z^{1} + 2z^{0} + z^{-1}$$

$$= 8z^{2} + 3z + 2 + \frac{1}{z}$$

ROC $\Rightarrow$ $|z| \neq \{0, \infty\}$

$$x[n] = 4\mu[n] + 19\,\delta[n]$$

$$= 4\sum_{n=0}^{\infty} z^{-n} + 19\sum_{n=0}^{0} z^{-n}$$

$$= 4\,\frac{1}{1-z^{-1}} + 19$$

$$= \frac{4z}{z-1} + 19$$

ROC $|z| > 1$

ROC Properties

sequence is	Finite Duration	Infinite Duration
Causal	$z \neq 0$	$\lvert z\rvert > r_a$
Anticausal	$\lvert z\rvert \neq \infty$	$\lvert z\rvert < r_b$
Two-Sided	$\lvert z\rvert \neq \{0, \infty\}$	$r_a < \lvert z\rvert < r_b$

$r_a \geq r_b$, then z-transform does not converge

For some radii, r_a r_b

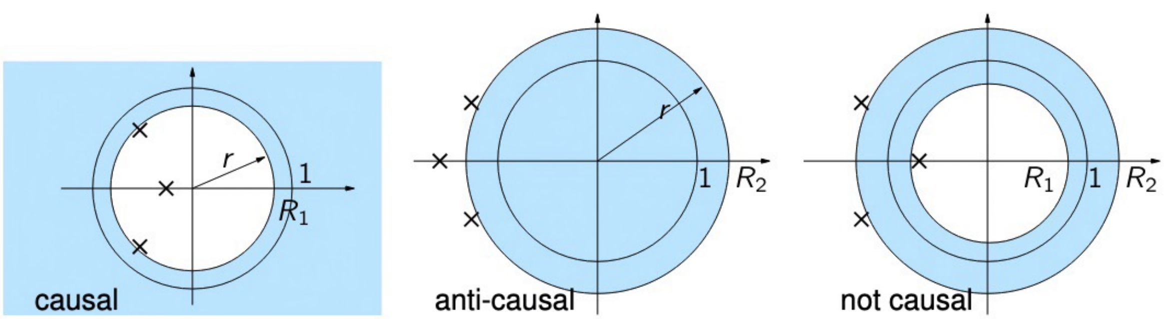


TABLE 3.3 SOME COMMON Z-TRANSFORM PAIRS

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

From Proakis and Manolakis

Z-Transform Properties

- **Linearity**

$$x[n] = a \cdot x_1[n] + b \cdot x_2[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (a x_1[n] + b \cdot x_2[n]) \cdot z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1[n] \, z^{-n} + b \sum_{n=-\infty}^{\infty} x_2[n] \, z^{-n}$$

$$\boxed{X(z) = a \cdot X_1(z) + b \cdot X_2(z)}$$

ROC is at least intersection of

ROC X_1 and ROC X_2

example

$x_1[n] = \mu[n]$ $\qquad$ $x_2[n] = \delta[n]$

$$X_1(z) = \frac{1}{1-z^{-1}}, \quad |z|>1$$

$$X_2(z) = 1, \quad \forall z$$

so, $X(z) = 1 + \dfrac{1}{1-z^{-1}}, \quad |z|>1$

$$= 1 + \frac{z}{z-1}$$

$$X(z) = \frac{z-1+z}{z-1} = \frac{2z-1}{z-1} \quad ; |z|>1$$

Time Shifting

$$y[n] = x[n-m]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-m]\, z^{-n}$$

Let, $l = n-m \rightarrow n = l+m$

$$Y(z) = \sum_{l+m=-\infty}^{\infty} x[l]\; z^{-(l+m)}$$

$$= \sum_{l=-\infty}^{\infty} x[l] \cdot z^{-l} \cdot z^{-m}$$

$$= z^{-m} \left[\sum_{l=-\infty}^{\infty} x[l] \cdot z^{-l} \right]$$

$$Y(z) = z^{-m} \cdot X(z) \begin{cases} \text{If } m>0 \text{, remove } z=0 \text{ from ROC} \\ \text{If } m<0 \text{, remove } |z|=\infty \text{ from ROC} \end{cases}$$

Time Reversal

$$y[n] = x[-n]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[-n]\, z^{-n}$$

let $m = -n$ then $n = -m$

$$Y(z) = \sum_{-m=-\infty}^{\infty} x[m]\, z^{m}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \left[z^{-1}\right]^{-m}$$

$$\boxed{Y(z) = X\left(\frac{1}{z}\right)}$$

If $ROC_X : r_a < |z| < r_b$

Then $ROC_Y : r_a < \left|\frac{1}{z}\right| < r_b$

or $\frac{1}{r_b} < |z| < \frac{1}{r_a}$

Convolution in Time

Let $x[n] = x_1[n] * x_2[n]$

$$= \sum_{m=-\infty}^{\infty} x_1[m]\, x_2[n-m]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} x_1[m]\, x_2[n-m] \right] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{n=-\infty}^{\infty} x_2[n-m]\, z^{-n}$$

let $l = n-m \Rightarrow n = l+m$

$$X(z) = \sum_{m=-\infty}^{\infty} x_1[m] \sum_{l+m=-\infty}^{\infty} x_2[l]\, z^{-[l+m]}$$

$$= \sum_{m=-\infty}^{\infty} x_1[m] \left[\sum_{l=-\infty}^{\infty} x_2[l]\, z^{-l}\, z^{-m} \right]$$

$$= \left[\sum_{m=-\infty}^{\infty} x_1[m]\, z^{-m} \right] \left[\sum_{l=-\infty}^{\infty} x_2[l]\, z^{-l} \right]$$

$$\boxed{X(z) = X_1(z) \cdot X_2(z)}$$

ROC at least intersection of $\{X_1(z), X_2(z)\}$ ROCs.

Initial Value Theorem

If $x[n] \rightarrow$ causal sequence

then $x[0] = \lim_{z \to \infty} X(z)$

Proof

for $x[n]$ causal sequenc.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \dots$$

Then $\lim_{z \to \infty} X(z)$

$$= \lim_{z \to \infty} \left[x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \dots \right]$$

$$= x[0]$$