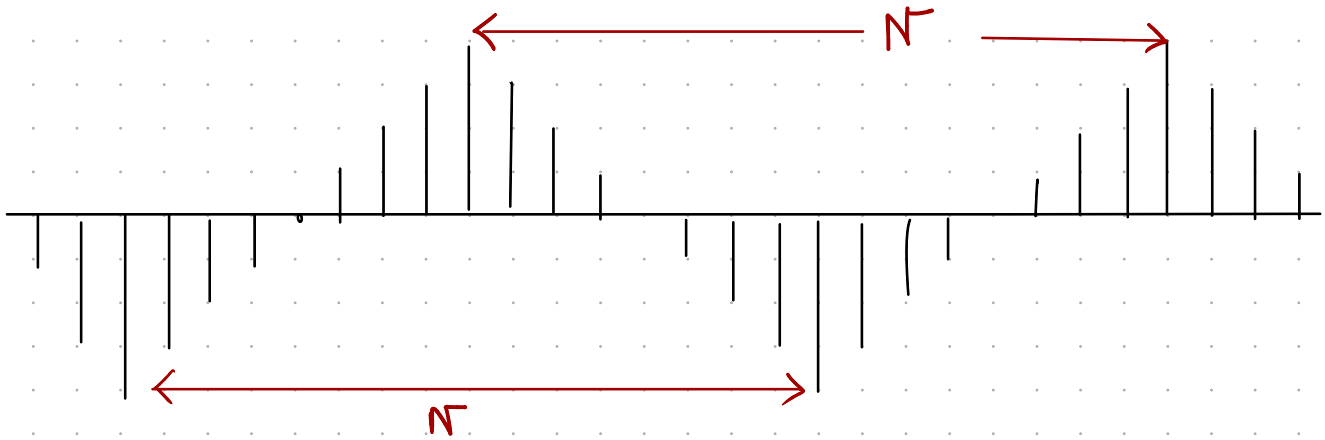
- A discrete time signal $x[n]$ is periodic with period N if

$$x[n] = x[n+N]$$



- The fundamental period is the smallest positive integer N for which the above equation holds and

$$\omega_0 = \frac{2\pi}{n} \rightarrow \text{fundamental frequency}$$

- Set of all discrete-time complex exponential signals that are periodic with period N:

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n} \quad ; \; k = 0, \pm 1, \pm 2, \cdots$$

- Discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical

$$\phi[n] = \phi_{k+rN}[n]$$

- Representation of more general periodic sequences in terms of linear combination of the sequence $\phi_k[n]$

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jw_0 n}$$

$$= \sum_k a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

k varies over a range of N successive integers

$$= \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Discrete-Time Fourier Series.

$$\underbrace{x[n]}_{\text{periodic signal}} = \sum_{k=0}^{N-1} \underbrace{a_k}_{\text{Fourier Coefficient}} e^{jk\left(\frac{2\pi}{N}\right)n} \rightarrow \text{Fourier Series Representation of a Periodic Signal}$$

$\frac{2\pi}{N}$ → frequency

N → Period

Fourier Coefficient → $a_k = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-jkw_0 n}$

- Discrete-Time Fourier series pair:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jkw_0 n} = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$ Synthesis equation.

$$a_k = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$ Analysis equation.

a_k: Complex number, True for all k → gives spectral representation.

- a_k repeats as the signal is periodic.

$$a_{k+N} = a_k$$

$\boxed{\text{proof}}$

$$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(k+N)\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \cdot e^{-j\frac{2\pi}{\cancel{N}}n.\cancel{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \cdot e^{-j2\pi n}$$

But, $e^{-j2\pi n} = \underbrace{\cos(-2\pi n)}_{1} + \underbrace{j\sin(-2\pi n)}_{0}$; $n = 0, 1, \ldots, [N-1]$

$= 1$

So, $$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= a_k$$

a_k is periodic in N

- Any N consecutive samples contain complete information.
- Usually select $k = 0$ to $N-1$ → frequencies 0 to 2π or 0 to f_s
  ↓ sampling freq.

Example

Consider the signal $x[n] = \sin \omega_0 n$. What are the fourier coefficients.

- $x[n]$ is periodic only if

$$\omega_0 = \frac{2\pi}{N} \qquad x[n] = \sin \frac{2\pi n}{N}$$

- expanding the signal as sum of two complex exponentials.

$$x[n] = \frac{e^{j\frac{2\pi n}{N}} - e^{-j\frac{2\pi n}{N}}}{2j}$$

$$\boxed{\sin x = \frac{e^{jx} - e^{-jx}}{2j}}$$

$$= \frac{1}{2j} e^{j\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j\left(\frac{2\pi}{N}\right)n}$$

The fourier co-efficients are.

$$a_1 = \frac{1}{2j} \qquad a_{-1} = \frac{-1}{2j}$$

Example

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \pi/2\right)$$

find a_k values.

- expand as sum of two exponentials.

$$x[n] = 1 + \frac{1}{2j}\left[e^{j\left(\frac{2\pi}{N}\right)n} - e^{-j\left(\frac{2\pi}{N}\right)n}\right] + \frac{3}{2}\left[e^{j\left(\frac{2\pi}{N}\right)n} + e^{-j\left(\frac{2\pi}{n}\right)n}\right]$$
$$+ \frac{1}{2}\left[e^{j\left(\frac{4\pi n}{N} + \pi/2\right)} + e^{-j\left(\frac{4\pi n}{N} + \pi/2\right)}\right]$$

$$= 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j\left(\frac{2\pi}{N}\right)n} + \left(\frac{3}{2} + \frac{-1}{2j}\right)e^{-j\left(\frac{2\pi}{N}\right)n}$$
$$+ \frac{1}{2}e^{j2\left(\frac{2\pi}{N}\right)n} \cdot \underbrace{e^{j\pi/2}}_{j} + \frac{1}{2}e^{-j2\left(\frac{2\pi}{N}\right)n} \cdot \underbrace{e^{-j\pi/2}}_{-j}$$

$$\boxed{\begin{array}{l} \sin x = \dfrac{e^{jx} - e^{-jx}}{2j} \\ \cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right) \end{array}}$$

$$a_0 = 1$$

$$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} + \frac{j}{2j^2} = \frac{3}{2} - \frac{1}{2}j$$

$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} - \frac{j}{2j^2} = \frac{3}{2} + \frac{1}{2}j$$

$$a_2 = \frac{1}{2}j \qquad a_{-2} = -\frac{1}{2}j$$