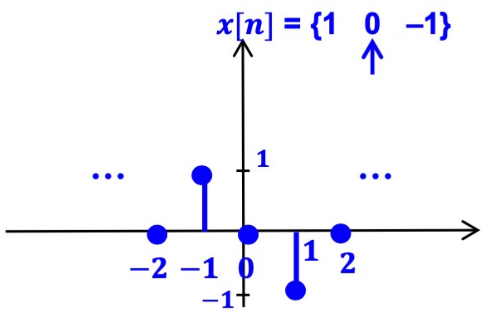
Graphical Convolution

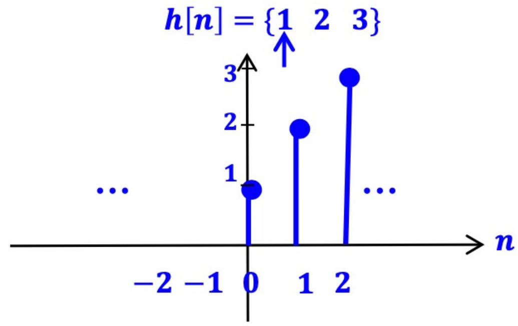
To implement convolution at $n = n_0$

1. Fold / Reflect $h[k]$ about $k=0$ $\Rightarrow h[-k]$
2. Delay / Advance $h[-k]$ by $n_0 \Rightarrow h[n_0 - k]$
3. multiply $x[k]$ by $h[n_0 - k]$ point by point
4. sum the product $y[n_0]$

$\boxed{\text{For another } n_0\text{, shift (delay/advance) again}}$

Example



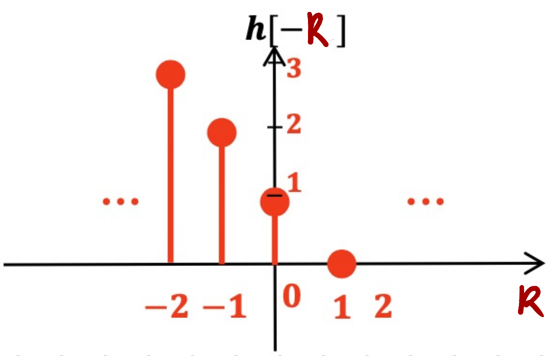


$$x[n] = \begin{cases} 1, & n = -1 \\ -1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 3, & n = 2 \\ 0, & \text{otherwise} \end{cases} = \{1, \ 2, \ 3\}$$

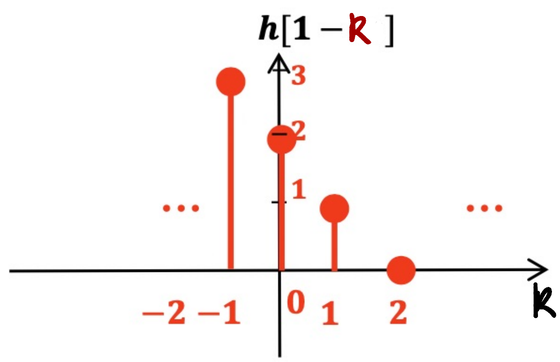
- **Find** $y[n] = x[n] * h[n]$ **for** $n = 1$

① Reflect

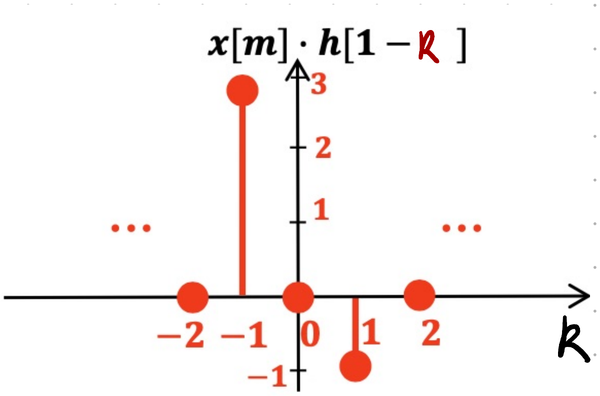


② Shift by $n_0 = 1$

[positive $n_0 \rightarrow$ shift right for $h[-k]$]



③ multiply $x[k]$ by $h[n_0 - k]$



④ Sum

$y[1] = 3 + 0 + (-1) = 2 = y[1]$

CHECK $\quad y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

so

$y[1] = x[-1] \cdot h[1-(-1)] + x[0] \cdot h[1-0] + x[1] \cdot h[1-1]$

$= x[-1] \cdot h[2] + x[0] \cdot h[1] + x[1] \cdot h[0]$

$= (1 \cdot 3) + (0 \cdot 2) + (-1 \cdot 1) = 3 + 0 - 1$

$= 2 = y[1]$

manual convolution computation

Convolve: $x[n] = \{1 \;\underset{\uparrow}{2}\; 3\}$ with $h[n] = \{\underset{\uparrow}{-1}\; 4\; 2\; 3\}$

Solution

$x[k] = \{1\; \underset{\uparrow}{2}\; 3\}$ $\quad h[-k] = \{3\; 2\; 4\; \underset{\uparrow}{-1}\}$

arrow indicate $n=0$

First overlap $(n=-1)$

$$\begin{array}{lccccccc} x[k] & & & & 1 & \underset{\uparrow}{2} & 3 \\ h[-1-k] & 3 & 2 & 4 & -1 & & \\ \hline \text{mul} & & & & -1 & & \end{array}$$

$\sum_{-1} = -1$

Second Overlap $(n=0)$

$$\begin{array}{lcccccc} x[k] & & & 1 & \underset{\uparrow}{2} & 3 \\ h[0-k] & 3 & 2 & 4 & -1 & \\ \hline & & & 4 & -2 & \end{array}$$

$\sum_{0} = 2$

Third Overlap $(n=1)$

$$\begin{array}{lcccccc} x[k] & & 1 & \underset{\uparrow}{2} & 3 \\ h[1-k] & 3 & 2 & 4 & -1 \\ \hline & & 2 & 8 & -3 \end{array}$$

$\sum_{1} = 7$

Forth overlap $(n=2)$ $x[k] = \{1 \ \underset{\uparrow}{2} \ 3\}$ $h[-k] = \{3 \ 2 \ \underset{\uparrow}{4} \ -1\}$

$x[k]$	1	$\underset{\uparrow}{2}$	3	
$h[2-k]$	3	2	4	-1
	3	4	12	

$\Sigma_2 = 19$

Fifth overlap $(n=3)$

$x[k]$	1	$\underset{\uparrow}{2}$	3		
$h[3-k]$		3	2	4	-1
		6	6		

$\Sigma_3 = 12$

Sixth Overlap $(n=4)$

$x[k]$	1	$\underset{\uparrow}{2}$	3			
$h[4-k]$			3	2	4	-1
			9			

$\Sigma_4 = 9$

Resut: $x[n] * h[n] = \{-1 \ 2 \ 7 \ 19 \ 12 \ 9\}$

Sequence Duration of Convolution Sum

IF $x[n] \rightarrow$ finite duration of length : L_x

$h[n] \rightarrow$ finite duration of length : L_h

THEN $x[n] * h[n] \rightarrow$ finite duration of length : $(L_x + L_h - 1)$

$$x[n] = \{x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{L_x}\}$$

$$h[n] = \{h_1 \quad h_2 \quad h_3 \quad \cdots \quad h_{L_h}\}$$

$$x[n] * h[n] = \{y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_{L_x + L_h - 1}\}$$

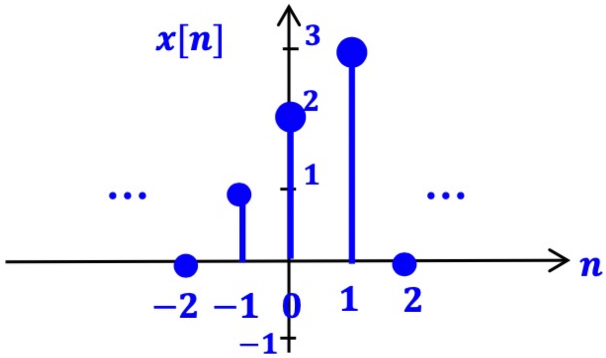
From previous example

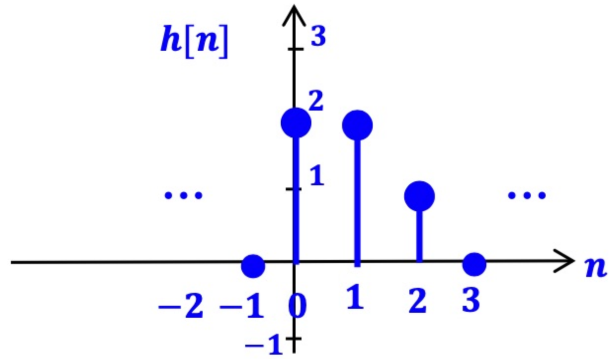
$$x[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\} \qquad h[n] = \{\underset{\uparrow}{-1} \quad 4 \quad 2 \quad 3\}$$

$$L_x = 3 \qquad L_h = 4$$

$$L_y = (3 + 4 - 1) = 6$$

example 2





$$x[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\}$$

$$h[n] = \{\underset{\uparrow}{2} \quad 2 \quad 1\}$$

Solution

$$y[n] = x[n] * h[n]$$

$$h[-k] = \{1 \quad 2 \quad \underset{\uparrow}{2}\}$$

$$L_y = (3+3-1) = 5 \text{ overlaps.}$$

y[-1]

$$\begin{array}{ccccc} & & 1 & \underset{\uparrow}{2} & 3 \\ 1 & 2 & \underset{\uparrow}{2} & & \\ \hline & & 2 & & \end{array} \Rightarrow \Sigma_{-1} = 2$$

y[0]

$$\begin{array}{cccc} & 1 & \underset{\uparrow}{2} & 3 \\ 1 & 2 & \underset{\uparrow}{2} & \\ \hline & 2 & 4 & \end{array} \Rightarrow \Sigma_0 = 6$$

y[1]

$$\begin{array}{ccc} 1 & \underset{\uparrow}{2} & 3 \\ 1 & 2 & \underset{\uparrow}{2} \\ \hline 1 & 4 & 6 \end{array} \Rightarrow \Sigma_1 = 11$$

y[2]

$$\begin{array}{cccc} 1 & \underset{\uparrow}{2} & 3 & \\ & 1 & 2 & \underset{\uparrow}{2} \\ \hline & 2 & 6 & \end{array} \Rightarrow \Sigma_2 = 8$$

y[3]

$$\begin{array}{ccccc} 1 & \underset{\uparrow}{2} & 3 & & \\ & & 1 & 2 & \underset{\uparrow}{2} \\ \hline & & 3 & & \end{array} \quad \Sigma_3 = 3$$

$$y[n] = \{2 \quad 6 \quad 11 \quad 8 \quad 3\}$$

example3 $\quad x[n] = \{-1 \quad \underset{\uparrow}{0} \quad 1\} \qquad h[n] = \{\underset{\uparrow}{2} \quad 2\}$

solution: $y[n] = x[n] * h[n]$

$h[-k] = \{2 \quad \underset{\uparrow}{2}\} \qquad L_y = L_x + L_h - 1 = 3+2-1 = 4$

$y[-1]$:

$$\begin{array}{cccc} -1 & \underset{\uparrow}{0} & 1 & \\ 2 \quad \underset{\uparrow}{2} & & & \\ \hline -2 & \Rightarrow \sum_{-1} = -2 & & \end{array}$$

$y[0]$:

$$\begin{array}{ccc} -1 & \underset{\uparrow}{0} & 1 \\ 2 & \underset{\uparrow}{2} & \\ \hline -2 & 0 & \Rightarrow \sum_{0} = -2 \end{array}$$

$$\boxed{y[n] = \{-2 \quad -2 \quad 2 \quad 2\}}$$

$y[1]$

$$\begin{array}{ccc} -1 & \underset{\uparrow}{0} & 1 \\ & 2 & \underset{\uparrow}{2} \\ \hline & & 2 \Rightarrow \sum_{1} = 2 \end{array}$$

$y[2]$

$$\begin{array}{cccc} -1 & \underset{\uparrow}{0} & 1 & \\ & & \underset{\uparrow}{2} & 2 \\ \hline & & 2 & \sum_{2} = 2 \end{array}$$

example 4

$$y[n] = \mu[n] * \mu[n]$$

Solution

$$\mu[k] = \{ \underset{\uparrow}{1} \quad 1 \quad 1 \quad 1 \cdots \}$$

$$\mu[-k] = \{ \cdots \cdots \underset{\uparrow}{1} \}$$

For $n < 0$ → $y[n] = 0$ ⇒ no overlap of non-zero numbers.

For $n = 0$ → $y[0]$

$$\{ \underset{\uparrow}{1} \quad 1 \quad 1 \cdots \}$$

$$\{ \cdots 1 \quad 1 \quad \underset{\uparrow}{1} \}$$

$$1 \Rightarrow \Sigma_0 = 1$$

For $n = 1$ → $y[1]$

$$\{ \underset{\uparrow}{1} \quad 1 \quad 1 \cdots \}$$

$$\{ \cdots 1 \quad 1 \quad \underset{\uparrow}{1} \}$$

$$1 \quad 1 \Rightarrow \Sigma_1 = 2$$

IN GENERAL

$$\boxed{y[n] = \{ \underset{\uparrow}{1} \quad 2 \quad 3 \quad 4 \cdots \} = n \cdot \mu[n]}$$