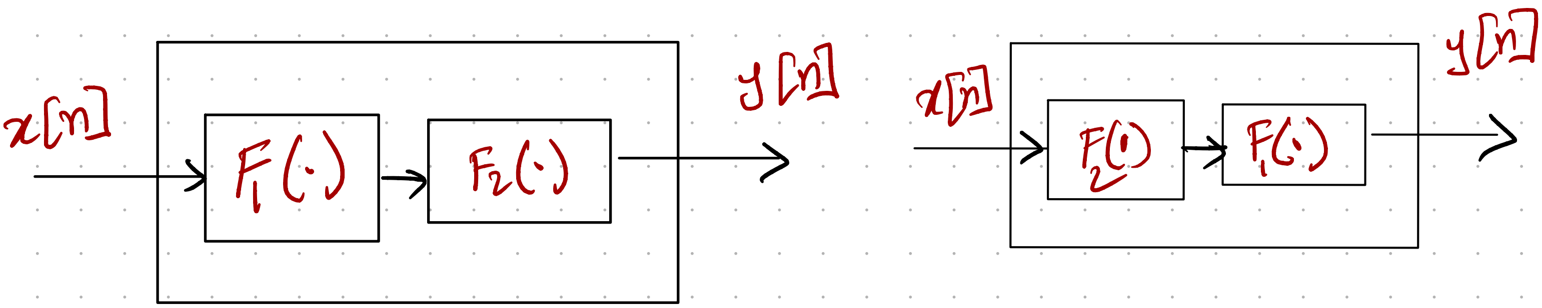
Convolution Sum

Decomposition

linearity is commutative → order of cascading system can be rearranged without affecting the characteristics of overall combination.



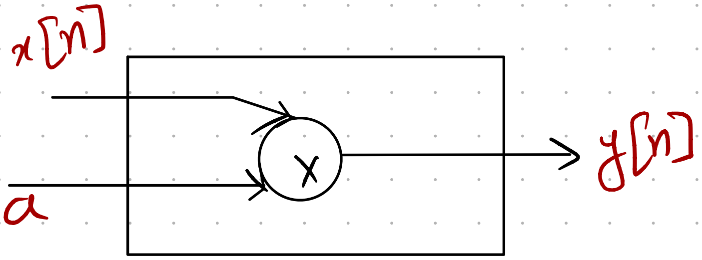
Superposition

↓

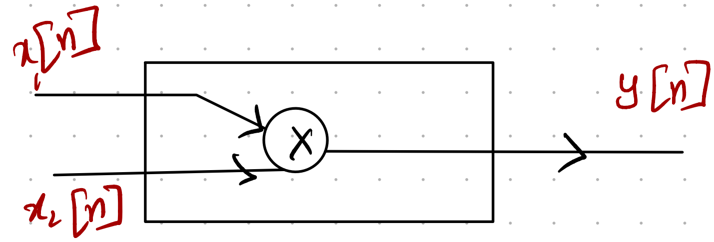
when two signals are added together & fed to a system the output is the same as if one had put each signal through the system seperately & then added the output

Multiplication in Linear System

multiplication in linear system → linear / non-linear } based on the signal it is multiplied by



linear



non-linear

Imagine a sinusoid multiplied with another sinusoid.

↓

resulting waveform not sinusoid.

Systhesis → the process of combining signals through scaling & addition

$$a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] = a_4 x_4[n]$$

Decomposition → a single signal is broken into two or more additive components.

opposite of synthesis.

Decomposition is more involved than synthesis, as there are infinite possible decomposition for any given signal

$15 + 25 = 40$

$40 \rightarrow 39 + 1$

$40 \rightarrow 38 + 2$

. . . .

$20 + 20$

$19 + 21$

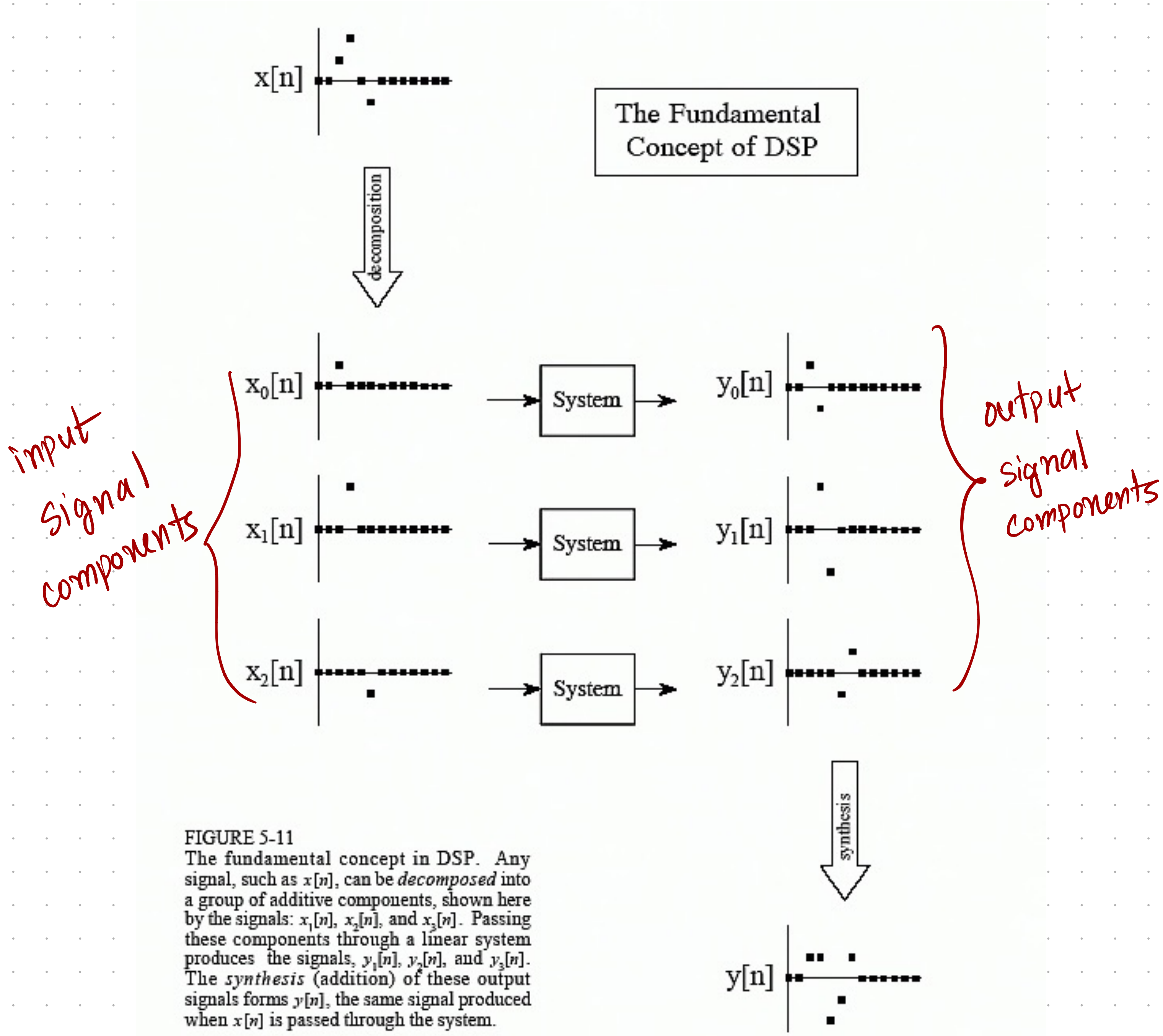


FIGURE 5-11
The fundamental concept in DSP. Any signal, such as $x[n]$, can be *decomposed* into a group of additive components, shown here by the signals: $x_1[n]$, $x_2[n]$, and $x_3[n]$. Passing these components through a linear system produces the signals, $y_1[n]$, $y_2[n]$, and $y_3[n]$. The *synthesis* (addition) of these output signals forms $y[n]$, the same signal produced when $x[n]$ is passed through the system.

The output signal obtained by this method is identical to the one produced by directly passing the input signal through the system.

Decomposition

impulse

step

even/odd

interlaced

fourier

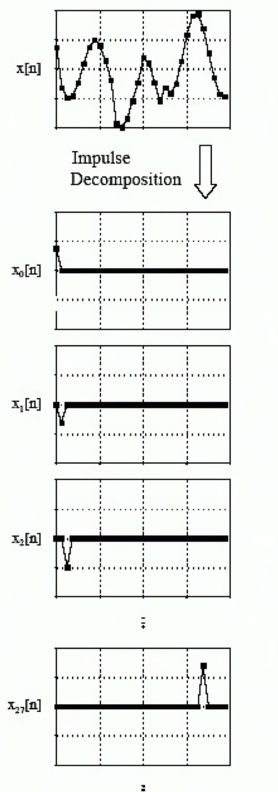


FIGURE 5-12
Example of impulse decomposition. An *N* point signal is broken into *N* components, each consisting of a single nonzero point.

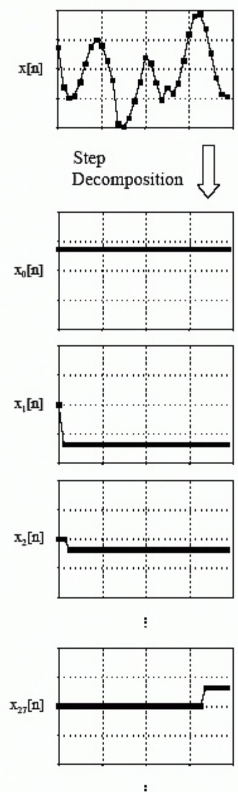


FIGURE 5-13
Example of step decomposition. An *N* point signal is broken into *N* signals, each consisting of a step function

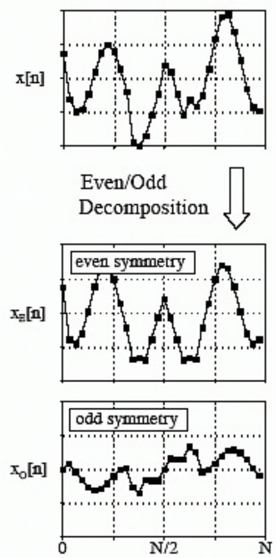


FIGURE 5-14
Example of even/odd decomposition. An *N* point signal is broken into two *N* point signals, one with even symmetry, and the other with odd symmetry.

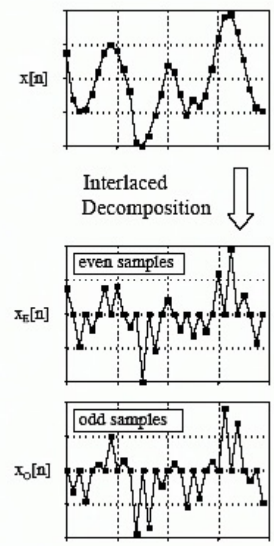


FIGURE 5-15
Example of interlaced decomposition. An *N* point signal is broken into two *N* point signals, one with the odd samples set to zero, the other with the even samples set to zero.

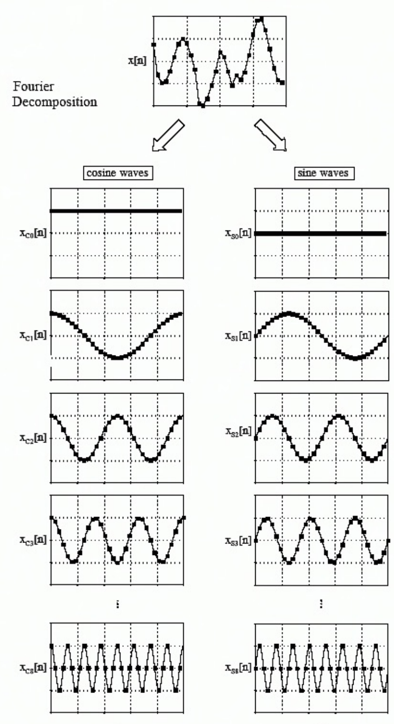


FIGURE 5-16
Illustration of Fourier decomposition. An *N* point signal is decomposed into *N*+2 signals, each having *N* points. Half of these signals are cosine waves, and half are sine waves. The frequencies of the sinusoids are fixed; only the amplitudes can change.

Convolution

Mathematical way of combining two signals to form a third signal

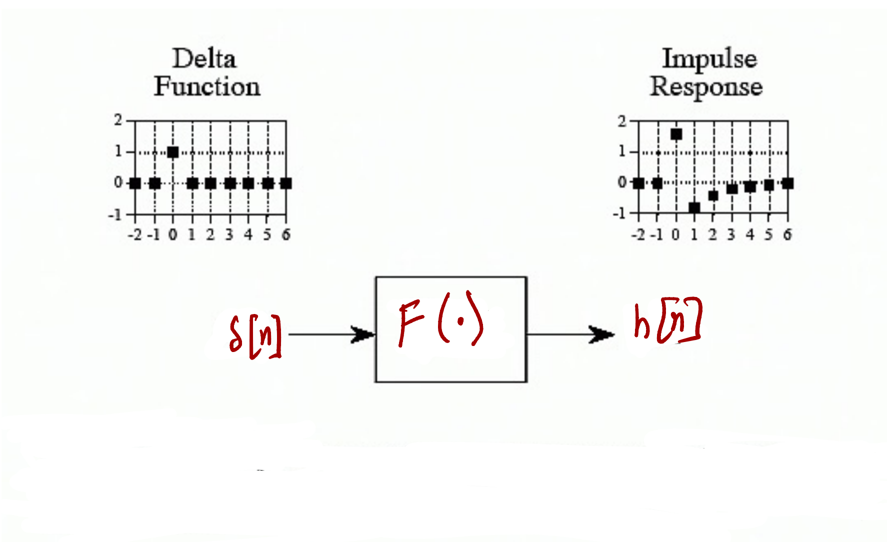
Use the strategy of impulse decomposition → impulse response

↓

way to analyse signals one sample at a time

When impulse decomposition is used, the procedure can be described by the mathematical operation convolution

$\delta[n] \rightarrow$ unit impulse



$$\delta[n] \rightarrow \boxed{F_1(\cdot)} \rightarrow h_1[n]$$

$$\delta[n] \rightarrow \boxed{F_2(\cdot)} \rightarrow h_2[n]$$

if $F_1(\cdot) \equiv F_2(\cdot)$

then $h_1[n] \equiv h_2[n]$

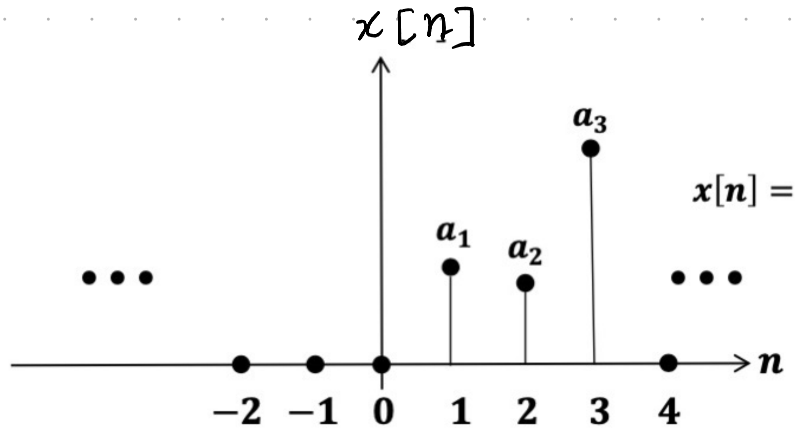
any impulse can be represented as a shifted and scaled unit impulse.

Signal $x[n] \rightarrow$ Composed of all zeros except sample number 8, which has value -3 $\quad x[n] = -3\,\delta[n-8] \rightarrow R$

Can we show $\quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k] \quad -\infty \leq n \leq \infty$

each impulse in sum is nonzero for one value

sum of scaled, delayed unit impulses



$$x[n] = \begin{cases} a_1, & n=1 \\ a_2, & n=2 \\ a_3, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

$x[n] = a_1\, \delta[n-1] + a_2\, \delta[n-2] + a_3\, \delta[n-3] \; \ldots$

Each $\delta[\cdot]$ is infinite duration sequence

Impulse Response of LTI System

System output (response) if input $= \delta[n-k]$

- General definition of impulse resp se

$$\underbrace{h[n, k]}_{\text{Time(Sample) Index}} = F(\delta[n-k])$$

k, $[n-k]$ → Sample when impulse value → 1

If sample is shift-invariant

$$h[\underbrace{n-k}_{}] = F(\delta[n-k])$$

↓ function of 1 variable which is the time difference between current time & when input equaled 1.

$\delta[n-k] \longrightarrow \boxed{F(\cdot)} \longrightarrow$ Impulse Response.

For filters impulse response → filter kernel
convolution kernel
kernel

Image Processing → point spread function

How a linear system change the input signal to an output signal?

input signal

↓

decompose into set of impulses
(shifted and scaled unit impulse)

↓

$F(\cdot)$

↓

output resulting from each input impulse is scaled & shifted input impulse

↓

add these scaled & shifted impulse responsed

↓

output signal

In linear system convolution is used to describe the relation between three signals → input signal
→ impulse signal
→ output signal

$$x[n] \longrightarrow \boxed{\begin{matrix} F(\cdot) \\ h[n] \end{matrix}} \longrightarrow y[n]$$

$$x[n] * h[n] = y[n]$$

Convolution is used as low-pass filter, high pass filter, inverting attenuator, discrete derivative

Derivation of LTI convolution sum

- General discrete system : $y[n] = F(x[n])$
- Rewrite $x[n]$ as a sum of δ's : $y[n] = F\left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right)$

- Assume linearity

linear, so sum comes out of $F(\cdot)$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot F(\underbrace{\delta[n-k]}_{\text{impulse response}})$$

just scaling values so comes out of F

y[n] is not a function of k

- Superposition summation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n,k]$$

n: time index

k: location of the impulse

as it is time invariant h is no longer a function of n & k but just their difference.

- Assuming Time-Invariance

$$\underbrace{y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]}_{\text{Convolution sum of LTI system}} \equiv x[n] * h[n]$$

Thus, LTI system fully characterized by impulse resorse : $h[n-k]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$-k \rightarrow$ any instance in time n we are taking the impulse response of the function & because there is a $-k$ we are flipping it around the y axis

$n-k \rightarrow$ shift that reversed impulse response by m

then multiply point by point x event & sum it over time.

9-point input 4 point impulse 9+4-1 = 12 point output

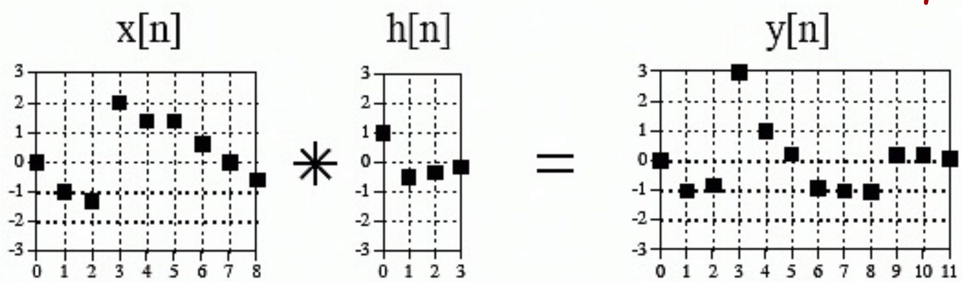


FIGURE 6-5
Example convolution problem. A nine point input signal, convolved with a four point impulse response, results in a twelve point output signal. Each point in the input signal contributes a scaled and shifted impulse response to the output signal. These nine scaled and shifted impulse responses are shown in Fig. 6-6.

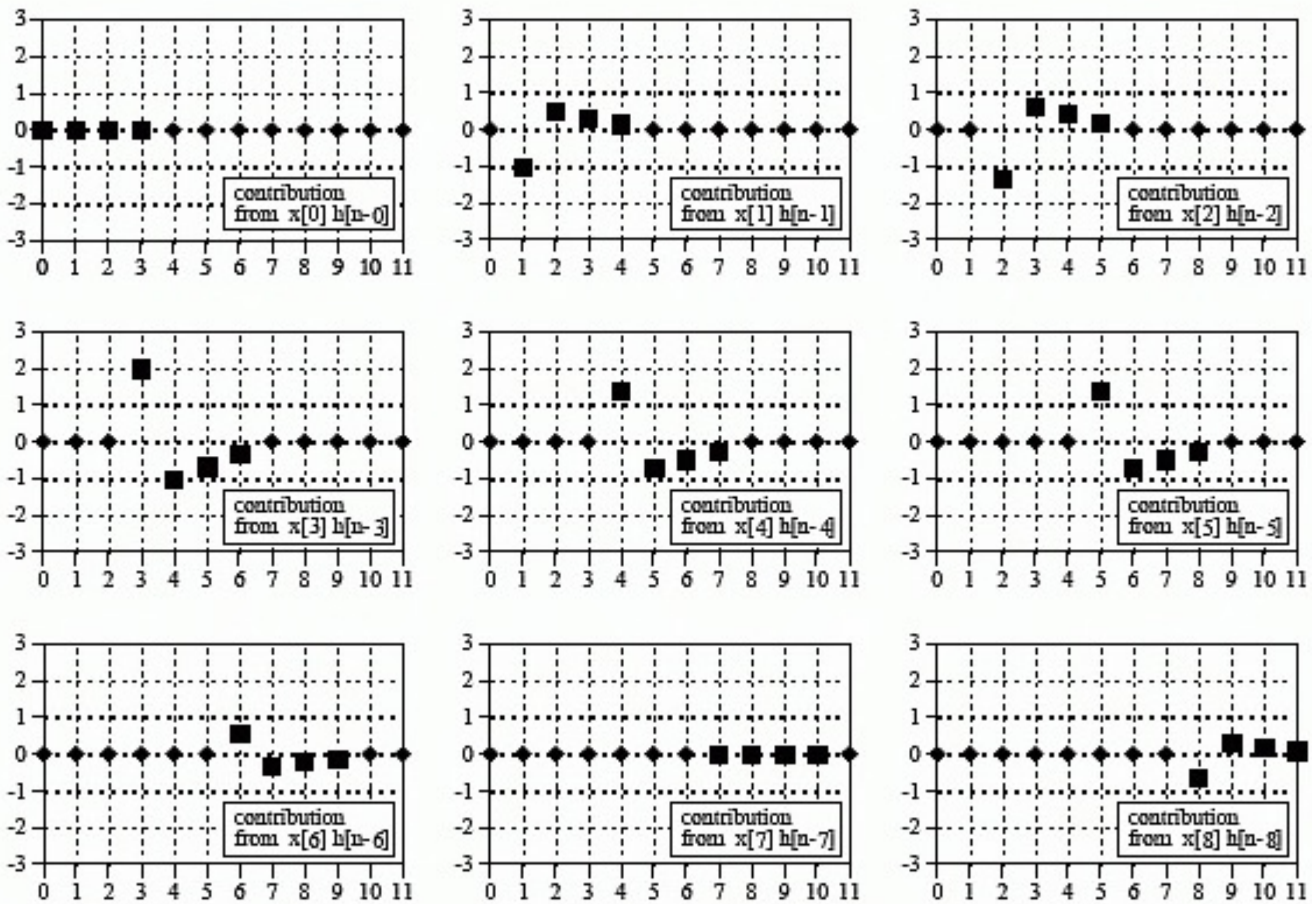


FIGURE 6-6
Output signal components for the convolution in Fig. 6-5. In these signals, each point that results from a scaled and shifted impulse response is represented by a square marker. The remaining data points, represented by diamonds, are zeros that have been added as place holders.

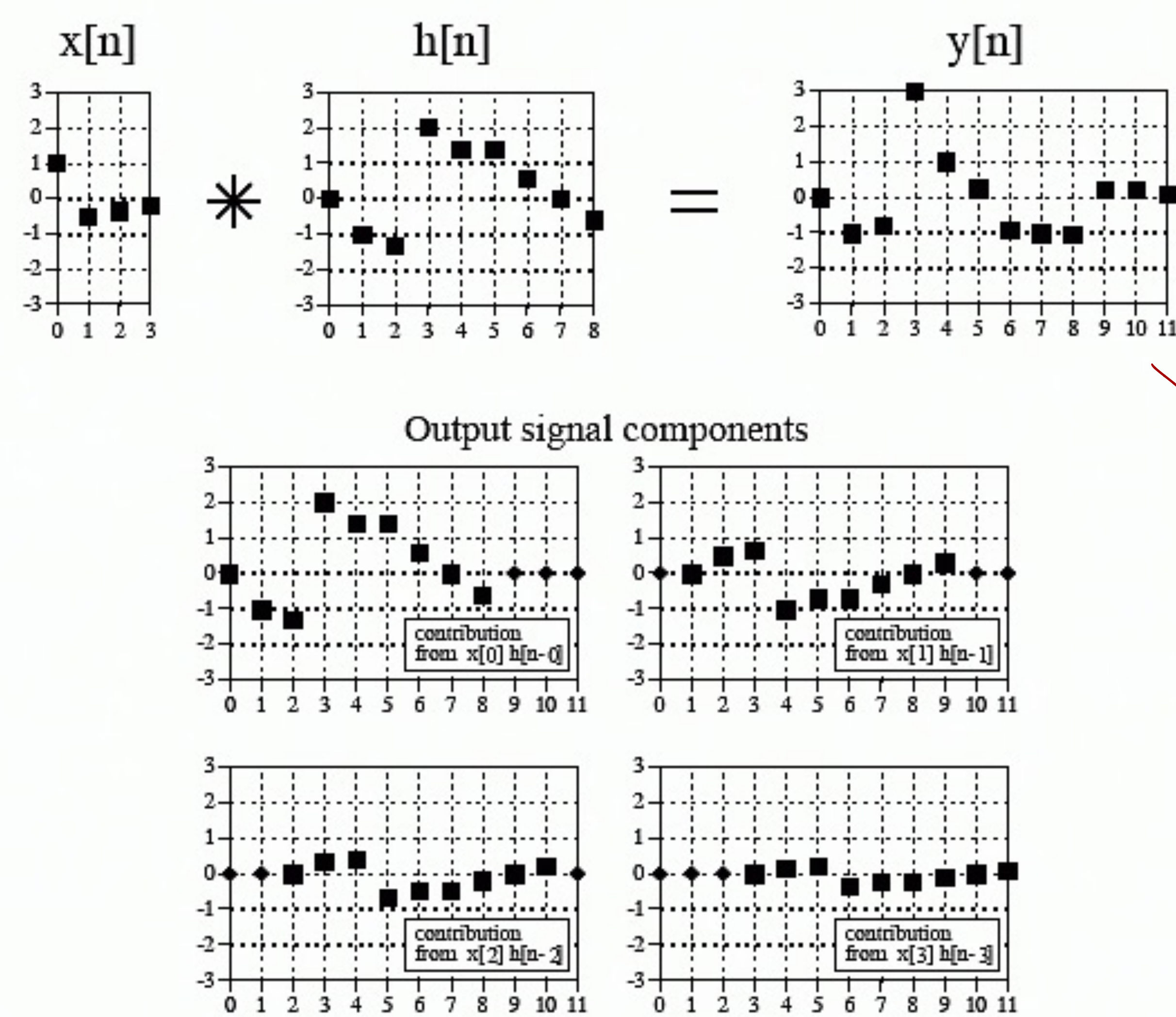


FIGURE 6-7
A second example of convolution. The waveforms for the input signal and impulse response are exchanged from the example of Fig. 6-5. Since convolution is commutative, the output signals for the two examples are identical.

identical

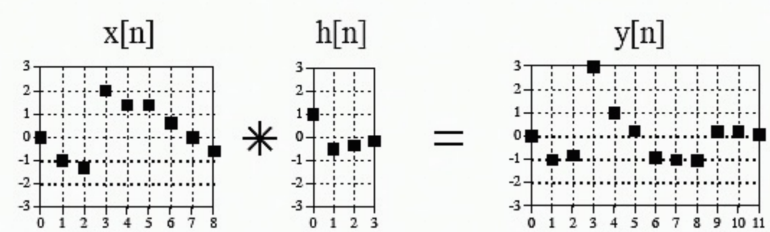


FIGURE 6-5
Example convolution problem. A nine point input signal, convolved with a four point impulse response, results in a twelve point output signal. Each point in the input signal contributes a scaled and shifted impulse response to the output signal. These nine scaled and shifted impulse responses are shown in Fig. 6-6.

Convolution is commutative $\Rightarrow$ $a[n] * b[n] = b[n] * a[n]$

Next $\rightarrow$ Graphical Convolution.