Linear Time-Invariant System

Discrete-Time System / Discrete System

A discrete-time system transforms discrete-time inputs into discrete-time outputs

$$x[n] \rightarrow y[n] = F(x[n])$$

$$x[n] \longrightarrow \boxed{F(\cdot)} \longrightarrow y[n]$$

Example → Balance Bank Account from Month to Month

$$y[n] = \underbrace{1.01\, y[n-1]}_{} + x[n]$$

→ models the fact that we accure 1% interest each month

$x[n] \rightarrow$ net deposit during n^{th} month

Basic System Properties

A system is memoryless if its output for each value of the independant variable at given time is dependant only on the input at the same time.

$$y[n] = \left(2x[n] - x^2[n]\right)^2 \longrightarrow \text{memoryless}$$

$$y[n] = y[n-1] + x[n] \longrightarrow \text{system with memory}$$

A system is invertible if distict inputs lead to distict outputs

$$y[n] = \sum_{k=-\infty}^{n} x[k] \longrightarrow \underbrace{x[n] = y[n] + y[n-1]}_{\text{Inverse system of the accumulator}}$$

A system is causal if the output at any time depends only on the input at the present time & past

All memoryless systems are causal. Why?

$$\underbrace{y[n] = \sum_{k=-\infty}^{n} x[k]}_{\text{Causal}} \qquad \underbrace{y[n] = x[n] - x[n+1]}_{\text{non-causal}}$$

A system is stable if small inputs lead to responses that do not diverge

accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

input $\rightarrow$ unit step signal.

$$y[n] = \sum_{k=-\infty}^{n} \mu[k] \rightarrow y[n] = (n+1)\mu[n]$$

$\downarrow$

the output grows without bound

$\downarrow$

Unstable

A system is <u>linear</u> if the following two properties hold.

Additive Property

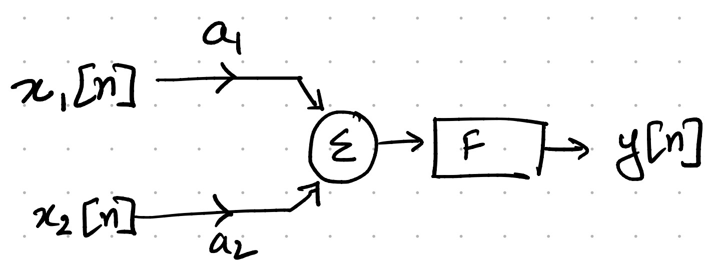
$$x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$$

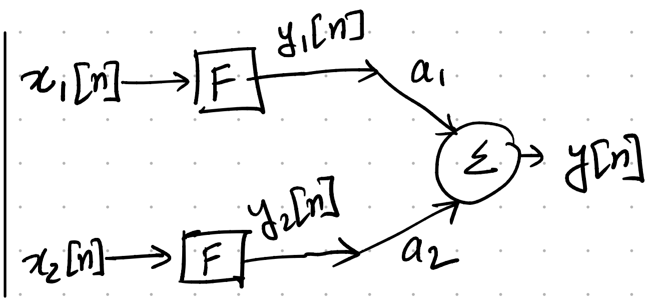
Scaling or Homogenity Property

$$ax[n] \rightarrow ay[n]$$

These two properties can be written as.

$$F(a_1 x_1[n] + a_2 x_2[n]) = a y_1[n] + a y_2[n]$$





$$\boxed{\text{Linear System} \Longleftrightarrow \text{Superposition}}$$

$$x_1[n] \rightarrow \boxed{F} \rightarrow y_1[n]$$
$$x_2[n] \rightarrow \boxed{F} \rightarrow y_2[n]$$

Test Linearity

① In system equation, <u>replace</u>:

$$x[n] \rightarrow a_1 x_1[n] + a_2 x_2[n]$$
$$y[n] \rightarrow a_1 y_1[n] + a_2 y_2[n]$$

② Equation still valid?

valid → linear

invalid → non-linear

Example 1 $\quad y[n] = 3x[n] + 2x[n-1]$

(i) Substituting

$$a_1 y_1[n] + a_2 y_2[n] = 3\left(a_1 x_1[n] + a_2 x_2[n]\right) + 2\left(a_1 x_1[n-1] + a_2 x_2[n-1]\right)$$

$$= a_1 \underbrace{\left(3x_1[n] + 2x_1[n-1]\right)}_{\equiv y_1[n]} + a_2 \underbrace{\left(3x_2[n] + 2x_2[n-1]\right)}_{\equiv y_2[n]}$$

$$= a_1 y_1[n] + a_2 y_2[n]$$

$\boxed{\text{Linear System}}$ Generally, sum of scaled, delayed/advanced inputs $\Rightarrow$ linear system

Example 2 $\quad y[n] = \alpha\, x[n] + \beta$

$$\left(a_1 y_1[n] + a_2 y_2[n]\right) = \alpha\left(a_1 x_1[n] + a_2 x_2[n]\right) + \beta$$

$$= a_1 \alpha x_1[n] + a_2 \alpha x_2[n] + \beta$$

$$= a_1\left(\alpha x_1[n] + \beta\right) - \beta a_1 + a_2\left(\alpha x_2[n] + \beta\right) - \beta a_2 + \beta$$

$$\neq a_1 y_1[n] + a_2 y_2[n] + \underbrace{\beta\left(1 - a_1 - a_2\right)}_{\text{not equal, unless } \beta = 0}$$

$\boxed{\text{non-linear system}}$

zero input $\longrightarrow$ zero output for linear system example 3

$$y[n] = n \cdot x[n] + n^2 x[n-1]$$

$$a_1 y_1[n] + a_2 y_2[n]$$

$$= n\left(a_1 x_1[n] + a_2 x_2[n]\right) + n^2\left(a_1 x_1[n-1] + a_2 x_2[n-1]\right)$$

$$= a_1\left(n x_1 + n^2 x[n-1]\right) + a_2\left(n x_2[n] + n^2 x_2[n-1]\right)$$

$$= a_1 y_1[n] + a_2 y_2[n]$$

$$\boxed{\text{Linear system}}$$

A system is time invariant if the behavior & characteristics of the system are fixed over time.

A system is time-invariant if a time shift in the input signal results in an identical time shift in the output signal.

$$x[n] \rightarrow y[n] \qquad x[n-n_0] \rightarrow y[n-n_0]$$

Time origin does not influence system transformation.

<u>example 1</u>

$$y[n] = x^2[n]$$

let's say the input is $x_1[n]$

① Apply function to input, then shift output.

$$y[n] = F(x_1[n]) = x_1^2[n]$$

$$out_1[n] = y[n-n_0]$$

$$= x_1^2[n-n_0]$$

Same

$\boxed{\text{Shift Invariant}}$

② Shift input, then apply function

shifted input $x_1[n-n_0]$

$$out_2[n] = F(x_1[n-n_0]) = (x_1[n-n_0])^2 = x_1^2[n-n_0]$$

Notes $y[n] = x^2[n]$

1. not linear
2. $\underbrace{\text{static non-linearities}}_{\text{depend only on current time}}$ are shift invariant

Dynamic → depends on past and/or future times.

Example 2 $y[n] = n \cdot x[n]$

input : $x_1[n]$

① $y[n] = n \cdot x_1[n]$

$out_1[n] = y[n-n_0] = [n-n_0]\, x_1[n-n_0]$

② shifted input $x_1[n-n_0]$

$out_2[n] = F(x_1[n-n_0])$

$= n\, x_1[n-n_0]$

not same

Time Varying

example 3

$$y[n] = \frac{x[n+1] - x[n-1]}{2}$$

1. input $x_1[n]$

$$y[n] = \frac{x_1[n+1] - x[n-1]}{2}$$

substitute $x_1[n]$ by $x_1[n-n_0]$

$$out_1[n] = y[n-n_0] = \frac{x_1[n-n_0+1] - x_1[n-n_0-1]}{2}$$

2. shifted input $x_1[n-n_0]$

$$out_2[n] = F(x_1[n-n_0])$$

$$= \frac{x_1[n-n_0+1] - x_1[n-n_0-1]}{2}$$

same

Time Invariant