Generalizations to $n \times n$ System

Recall from linear algebra that a system of equations with upper/lower triangular matrix representation is very easy to solve, just like the above. In general, consider the following triagnular system,

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ 0 & a_{22} & . & . & a_{2n} \\ . & 0 & * & * & * \\ . & . & 0 & * & * \\ 0 & . & . & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}.$$

We can read off the $x_n = \frac{b_n}{a_{nn}}$ immediately. Then,

$$x_{n-1} = \frac{b_{n-1} - a_{(n-1)n}x_n}{a_{(n-1)(n-1)}}$$

can be found since we know x_n already. Pushing this sequence backwards, we can find all x_i's sequentially, that is,

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$$

as we know about $x_{i+1}, \ldots x_n$ from previous steps. This procedure is called **back substitution**.

Therefore, it is very beneficial to reduce any augmented matrix $\tilde{A} = [A \mid b]$ into a upper triangular form considered above. In other words, we reduce the augmented system to **row echelon form**, that is, for the k^{th} row, the first nonzero entry is to the right of the k^{th} column. More precisely, we achieve this by the following **elementary row operations.**

Consider the system

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$$

which can be further represented by the **augmented** matrix,

$$\widetilde{A} = [A \mid b] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & . & . & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & . & . & a_{2n} & a_{2,n+1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} & a_{n,n+1} \end{array}\right]$$

where

$$a_{i,n+1} = b_i, \quad i = 1, 2, \ldots, n.$$

(1) Given that $a_{11} \neq 0$, construct the multipliers

$$m_{j1} = \frac{a_{j1}}{a_{11}}, \quad j = 2, 3, \ldots, n$$

for rows other than row 1. We call a_{11} the **pivot** of the first row.

(2) Eliminate the coefficient of x_1 in each row via:

$$\left(E_j - \frac{a_{j1}}{a_{11}}E_1\right) \to (E_j), \quad j = 2, 3, \ldots, n, \tag{0.1}$$

This will make the system look like

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & . & . & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & . & . & a_{2n} & a_{2,n+1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & a_{n-1}a_2 & . & . & a_{nn} & a_{n,n+1} \end{array}\right]$$

where the entries are NOT necessarily the same as before. This is simply showing the resulting structure after applying Eq.0.1.

(3) For each remaining coefficients, we perform, by keeping E_i intact but using it to modify other rows, i.e.

$$\left(E_j - \frac{a_{ji}}{a_{ii}} E_i\right) \to (E_j)\,, \quad j = i+1, i+2, \dots, n,$$

provided that $a_{ii} \neq 0$. This eliminates the coefficient of x_i in each row below the i^{th} for $i = 1, 2, \dots, n-1$.

Algorithm

(1) For $i = 1, \dots, n-1$
   (a) Let p be the smallest integer with $i \leq p \leq n$ and $a_{pi} \neq 0$
   If no integer p can be found
   then OUTPUT ('no unique solution exists');
   STOP.
   (This step is checking if we find an entire row of zeros).
   (b) If $p \neq i$, then perform $(E_p) \leftrightarrow (E_i)$.
   (Say, for $i = 1$, we found $p = 2$, meaning that $a_{11} = 0$ while $a_{12} \neq 0$. This row thus cannot be used to perform row operations since the multiplier requires that $a_{11} \neq 0$. However, since there is only one zero to the left of the leading entry a_{12}, it would be nice if it is moved to the second row, such that it is "nicely reduced" already.)
   (c) For $j = i+1, \dots, n$
      (i) Set $m_{ji} = \frac{a_{ji}}{a_{ii}}$ (form multiplier used on E_i to modify E_j via **elementary row operations**);
      (ii) Perform $(E_j - m_{ji}E_i) \to (E_i)$ (**elementary row operations**).

(2) If $a_{nn} = 0$, then OUTPUT ('no unique solution exists');
STOP.

(3) Set $x_n = \frac{a_{n,n+1}}{a_{nn}}$. (Start back substitution)

(4) For $i = n-1, \dots, 1$, set

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}.$$

(5) OUTPUT $(x_1, \dots, x_n)$.