1. Preliminaries for Linear Systems

A system of equations

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2 \\ &. \\ &. \\ &. \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

can be represented by the matrix equation

$$Ax = b$$

where

$$A = \{a_{ij}\}_{i,j=1}^{n}, \quad x = \{x_i\}_{i=1}^{n}, \quad b = \{b_i\}_{i=1}^{n},$$

or

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}.$$

Your HW2 has a coding exercise that carries out this multiplication.

2. Direct Method: Gaussian Elimination

2.1. Simple system.

Example. Consider the 2×2 system:

$$\begin{aligned} 3x_1 + 2x_2 &= 8 \\ 2x_1 - 3x_2 &= 1 \end{aligned}$$

or in matrix form

$$\begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$
$$Ax = b$$

(By inspection, the true solution is $x_1 = 2$ and $x_2 = 1$.)

(1) Naive substitution ($O\left(n^4\right)$ method, avoid at all cost). See notes. Optional reading.
(2) Gaussian elimination via **elementary row operations** and back substitution.

Solution. First, we form the augmented matrix by appending b to the right hand side of A.

$$[A \mid b] = \left[\begin{array}{cc|c} 3 & 2 & 8 \\ 2 & -3 & 1 \end{array}\right].$$

We multiply the first row by $2/3$ so that the first entry of each row matches. This modifies the above into

$$\left[\begin{array}{cc|c} 3 & 2 & 8 \\ 2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & \frac{4}{3} & \frac{16}{3} \\ 2 & -3 & 1 \end{array}\right].$$

Now, we create a new second row by subtracting the first from the second,

$$\left[\begin{array}{cc|c} 2 & \frac{4}{3} & \frac{16}{3} \\ 2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} \boxed{2} & \frac{4}{3} & \frac{16}{3} \\ 0 & \boxed{\dfrac{13}{3}} & \frac{13}{3} \end{array}\right].$$

This transformed matrix is now in **row echelon form**, meaning that the leading entry (the first nonzero entry) is to the right of the leading entry of the previous rows (see the boxed entries).

From the last row, we can immediately read that

$$x_2 = \frac{13/3}{13/3} = 1.$$

Then, back substitution into the first equation gives

$$x_1 = \frac{16/3 - \frac{4}{3}x_2}{2} = \frac{4}{2} = 2.$$

Remark. The most important takeaway here is that the procedures in the first method CANNOT be represented by **elementary row operations** and hence by updating the matrix representation of the system. The second method, however, always admits a matrix representation.

2.2. **Larger System.**

Example. (4×4 system)

$$\begin{aligned} E_1 &: \quad x_1 + x_2 \quad + 3x_4 = 4, \\ E_2 &: \quad 2x_1 + x_2 - x_3 + x_4 = 1, \\ E_3 &: \quad 3x_1 - x_2 - x_3 + 2x_4 = -3, \\ E_4 &: \quad -x_1 + 2x_2 + 3x_3 - x_4 = 4. \end{aligned}$$

We keep the first equation intact. Perform

$$\begin{aligned} (E_2 - 2E_1) &\to (E_2) \\ (E_3 - 3E_1) &\to (E_3) \\ (E_4 + E_1) &\to (E_4) \end{aligned}$$

to obtain

$$\begin{aligned} E_1 &: \quad x_1 + x_2 \quad + 3x_4 = 4, \\ E_2 &: \quad -x_2 - x_3 + x_4 = -7, \\ E_3 &: \quad -4x_2 - x_3 - 7x_4 = -15, \\ E_4 &: \quad 3x_2 + 3x_3 + 2x_4 = 8. \end{aligned}$$

Now, we start with the second row, and try to eliminate the coeffcients of x_2 in row 3 and row 4. Perform

$$\begin{aligned} (E_3 - 4E_2) &\to (E_3) \\ (E_4 + 3E_2) &\to (E_4) \end{aligned}$$

to obatin

$$\begin{aligned}
E_1 &: \quad x_1 + x_2 \qquad + \ 3x_4 = \ \ 4,\\
E_2 &: \quad -x_2 - x_3 + x_4 = -7,\\
E_3 &: \quad 3x_3 + 13x_4 = \ 13,\\
E_4 &: \quad -13x_4 = \ 13.
\end{aligned}$$

In matrix form, this looks like

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 13 \\ -13 \end{bmatrix}$$

where we observe the matrix is **upper triangular**. The unknowns now can be solved backwards starting with x_4. Indeed,

$$x_4 = \frac{-13}{-13} = 1,$$

$$x_3 = \frac{13 - 13x_4}{3} = 0,$$

$$x_2 = \frac{-7 - (-5)\,x_4 - (-1)\,x_3}{-1} = 2,$$

$$x_1 = \frac{4 - 3x_4 - 0x_3 - x_2}{1} = -1.$$

2.3. **Errors Still Occur.**

Example. Consider the 3×3 system

$$\begin{aligned}
E_1 &: \quad -x_1 + 4x_2 + x_3 = 8\\
E_2 &: \quad \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 = 1\\
E_3 &: \quad 2x_1 + x_2 + 4x_3 = 11
\end{aligned}$$

or

$$\begin{bmatrix} -1 & 4 & 1 \\ \frac{5}{3} & \frac{2}{3} & \frac{2}{3} \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 11 \end{bmatrix}$$

or in augmented form

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ \frac{5}{3} & \frac{2}{3} & \frac{2}{3} & 1 \\ 2 & 1 & 4 & 11 \end{array}\right].$$

The true solution is

$$x_1 = -1, \quad x_2 = 1, \quad x_3 = 3.$$

Let's solve this system using 2-digit rounding arithmetic and Gaussian elimination with backward substitution. We begin with elementary row operations on E_2 and E_3 using E_1.

- $\left(E_2 + \frac{5}{3}E_1\right) \to (E_2)$

$\frac{5}{3}E_1$ has exact entries,

$$-\tfrac{5}{3} \quad \tfrac{20}{3} \quad \tfrac{5}{3} \;\Big|\; \tfrac{40}{3}$$

We need to 2-digit round here into

$$-1.7 \quad 6.7 \quad 1.7 \;\Big|\; 13$$

We also round the second row as

$$\begin{array}{cccc} -1.7 & 6.7 & 1.7 & 13 \\ +)1.7 & 0.67 & 0.67 & 1 \\ \hline 0 & 7.37 \to 7.4 & 2.37 \to 2.4 & 14 \end{array}$$

- $(E_3 + 2E_1) \to (E_3)$

We are lucky that E_3 does not require rounding initially. Nor does $2E_1$

$$-2 \quad 8 \quad 2 \;\Big|\; 16$$

$$\begin{array}{ccc|c} -2 & 8 & 2 & 16 \\ +)2 & 1 & 4 & 11 \\ \hline 0 & 9 & 6 & 27 \end{array}$$

After these two elementary row operations, we have

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.4 & 2.4 & 14 \\ 0 & 9 & 6 & 27 \end{array}\right].$$

Now, we need to use E_2 to modify E_3 by killing the 9. We do

$$\left(E_3 - \frac{9}{7.4}E_2\right) \to (E_3)$$

Note that, with rounding, $9/7.4 = 1.2$, but this will NOT be applied to the first nonzero term because $7.4 \times \frac{9}{7.4} = 9$.

$$\frac{9}{7.4}E_2 = \left[\begin{array}{cccc} 0 & \frac{9}{7.4} \times 9 & 2.4 \times \cancelto{1.2}{\frac{9}{7.4}} & 14 \times \cancelto{1.2}{\frac{9}{7.4}} \end{array}\right] = \left[\begin{array}{cccc} 0 & 9 & 2.88 \to 2.9 & 16.8 \to 17 \end{array}\right]$$

Then,

$$E_3 - \frac{9}{7.4}E_2 = \begin{array}{ccc|c} 0 & 9 & 6 & 27 \\ -)0 & 9 & 2.9 & 17 \\ \hline 0 & 0 & 3.1 & 10 \end{array}$$

and we arrive at row echelon form

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.4 & 2.4 & 14 \\ 0 & 0 & 3.1 & 10 \end{array}\right].$$

With back substitution, we have

$$x_3 = \frac{10}{3.1} = 3.2258 \to 3.2.$$

$$x_2 = \frac{14 - 2.4 \times 3.2}{7.4} = \frac{14 - 7.68 \to 7.7}{7.4} = \frac{6.3}{7.4} = 0.85135 \to 0.85.$$

$$x_1 = \frac{8 - x_3 - 4x_2}{-1} = \frac{8 - 3.2 - 4 \times 0.85}{-1} = \frac{8 - 3.2 - 3.4}{-1} = -1.2.$$

Compare to the true solution

	2-digit rounding	True
x_1	-1.2	-1
x_2	0.85	1
x_3	3.2	3