## 1. Rates of Convergence

Consider two functions $f(x) = \frac{1}{x}$ and $g(x) = e^{-x}$. Send $x$ to $\infty$. We know both limits are zero. But, which one does it faster? We actually compare their rates via L'Hôpital,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x}{e^{-x}} = \lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

So, the quotient blows up, meaning that $f(x)$ is running away faster than $g(x)$ – or in other words, $g(x) \to 0$ faster than $f(x)$ does.

Can we quantify this "rate" when we no longer have differentiability to help us, i.e. bye-bye L'Hôpital?

How about first just a sequence of numbers? For example, compare the rate at which

$$\alpha_n = \frac{1}{n}, \quad \gamma_n = \frac{1}{n^2}$$

converge to 0? We anticipate that $\gamma_n$ goes to zero faster, in the sense that we can compute

$$\lim_{n\to\infty} \frac{\alpha_n}{\gamma_n} = \lim_{n\to\infty} \frac{1/n}{1/n^2} = \infty$$

which means $\alpha_n$ is much larger than $\gamma_n$ when $n$ is large – $\gamma_n$ is much closer to zero than $1/n$ and thus converges faster! Now, the task is to quantify the rate of convergence of the sequences.

**Definition.** ("big O notation") Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence known to converge to zero, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number $\alpha$. If a positive constant $K$ exists with

$$|\alpha_n - \alpha| \le K\beta_n, \quad \text{for large } n,$$

then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to $\alpha$ with **rate**, or **order**, **of convergence** $O(\beta_n)$ ("big O of $\beta_n$"). We also write this as

$$\alpha_n = \alpha + O(\beta_n), \quad n \to \infty.$$

Though $\{\beta_n\}$ is an arbitrary sequence that goes to zero, we almost always use the form

$$\beta_n = \frac{1}{n^p}, \quad p > 0.$$

We are interested in the largest value of $p$ such that $\alpha_n = \alpha + O(1/n^p)$. For example, if $p = 1$, we say it is 1st-order/linear convergence; $p = 2$, 2nd order/quadratic convergence, and so on.

**Example.** $1/n^p = O(1/n^p)$ certainly. How about $\frac{n}{n^2+1}$? It converges to 0 but at which order? One may suspect 1st order because of the power difference. Let's confirm it.

$$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} \le \frac{n}{n^2} = \frac{1}{n}$$

where the constant $K = 1$, and $\beta_n = \frac{1}{n}$. Yes, indeed, 1st order.

**Definition.** A little more about "big O" notation here: instead of sequences, we may also say certain function is "big O" of some other function, usually to capture the order of growth/decay. The definition in this case is very similar to that for sequences. If we can find a positive real number $M$ and a threshold $x_0$ such that

$$|f(x)| \le Mg(x), \quad \forall x \ge x_0,$$

then we say that

$$f(x) = O(g(x)), \quad x \to \infty.$$

**Example.** Consider $f(x) = 6x^4 - 2x^2 + 5$. We want to capture its behaviour as $x \to \infty$.

$$\left|6x^4 - 2x^2 + 5\right| \le 6x^4 + 2x^2 + 5 \le 6x^4 + 2x^4 + 5x^4 = 13x^4, \quad \forall x \ge x_0$$

(in fact, regardless of what $x_0$ is) where we identified the constant $K = 13$ and $g(x) = x^4$. Thus, we say $f(x) = O\left(x^4\right)$ as $x \to \infty$.

**Definition.** Suppose we are no longer interested in sending $x \to \infty$ but rather $x \to a$, then the above condition is modified: if we can furnish a positive real number $M$ and an interval size $\delta$ such that

$$|f(x)| \le Mg(x), \quad 0 < |x - a| < \delta$$

then we say

$$f(x) = O(g(x)), \quad x \to a.$$

**Example.** Consider the same function $f(x) = 6x^4 - 2x^2 + 5$, but now we are interested in the asymptotic behaviour as $x \to 1$. Given $\delta > 0$, we look at the points in the neighborhood of $x = 1$, i.e. the points $x$ that satisfy $0 < |x - 1| < \delta$ (need to utilize this bound on $x$).

Now, these points obviously satisfy

$$-\delta < x - 1 < \delta$$

which is equivalent to

$$1 - \delta < x < 1 + \delta.$$

Thus,

$$\begin{aligned} |f(x)| &= \left|6x^4 - 2x^2 + 5\right| \\ &\le \left|6(1+\delta)^4 + 2(1+\delta)^2 + 5\right| \\ &= 6(1+\delta)^4 + 2(1+\delta)^2 + 5. \end{aligned}$$

The RHS here is just a number. You choose how close you want to be to $x = 1$ (via $\delta$), and you can always bound the function by the constant above. Therefore, identify

$$M = 6(1+\delta)^4 + 2(1+\delta)^2 + 5, \quad g(x) = 1.$$

We say that

$$f(x) = O(1), \quad x \to 1.$$

In practice, we sometimes care about $x \to 0$, or better if we change the variable to $h \to 0$, where $h$ may represent the interval size of a numerical integral, i.e. the step size of a Riemann sum. We wish to approximate some integral using areas of small rectangles spanned by small subintervals. Another example pertains to solutions of ODEs, where we can only take finite but small time steps to approximate solutions to $\frac{d^2y}{dt^2} = -ky$ (Hooke's law).

**Definition.** Suppose that $\lim_{h\to 0} G(h) = 0$ and $\lim_{h\to 0} F(h) = L$. If a positive constant $K$ exists with

$$|F(h) - L| \le K|G(h)|$$

for sufficiently small $h$, then we say

$$F(h) = L + O(G(h)).$$

Per usual, we care about $G(h) = h^p$ for $p > 0$. We are interested in the largest value of $p$ for which $F(h) = L + O(h^p)$.

**Example.** Use the Maclaurin series of $\cos(h)$ up to three terms to show that $\cos(h) + \frac{1}{2}h^2 = 1 + O\left(h^4\right)$.

*Proof.* Let $f(h) = \cos(h)$. Then we need derivatives up to the fourth order since two of them evaluate to zero at $h = 0$.

$$f'(h) = -\sin(h), \quad f''(h) = -\cos(h), \quad f'''(h) = \sin(h), \quad f^{(4)}(h) = \cos(h).$$

and thus about $h = 0$, we have

$$\cos(h) = \cos(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \frac{h^3}{3!}f'''(0) + \frac{h^4}{4!}f^{(4)}(\xi)$$

where $\xi \in (0, h)$ (last step is the error term of Taylor series).

$$\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!}\cos(\xi(h))$$

where in the last term, we must write that this $\xi$ depends on $h$. Moving terms around, we have

$$\cos(h) + \frac{1}{2}h^2 = 1 + \frac{h^4}{4!}\cos(\xi(h)).$$

Furthermore, identify that $F(h) = \cos(h) + \frac{1}{2}h^2$, $L = 1$ (clearly it is the limit of $F(h)$ as $h \to 0$).

$$\left|\cos(h) + \frac{1}{2}h^2 - 1\right| = \left|\frac{h^4}{4!}\cos(\xi(h))\right| \leq \frac{1}{24}h^4$$

which we further identify $K = 1/24$ and $G(h) = h^4 \to 0$ as $h \to 0$ (required by definition). Thus, we can say that

$$\cos(h) + \frac{1}{2}h^2 = 1 + O\left(h^4\right), \quad h \to 0.$$

□

**Example.** Consider $f(x) = \sin(x)$. We want to capture its behaviour as $x \to 0$. Given $\delta > 0$ such that $0 \leq |x| < \delta$, we have

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!}f^{(3)}(\xi) \\
\implies x - \sin x &= \frac{x^3}{3!}f^{(3)}(\xi) \\
\implies |\sin x - x| &= \left|\frac{x^3}{3!}f^{(3)}(\xi)\right| \\
&= \left|\frac{x^3}{3!}(-\cos(\xi))\right| \\
&\leq \frac{x^3}{6}
\end{aligned}$$

where $\xi \in (0, \delta)$ (think Mean Value Theorem applied to Taylor polynomials, or Taylor's Theoerem, Section 1.1, Theorem 1.14). We identify $M = 1/6$ and $g(x) = x^3$. In other words,

$$f(x) = \sin(x) = x + O\left(x^3\right), \quad x \to 0.$$

*Remark.* It is very important that one clarifies where the point of interest is when using big oh notation, i.e. it is a characterization of the **asymptotic behaviour** of $f(x)$, and when something is asymptotic, you state where the independent variable is limiting. More precisely, when one says $f(x) = O\left(x^2\right)$, we must specify where $x$ is going. For example, for the same function $f(x) = x^2 + 1$, $f(x) = O\left(x^2\right)$ as $x \to \infty$, but $f(x) = O(1)$ as $x \to 0$, because given $\delta$ such that $0 < |x| < \delta$, we have

$$|f(x)| = \left|x^2 + 1\right| \leq \delta^2 + 1$$

which is a constant on the RHS (identify $K = \delta^2 + 1$ and $g(x) = 1$), and thus we can say

$$f(x) = O(1), \quad x \to 0.$$