## Newton's Method

In the last section, we saw first how we can turn any root-finding problem to a fixed-point finding problem, and then utilize the fixed-point iteration to find the fix point. However, fixed-point iteration can become very slow when the iterated function does not have a derivative with small magnitude (one of the sufficient conditions to converge). Newton's method utilizes the slope of the tangent (regardless of its magnitude) to construct linear approximations of the function at each point of the iteration, and finally arrive at the zero of a function without relying on the magnitude of the derivative being smaller than 1.

Consider a function $f(x)$ where we know $f(p) = 0$ for $p \in [a, b]$. We start with an initial guess $p_0 \in [a, b]$ such that $f(p_0) \neq 0$. Then, if $f \in C^1[a, b]$, we can compute the equation of the tangent line at $p_0$,

$$y = f(p_0) + f'(p_0)(x - p_0),$$

i.e., the linearization of $f$ at $x = p_0$.

This tangent line is an approximation of $f$, which may motivate us to speculate that the $x$-intercept of this line is close to the zero of $f$. We see that the $x$-intercept of the line, $p_1$, is no other than

$$0 = f(p_0) + f'(p_0)(p_1 - p_0) \implies p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}.$$

Then, we use $p_1$ as our "guess" for the next approximation. We compute the equation of the tangent line at $x = p_1$ now, and use the $x$-intercept of this tangent as an approximation $p_2$. More precisely,

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}.$$

In general, we continue to use the $x$-intercept of the tangent line at a sequence of points to approximate the true "$x$-intercept" of $f$, in the iterative fashion as

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}.$$

This method is called **Newton-Raphson method**, or sometimes in short, **Newton's method**.

The stopping criterion of this method is the sequential absolute error

$$|p_N - p_{N-1}| < \epsilon$$

for some prescribed tolerance level $\epsilon$, or in sequential relative error,

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon, \quad p_N \neq 0,$$

or in function value

$$|f(p_N)| < \epsilon$$

since we want $f(p_N) \approx 0$.

### Relationship to Fixed-Point Iteration and Convergence

Note that the Newton's method

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

is a functional iteration technique with $p_n = g(p_{n-1})$ where

$$g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.$$

**Example.** Consider the function $f(x) = \cos(x) - x$. Approximate a root of $f$ using both a fixed-point method and then Newton's method. Which one converges faster?

**Solution.** The fixed-point method requires a function $g_{\text{fix}}$ such that the fixed point of $g$ is a root of $f$. Clearly, $g_{\text{fix}}(x) = \cos(x)$ works.

Newton's method requires that we identify the iterative function $g_{\text{newt}}(x)$ such that

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

In this problem, we have

$$g(x) = x - \frac{\cos(x) - x}{-\sin(x) - 1} = x + \frac{\cos(x) - x}{\sin(x) + 1}.$$

Thus, to study the convergence framework of Newton's method, we may equivalently study that of fixed-point iterative algorithms.

**Theorem.** *Let $f \in C^2[a,b]$. If $p \in (a,b)$ is a root of $f$ such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to $p$ for any initial approximation $p_0 \in [p-\delta, p+\delta]$.*

*Proof.* Consider the iterative scheme $p_n = g(p_{n-1})$ where

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Then, it is equivalent to show that $g$ has a unique fixed point in $[p-\delta, p+\delta]$. This requires two conditions on $g$.

(1) $g$ maps $[p-\delta, p+\delta]$ to $[p-\delta, p+\delta]$.
(2) $|g'(x)| \leq k < 1$ for $x \in [p-\delta, p+\delta]$.

The details of the proof are in Theorem 2.6. Very interesting read with very straightforward computations. □

## Secant Method

In practice, we aren't always blessed with smooth functions. First-order derivatives are commonly approximated by a finite difference instead of being pinpointed by an exact value. Therefore, we consider a more practical version of Newton's method.

We write the derivative as a finite difference

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

If $x = p_{n-2}$ is very close to $p_{n-1}$, then we may write

$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

Replacing the derivative in Newton's method, we then have the **Secand Method**, an iteration on

$$p_n = p_{n-1} - f(p_{n-1}) \frac{p_{n-1} - p_{n-2}}{f(p_{n-1}) - f(p_{n-2})}.$$

Note that this iterative scheme is a **second-order recursion**, which means it will requires **two initial guesses** $p_0$ and $p_1$.