Jacobi Iteration

For a linear system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n,$$

we can express each component of the solution $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ in terms of other components. More precisely, by isolating x_i in the i^{th} equation, we have

$$\begin{aligned} x_1 &= \frac{b_1 - (a_{12}x_2 + \cdots + a_{1n}x_n)}{a_{11}}, \\ x_2 &= \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n)}{a_{22}}, \\ &\vdots \\ x_n &= \frac{b_n - \left(a_{n1}x_1 + \cdots + a_{(n-1)n}x_{n-1}\right)}{a_{nn}}. \end{aligned}$$

This looks a lot like the "naive substitution" technique introduced before Gaussian elimination. Indeed, this is a costly algorithm. However, some two hundred years ago, Carl Gustav Jacob Jacobi (also responsible for the Jacobian matrix for multidimensional change of variable) found that you can start with some guesses

$$\boldsymbol{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}\right)$$

and plug them into the RHS. We then obtain a new vector

$$\boldsymbol{x}^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\right).$$

Now, chuck this into the RHS, and continue iterating, we somehow obtain the true solution $\boldsymbol{x}$ by approximating with

$$\boldsymbol{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right).$$

This iteration ends when the error of the approximation is below some prescribed tolerance, i.e.,

$$\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right\| < \epsilon$$

under various notions of vector norms (and also various notions of error). Other forms of stopping criterion may be

$$\frac{\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right\|}{\left\|\boldsymbol{x}^{(k)}\right\|} < \epsilon$$

which is well-defined because $\left\|\boldsymbol{x}^{(k)}\right\| \neq 0$ for $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} \neq \boldsymbol{0}$. The criterion can depend on the problem at hand.

In component form, this method can be written as

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n.$$

Example

Example. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Let's solve $A\boldsymbol{x} = b$ using Jacobi iteration, with an initial guess $\boldsymbol{x}^{(0)} = (0, 0)^{\mathrm{T}}$. Note that the true solution is $\boldsymbol{x} = (1, 1)^{\mathrm{T}}$.

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)}}{a_{11}} = \frac{3 - 1x_2^{(k)}}{2},$$
$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k)}}{a_{22}} = \frac{3 - 1x_1^{(k)}}{2}.$$

With this formula, we have

$$\boldsymbol{x}^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right)^{\mathrm{T}} = \left(\frac{3}{2}, \frac{3}{2}\right)$$
$$\boldsymbol{x}^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right)^{\mathrm{T}} = \left(\frac{3-\frac{3}{2}}{2}, \frac{3-\frac{3}{2}}{2}\right) = \left(\frac{3}{4}, \frac{3}{4}\right)$$
$$\boldsymbol{x}^{(3)} = \left(x_1^{(3)}, x_2^{(3)}\right)^{\mathrm{T}} = \left(\frac{3-\frac{3}{4}}{2}, \frac{3-\frac{3}{4}}{2}\right) = \left(\frac{9}{8}, \frac{9}{8}\right)$$
$$\boldsymbol{x}^{(4)} = \left(x_1^{(4)}, x_2^{(4)}\right)^{\mathrm{T}} = \left(\frac{3-\frac{9}{8}}{2}, \frac{3-\frac{9}{8}}{2}\right) = \left(\frac{15}{16}, \frac{15}{16}\right)$$

Matrix Form

Let us revisit the overall structure of the Jacobi method.

$$x_1 = \frac{b_1 - (a_{12}x_2 + \cdots + a_{1n}x_n)}{a_{11}},$$
$$x_2 = \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n)}{a_{22}},$$
$$.$$
$$.$$
$$x_n = \frac{b_n - \left(a_{n1}x_1 + \cdots + a_{(n-1)n}x_{n-1}\right)}{a_{nn}}.$$

Here, we see that each row involves other variables. Rearranging a little, we see that

$$a_{11}x_1 = b_1 - (a_{12}x_2 + \cdots + a_{1n}x_n)$$

More precisely, let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, we observe

$$a_{11}x_1 = b_1 - (0, a_{12}, a_{13}, \ldots, a_{1n})\,\boldsymbol{x}.$$

In general, we have

$$\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & . & & \\ & & & . & \\ & & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & . & . & a_{1n} \\ a_{21} & 0 & . & . & a_{2n} \\ . & & 0 & & . \\ . & & & 0 & . \\ a_{n1} & & & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}.$$

Define the diagonal, upper triangular and lower triangular part of the matrix $A = D + L + U$,

$$D = \begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & . & & \\ & & & . & \\ & & & & a_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & . & . & 0 \\ a_{21} & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & a_{n,n-1} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & . & . & a_{1n} \\ 0 & 0 & . & . & a_{2n} \\ . & . & 0 & . & . \\ . & . & . & 0 & a_{n-1,n} \\ 0 & . & . & . & 0 \end{bmatrix},$$

we can rewrite the system as

$$D\boldsymbol{x} = \boldsymbol{b} - (L + U)\,\boldsymbol{x}.$$

Multiplying both sides of the equation by D^{-1}, which exists (just a diagonal matrix of the reciprocals of D), we have

$$\boldsymbol{x}^{(k+1)} = -D^{-1}(L + U)\,\boldsymbol{x}^{(k)} + D^{-1}b.$$

Note that in component form, this is exactly the same formula as we had in the last section,

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n,$$

since $D^{-1} = \left\{ \frac{1}{a_{ii}} \right\}_{i=1}^{n}$.