Exam Review: Exercise

Example. Consider the sequence

$$\boldsymbol{x}^{(k)} = \left(1, 2+\frac{1}{k}, \frac{3}{k^3}, e^{-k}\sin k\right)^{\mathrm{T}} \in \mathbb{R}^4.$$

Find the limit of $\left\{\boldsymbol{x}^{(k)}\right\}$ in l^{∞}-norm and l^2-norm respectively.

Solution. l^{∞}-norm is straightforward. Theorem 1 tells us that the l^{∞} limit of a sequence, if exists, is the limit of individual components. Thus,

$$\begin{aligned}\lim_{k\to\infty} 1 &= 1;\\ \lim_{k\to\infty} 2+\frac{1}{k} &= 2;\\ \lim_{k\to\infty} \frac{3}{k^2} &= 0;\\ \lim_{k\to\infty} e^{-k}\sin(k) &= 0,\end{aligned}$$

and therefore, $\boldsymbol{x}^{(k)}$ converges to $(1,2,0,0)^{\mathrm{T}}$ with respect to the l^{∞}-norm.

To show that $\boldsymbol{x}^{(k)}$ converges to the same limit in l^2, we use Theorem 2 and the definition of convergence. Since $\boldsymbol{x}^{(k)} \to \boldsymbol{x} = (1,2,0,0)^{\mathrm{T}}$ in l^{∞}, for any $\epsilon > 0$, we can furnish an integer $N(\epsilon)$ such that

$$\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\|_{\infty} < \frac{\epsilon}{2}$$

whenever $k \geq N(\epsilon)$. By the equivalent theorem (Theorem 2), for exactly the same $N(\epsilon)$, we have

$$\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\|_{2} \overset{\text{Theorem 2 for } n=4}{\leq} \sqrt{4}\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\|_{\infty} \leq 2\left(\frac{\epsilon}{2}\right) = \epsilon$$

for $k \geq N(\epsilon)$. Thus $\boldsymbol{x}^{(k)}$ also converges to $\boldsymbol{x}$ in l^2-norm.

Spectral Radius

Definition. The **spectral radius** $\rho(A)$ of a matrix A defined by

$$\rho(A) = \max|\lambda|$$

where λ is an eigenvalue of A. If λ is complex, i.e., $\lambda = \alpha + i\beta$, define

$$|\lambda| = \sqrt{\alpha^2+\beta^2}.$$

As it turns out, the spectral radius can be linked to the matrix norm induced by l^2-norm.

Theorem. *If A is an $n\times n$ matrix, then*

(1) $\|A\|_2 = \left[\rho\left(A^{\mathrm{T}}A\right)\right]^{1/2}$,
(2) $\rho(A) \leq \|A\|$, *for any induced norm* $\|\cdot\|$ *(also known as natural norm).*

Let us use the first formula to compute the 2-norm of a matrix.

Example. Determine the l^2-norm of

$$A = \begin{bmatrix} 1 & 1 & 0\\ 1 & 2 & 1\\ -1 & 1 & 2\end{bmatrix}.$$

Solution. We need $\rho\left(A^{\mathrm{T}}A\right)$. But first, we compute $A^{\mathrm{T}}A$.

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & -1\\ 1 & 2 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix} 1 & 1 & 0\\ 1 & 2 & 1\\ -1 & 1 & 2\end{bmatrix} = \begin{bmatrix} 3 & 2 & -1\\ 2 & 6 & 4\\ -1 & 4 & 5\end{bmatrix} = B.$$

Then, we find the eigenvalues of B.

$$\begin{aligned}\det(B-\lambda I) &= \det\begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix} \\ &= -\lambda^3 + 14\lambda^2 - 42\lambda \\ &= -\lambda\left(\lambda^2 - 14\lambda + 42\right).\end{aligned}$$

Thus, $\lambda = 0$, or $\lambda = 7 \pm \sqrt{7}$.

$$\|A\|_2 = \sqrt{\rho\left(A^{\mathrm{T}}A\right)} = \sqrt{\max\left\{0, 7-\sqrt{7}, 7+\sqrt{7}\right\}} = \sqrt{7+\sqrt{7}}.$$

$\|A\|_\infty$

We first provide a formula for the l^∞-norm of a matrix, as it comes in quite handy when we need to compute any induced matrix norms.

Theorem. *Let A be an $n \times n$ square matrix. Then,*

$$\|A\|_\infty \overset{\textit{definition}}{=} \max_{\|\boldsymbol{x}\|\neq 0} \|A\boldsymbol{x}\| = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|,$$

namely, the maximum row sum of absolute values of the entries.

Proof. See Theorem 7.11 in textbook. The proof involves establishing $\|A\|_\infty \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$ and $\|A\|_\infty \geq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$. □

Condition Number of a Matrix (Section 7.5)

With a formula to compute $\|A\|_2$ via the spectral radius, we can then define the condition number of a matrix $\kappa(A)$. Before that, we consider a simple system of equations to showcase the importance of knowing the condition numbers, before we use the matrix to solve any problems.

Example. Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$ given by

$$\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}$$

has the uniques solution $\boldsymbol{x} = (1,1)^{\mathrm{T}}$. Determine the residual vector for the poor approximation $\widetilde{\boldsymbol{x}} = (3, -0.0001)^{\mathrm{T}}$, define as $\boldsymbol{r} = \boldsymbol{b} - A\widetilde{\boldsymbol{x}}$.

Solution. We have

$$\boldsymbol{r} = \boldsymbol{b} - A\widetilde{\boldsymbol{x}} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -0.0001 \end{bmatrix} = \begin{bmatrix} 0.0002 \\ 0 \end{bmatrix}$$

and observe that $\|\boldsymbol{r}\|_\infty = 0.0002$, which is reasonably small. However, obviously the approximation is very poor,

$$\|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\|_\infty = 2.$$

In the previous example, we see $\|\boldsymbol{r}\|_\infty = \|A\boldsymbol{x} - A\widetilde{\boldsymbol{x}}\|_\infty$ as the "perturbation to input", while $\|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\|_\infty$ as perturbation to output. So, if we can somehow relate them, we have an estimate of the "condition number" of the matrix A.

Theorem. *Suppose $\widetilde{\boldsymbol{x}}$ is an approximation to the solution of $A\boldsymbol{x} = \boldsymbol{b}$, where A is nonsingular, and $\boldsymbol{r}$ is the residual vector for $\tilde{\boldsymbol{x}}$. Then, for any induced norm, for $\boldsymbol{x} \neq 0$ and $\boldsymbol{b} \neq 0$, we have*

$$\frac{\|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|} \leq \|A\| \left\|A^{-1}\right\| \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|}.$$

Remark. We skip the proof (See Theorem 7.27). However, this important result leads us to the definition of the condition number of a matrix $\kappa(A)$. Note that the LHS $\frac{\|\boldsymbol{x}-\widetilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|}$ is **relative error in output** (solution), and the RHS is

$$\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|}=\frac{\|\boldsymbol{b}-A\widetilde{\boldsymbol{x}}\|}{\|\boldsymbol{b}\|}$$

can be viewed as a **relative error in input**. So, we may define the **relative condition number** of A as

$$\kappa(A)=\|A\|\left\|A^{-1}\right\|.$$

By a property of the matrix norm (sub-multiplicativity), we have

$$\kappa(A)=\|A\|\left\|A^{-1}\right\|\geq\left\|AA^{-1}\right\|=\|I\|=1.$$

We say that a matrix is **well-condition** if $\kappa(A)$ is close to 1, and is **ill-conditioned** when $\kappa(A)$ is significantly greater than 1.

Example. Determine the condition number of the matrix in the last example.

Solution. We are allowed to use any induced norm, so let's do the easiest one, $\|A\|_\infty$.

$$\|A\|_\infty=\max\{|1|+|2|,|1.0001|+|2|\}=3.0001$$

which is not too big. However,

$$A^{-1}=\begin{bmatrix}-10000 & 10000\\ 5000.5 & -5000\end{bmatrix}$$

and thus

$$\left\|A^{-1}\right\|_\infty=20000.$$