Iterative Methods: an Introduction

In the previous section, we have studied direct methods of solving linear systems, in the sense that the error in our numerical solution arises purely from round-off errors. In this section, we study iterative methods, namely, approximating the true solution closer and closer, but only get close enough with a prescribed tolerance level.

Example. Suppose an iterative algorithm produces $x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)$ as an approximate solution to a linear system at the k^{th} iteration.

If we know the true solution is x, we want to see how close we are. We need a measure of the "difference" $\left(x^{(k)} - x\right)$.

If we don't know the true solution (almost always), we want to see how much we have improved from the previous step, that is, we want to look at $\left(x^{(k)} - x^{(k-1)}\right)$ and see how this "difference" is changing as k changes. Certainly, if this "difference" becomes smaller and smaller, and if we can also prove (in mathematical analysis) that this "difference" goes to 0 as $k \to \infty$, then we know the sequence is **convergent**.

Definition. A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ to $\mathbb{R}$ with the following properties:

(1) (nonnegativity) $\|\boldsymbol{x}\| \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$,
(2) (positive definiteness/point-separating) $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$,
(3) (absolute homogeneity) $\|\alpha \boldsymbol{x}\| = |\alpha| \, \|\boldsymbol{x}\|$ for all $\alpha \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$,
(4) (triangle inequality/subadditivity) $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

We have a convention that a vector in $\mathbb{R}^n$ is a **column vector**. Then a **row vector** may be represented by the **transpose** of a vector, that is,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_5 \end{bmatrix} = (x_1, \ldots, x_5)^{\mathrm{T}}$$

where transpose simply means switching the dimension of the array.

Definition. The l^2 and l^∞ norms for the vector $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ are defined by

$$\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \quad \|\boldsymbol{x}\|_\infty = \max_{1 \leq i \leq n} |x_i| \, .$$

The l^2-norm is also called **Euclidean norm** because it represents the usual notion of distance from the origin. You may deduce that the vectors that satisfies $\|\boldsymbol{x}\|_2 \leq 1$ cover a circle of radius 1 ($x_1^2 + x_2^2 \leq 1$), in 2D; or a sphere of radius 1 ($x_1^2 + x_2^2 + x_3^2 \leq 1$), in 3D. One should also note that the dot product of a vector to itself is

$$\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} = \sum_{i=1}^{n} x_i^2 = \|\boldsymbol{x}\|_2^2 \, .$$

The l^∞-norm also has a geometric feature. It represents squares in 2D and cubes in 3D.

Example. Compute the l^2 and l^∞-norm of the vector $\boldsymbol{x} = (-1, 1, -2)^{\mathrm{T}}$.

$$\|\boldsymbol{x}\|_2 = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6},$$

and

$$\|\boldsymbol{x}\|_\infty = \max\{|-1|, |1|, |-2|\} = 2.$$

In the definition of l^2 and l^∞-norm, we haven't really proved that they are indeed norms, that is, they satisfy the four properties given in the first definition. In fact, showing that l^∞ is a norm is not hard.

Proposition. *l^∞ is a norm.*

Proof. We check the four criteria.

(1) (nonnegativitiy) For $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|\boldsymbol{x}\|_\infty = \max_{1\le i\le n} \{|x_i|\} \ge 0$$

since the absolute value is always nonnegative.

(2) (positive definiteness) For $\boldsymbol{x} = \mathbf{0}$, then clearly, $\|\boldsymbol{x}\|_\infty = \|\mathbf{0}\|_\infty = \max_{1\le i\le n} \{|0|\} = 0$. Suppose now $\|\boldsymbol{x}\|_\infty = 0$, then $\max_{1\le i\le n} \{|x_i|\} = 0$, which implies $x_i = 0$ for all $i = 1, \ldots, n$.

(3) (absolute homogeneity) For $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|\alpha\boldsymbol{x}\|_\infty = \max_{1\le i\le n} \{|\alpha x_i|\} = |\alpha| \max_{1\le i\le n} \{|x_i|\} = |\alpha| \, \|\boldsymbol{x}\|_\infty .$$

(4) (triangle inequality)

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|_\infty &= \max_{1\le i\le n} \{|x_i + y_i|\} \\
&\le \max_{1\le i\le n} (|x_i| + |y_i|) \\
&\le \max_{1\le i\le n} |x_i| + \max_{1\le i\le n} |y_i| \\
&= \|\boldsymbol{x}\|_\infty + \|\boldsymbol{y}\|_\infty
\end{aligned}$$

$\square$