Operation Counts

Suppose, we are trying to use the i^{th} row to remove the coefficient of x_i in all subsequent rows ($i+1, i+2$ up to n), what does the "modified" matrix look like before we begin?

$$\widetilde{A}^{(i)} = \left[\begin{array}{cccccccccccccc|c}
a_{11} & a_{12} & a_{13} & . & . & a_{1,i-1} & a_{1i} & . & . & . & . & . & a_{1n} & a_{1,n+1} \\
0 & a_{22} & a_{23} & . & . & a_{2,i-1} & a_{2i} & . & . & . & . & . & a_{2n} & a_{2,n+1} \\
0 & 0 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & a_{i-1,i-1} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 0 & \boxed{a_{ii}} & a_{i,i+1} & . & . & . & . & a_{in} & a_{i,n+1} \\
. & . & . & . & . & 0 & a_{i+1,i} & a_{i+1,i+1} & . & . & . & . & a_{i+1,n} & a_{i+1,n+1} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & . & . & . & . & 0 & a_{ni} & a_{n,i+1} & . & . & . & . & a_{nn} & a_{n,n+1}
\end{array}\right].$$

So, in the i^{th} row, we have $(n+1)-(i-1) = n-i+2$ entries (assuming $a_{ii} \neq 0$); a sanity check for whether this formula is correct is by checking $i = 1$ – we must have $n-1+2 = n+1$ entries to begin with.

For each row E_j below, we need to find the multiplier,

$$m_{ji} = \frac{a_{ji}}{a_{ii}}, \quad j = i+1, i+2, \ldots, n,$$

that is, a total of $(n-i)$ divisions. Carrying out the multiplications for each j, i.e.

$$m_{ji}E_i,$$

we performed $(n-i+1)$ multiplications, where we do not count the first entry – it is always going to be a_{ji}. To perform this multiplication for each j, we have done

$$(n-i)(n-i+1)$$

multiplications. Therefore, to finish Gaussian elimination pivoted at a_{ii}, we perform a total of multiplication/divisions

$$(n-i) + (n-i)(n-i+1) = (n-i)(n-i+2)$$

times.

Then, $E_j - m_{ji}E_i$ does $(n-i+1)$ subtractions in each row, and does this $(n-i)$ times, that is,

$$(n-i+1)(n-i)$$

additions/subtractions.

Thus, to find the total number of multiplications and divisions, we add for $i = 1, \ldots, n-1$, in the sum

$$\begin{aligned}
\sum_{i=1}^{n-1} (n-i)(n-i+2) &= \sum_{i=1}^{n-1} (n-i)^2 + 2\sum_{i=1}^{n-1} (n-i) \\
&= \sum_{i=1}^{n-1} i^2 + 2\sum_{i=1}^{n-1} i \\
&= \frac{(n-1)\,n\,(2n-1)}{6} + 2\frac{(n-1)\,n}{2} \\
&= \frac{2n^3 + 3n^2 - 5n}{6}.
\end{aligned}$$

Similarly, total number of additions/subtractions is

$$\begin{aligned}\sum_{i=1}^{n-1}(n-i)(n-i+1) &= \sum_{i=1}^{n-1}(n-i)^2+\sum_{i=1}^{n-1}(n-i)\\ &= \sum_{i=1}^{n-1}i^2+\sum_{i=1}^{n-1}i\\ &= \frac{(n-1)\,n\,(2n-1)}{6}+\frac{(n-1)\,n}{2}\\ &= \frac{n^3-n}{3}.\end{aligned}$$

In backward substitution, the n^{th} requires one mere division, $x_{nn}=\frac{a_{n,n+1}}{a_{nn}}$. For the i^{th} row, we perform

$$x_i=\frac{a_{i,n+1}-\sum_{j=i+1}^{n}a_{ij}x_j}{a_{ii}}$$

which incurs $(n-(i+1)+1)=n-i$ multiplications to carry out $a_{ij}x_j$, $(n-(i+1))$ additions in the sum, one subtraction and then one division. We perform i from $n-1$ down to 1.

More precisely, in the backward substitution step, the total number of multiplication is

$$\begin{aligned}1+\sum_{i=1}^{n-1}(n-i+1) &= 1+\sum_{i=1}^{n-1}(n-i)+(n-1)\\ &= n+\sum_{i=1}^{n-1}i\\ &= n+\frac{n\,(n-1)}{2}\\ &= \frac{n^2+n}{2}.\end{aligned}$$

The total number of addition and subtractions is

$$\sum_{i=1}^{n-1}\left[(n-(i+1))+1\right]=\sum_{i=1}^{n-1}(n-i)=\sum_{i=1}^{n-1}i=\frac{n^2-n}{2}.$$

Altogether, with elimination AND substitution, we have

$$\begin{aligned}\text{(Multiplications/divisions)}\quad &\frac{2n^3+3n^2-5n}{6}+\frac{n^2+n}{2}=\frac{n^3}{3}+n^2-\frac{n}{3},\\ \text{(Additions/subtractions)}\quad &\frac{n^3-n}{3}+\frac{n^2-n}{2}=\frac{n^3}{3}+\frac{n^2}{2}-\frac{5n}{6}.\end{aligned}$$

We can see that for n large, total number of flops is $O\left(n^3\right)$.

Looking forward: The What-if's Fail

Numerical instability occurs when we divide by a small number. The large quotient, if used, produces large round-off errors. Consider the multiplier step in Gaussian elimination,

$$m_{ji}=\frac{a_{ji}}{a_{ii}}.$$

What if $a_{ii}\ll 1$, such that m_{ji} is very large. Then, m_{ji} is multiplied to each term of E_i which may result in large round-off error (remember, eps $\left(\text{single}\left(2^{40}\right)\right)$'s neighbor is some 10^5 units away). What's more, in back substitution, we have

$$x_i=\frac{a_{i,n+1}-\sum_{j=i+1}^{n}a_{ij}}{a_{ii}}$$

where a very small a_{ii} can lead to a large round-off error.

However, dividing by a **big** number is completely O.K. because the resulting small number is very fine machine representation, i.e., small round-off error. This desire to achieve numerical stability prompts the following modified version of Gaussian elimination.